Some New Trigonometric, Hyperbolic and Exponential Measures of Fuzzy Entropy and Fuzzy Directed Divergence.

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Abstract: New Trigonometric, Hyperbolic and Exponential Measures of Fuzzy Entropy and Fuzzy Directed Divergence are obtained and some particular cases have been discussed.

Index Terms: Fuzzy Entropy, Fuzzy Directed Divergence, Measures of Fuzzy Information.

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1. Introduction: Uncertainty and fuzziness are the basic nature of human thinking and of many real world objectives. Fuzziness is found in our decision, in our language and in the way we process information. The main use of information is to remove uncertainty and fuzziness. In fact, we measure information supplied by the amount of probabilistic uncertainty removed in an experiment and the measure of uncertainty removed is also called as a measure of information while measure of fuzziness is the measure of vagueness and ambiguity of uncertainties. Shannon [2] used "entropy" to measure uncertain degree of the randomness in a probability distribution. Let X is a discrete random variable with probability distribution $P = (p_{1,}p_{2,} \ldots\ldots, p_n)$ in an experiment. The information contained in this experiment is given by

$$H(P) = -\sum_{i=1}^{n} p_i log p_i \qquad (1)$$

Which is well known Shannon entropy.

The concept of entropy has been widely used in different areas, e.g. communication theory, statistical mechanics, finance, pattern recognition, and neural network etc. Fuzzy set theory developed by Lofti A. Zadeh [8] has found wide applications in many areas of science and technology, e.g. clustering, image processing, decision making etc. because of its capability to model non-statistical imprecision or vague concepts.

It may be recalled that a fuzzy subset A in U (universe of discourse) is characterized by a *membership function* $\mu_A: U \rightarrow [0,1]$ which represents the *grade of membership* of $x \in U$ *in* A as follows

$$\mu_A(x) = 0 \; if\; x\; does\; not\; belongs\; to\; A, \; and\; there\; is\; no\; uncertainty$$

$$= 1 \; if\; x\; belongs\; to\; A\; and\; there\; is\; no\; uncertainty$$

$$= 0.5 \; if\; maximum\; uncertainty$$

In fact $\mu_A(x)$ associates with each $x \in U$ a grade of membership in the set *A*. When $\mu_A(x)$ is valued in $\{0,1\}$ it is the characteristic function of a crisp (i.e. nonfuzzy) set. Since $\mu_A(x)$ and $1 - \mu_A(x)$ gives the same degree of fuzziness, therefore, corresponding to the entropy due to Shannon [2], De Luca and Termini [1] suggested the following measure of fuzzy entropy:

$$H(A) = -\left[\sum_{i=1}^{n} \mu_A(x_i) log \mu_A(x_i) + \sum_{i=1}^{n} \big(1 - \mu_A(x_i)\big) log\big(1 - \mu_A(x_i)\big)\right] \qquad (2)$$

De Luca and Termini introduced a set of properties and these properties are widely accepted as a criterion for defining any new fuzzy entropy. In fuzzy set theory, the entropy is a measure of fuzziness which expresses the amount of average ambiguity/difficulty in making a decision whether an element belongs to a set or not. So, a measure of average fuzziness in a fuzzy set should have at least the following properties to be valid fuzzy entropy:

i) $H(A) = 0$ when $\mu_A(x_i) = 0 \; or \; 1$.
ii) $H(A)$ increases as $\mu_A(x_i)$ increases from 0 to 0.5.
iii) $H(A)$ decreases as $\mu_A(x_i)$ increases from 0.5 to 1.
iv) $H(A) = H(\bar{A})$, i.e. $\mu_A(x_i) = 1 - \mu_A(x_i)$
v) $H(A)$ is a concave function of $\mu_A(x_i)$.

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Kullback and Leibler [7] obtained the measure of directed divergence of probability distribution $P = (p_1, p_2, \ldots\ldots, p_n)$ from the probability distribution $Q = (q_1, q_2, \ldots\ldots, q_n)$ as

$$D(P:Q) = \sum_{i=1}^{n} p_i \, log \frac{p_i}{q_i} \qquad (3)$$

Let A and B be two standard fuzzy sets with same supporting points $x_1, x_2, \ldots\ldots, x_n$ and with fuzzy vectors $\mu_A(x_1), \mu_A(x_2), \ldots\ldots, \mu_A(x_n)$ and $\mu_B(x_1), \mu_B(x_2), \ldots\ldots, \mu_B(x_n)$. The simplest measure of fuzzy directed divergence as suggested by Bhandari and Pal (1993), is

$$D(A:B) = \sum_{i=1}^{n} \mu_A(x_i) \, log \frac{\mu_A(x_i)}{\mu_B(x_i)} + \sum_{i=1}^{n} \big(1-\mu_A(x_i)\big) \, log \frac{\big(1-\mu_A(x_i)\big)}{\big(1-\mu_B(x_i)\big)} \qquad (4)$$

satisfying the conditions:

i) $D(A:B) \geq 0$

ii) $D(A:B) = 0 \; iff \; A = B$

iii) $D(A:B) = D(B:A)$

iv) $D(A:B)$ is a convex function of $\mu_A(x_i)$

later kapur [5],[6] introduced a number of trigonometric hyperbolic and exponential measures of fuzzy entropy and fuzzy directed divergence. In section 2 and 3 we introduce some new trigonometric, hyperbolic and exponential measures of fuzzy entropy and measures of fuzzy directed divergence.

2. New Measures of Fuzzy Entropy

2.1 Trigonometric Measure of Fuzzy Entropy

Consider the function $sin\pi x$ where $0 \leq x \leq 1$, is a convex function which gives us

$$H_1(A) = \sum_{i=1}^{n} sin\big(\pi\mu_A(x_i)\big) + \sum_{i=1}^{n} sin\Big(\pi\big(1-\mu_A(x_i)\big)\Big) \qquad (5)$$

is a new measure of fuzzy entropy.

in particular for $\beta \leq \pi$

$$H_2(A) = \sum_{i=1}^{n} sin\big(\beta\mu_A(x_i)\big) + \sum_{i=1}^{n} sin\Big(\beta\big(1-\mu_A(x_i)\big)\Big) - sin\beta \qquad (6)$$

is also a new measure of fuzzy entropy.

(5) is a special case of (6) when $\beta = \pi$.

Another special case of (6) arises when $\beta = \frac{\pi}{2}$ we get

$$H_3(A) = \sum_{i=1}^{n} sin\left(\frac{\pi}{2}\mu_A(x_i)\right) + \sum_{i=1}^{n} sin\left(\frac{\pi}{2}\big(1-\mu_A(x_i)\big)\right) - 1 \qquad (7)$$

Another trigonometric measure of fuzzy entropy is

$$H_4(A) = \sum_{i=1}^{n} sin(\beta\mu_A(x_i) + \alpha) + \sum_{i=1}^{n} sin\big(\beta\big(1-\mu_A(x_i)\big) + \alpha\big) - sin(\alpha+\beta) \qquad (8)$$

(8) reduces to (6) when $\alpha = 0$.

(8) reduces to (7) when $\alpha = 0, \beta = \frac{\pi}{2}$.

(8) reduces to (5) when $\alpha = 0, \beta = \pi$.

(8) is a 2-parameter measure of fuzzy entropy.

If we put $\propto = \frac{\pi}{2}$ we get

$$H_5(A) = \sum_{i=1}^{n} cos\big(\beta\mu_A(x_i)\big) + \sum_{i=1}^{n} cos\Big(\beta\big(1-\mu_A(x_i)\big)\Big) - cos\beta \qquad (9)$$

is a new measure of fuzzy entropy. Clearly above given measures of fuzzy entropy are satisfying all the properties which are given in section 1. So these are valid measures of fuzzy entropy.

2.1 Hyperbolic Measure of Fuzzy Entropy

$sinhx, coshx, tanhx$ where $0 \leq x \leq 1$ are all convex functions and gives us following valid measures of fuzzy entropy

$$H_6(A) = sinh\beta - \sum_{i=1}^{n} sinh\big(\beta\mu_A(x_i)\big) - \sum_{i=1}^{n} sinh\beta\big(1-\mu_A(x_i)\big) \quad (10)$$

$$H_7(A) = cosh\beta - \sum_{i=1}^{n} cosh\big(\beta\mu_A(x_i)\big) - \sum_{i=1}^{n} cosh\beta\big(1-\mu_A(x_i)\big) \quad (11)$$

$$H_8(A) = tanh\beta - \sum_{i=1}^{n} tanh\big(\beta\mu_A(x_i)\big) - \sum_{i=1}^{n} tanh\beta\big(1-\mu_A(x_i)\big) \quad (12)$$

Since $x^m sinhx, x^m coshx, x^m tanhx$ are also convex functions for $m \geq 1$, we get the following additional measures of fuzzy entropy.

$$H_9(A) = sinh\beta - \sum_{i=1}^{n} {\mu_A}^m(x_i) sinh\big(\beta\mu_A(x_i)\big) - \sum_{i=1}^{n}\big(1-\mu_A(x_i)\big)^m sinh\beta\big(1-\mu_A(x_i)\big) \quad (13)$$

$$H_{10}(A) = cosh\beta - \sum_{i=1}^{n} {\mu_A}^m(x_i) cosh\big(\beta\mu_A(x_i)\big) - \sum_{i=1}^{n}\big(1-\mu_A(x_i)\big)^m cosh\beta\big(1-\mu_A(x_i)\big) \quad (14)$$

$$H_{11}(A) = tanh\beta - \sum_{i=1}^{n} {\mu_A}^m(x_i) tanh\big(\beta\mu_A(x_i)\big) - \sum_{i=1}^{n}\big(1-\mu_A(x_i)\big)^m tanh\beta\big(1-\mu_A(x_i)\big) \quad (15)$$

2.2 Exponential Measures of Fuzzy Entropy

Since $x^m e^{ax}$ is a convex function when $m \geq 1, x > 0$ we get the measure of fuzzy entropy

$$H_{12}(A) = e^a - \sum_{i=1}^{n} {\mu_A}^m(x_i)\, e^{a\mu_A(x_i)} - \sum_{i=1}^{n}\big(1-\mu_A(x_i)\big)^m e^{a(1-\mu_A(x_i))} \quad (16)$$

3. New Measures of Fuzzy Directed Divergence

3.1 New Hyperbolic Measures of Fuzzy Directed Divergence

Using the convexity of $sinhx, coshx, tanhx$ we get the following measures of hyperbolic fuzzy

directed divergence.

$$D_1(A:B) = \sum_{i=1}^{n} \mu_B(x_i)\, sinh\left(\beta\frac{\mu_A(x_i)}{\mu_B(x_i)}\right) + \sum_{i=1}^{n}\big(1-\mu_B(x_i)\big)\, sinh\left(\beta\frac{\big(1-\mu_A(x_i)\big)}{\big(1-\mu_B(x_i)\big)}\right) - \mu_B(x_i) sinh\beta - \big(1-\mu_B(x_i)\big) sinh\beta \quad (17)$$

$$D_2(A:B) = \sum_{i=1}^{n} \mu_B(x_i)\, cosh\left(\beta\frac{\mu_A(x_i)}{\mu_B(x_i)}\right) + \sum_{i=1}^{n}\big(1-\mu_B(x_i)\big)\, cosh\left(\beta\frac{\big(1-\mu_A(x_i)\big)}{\big(1-\mu_B(x_i)\big)}\right) - \mu_B(x_i) cosh\beta - \big(1-\mu_B(x_i)\big) cosh\beta \quad (18)$$

$$D_3(A:B) = \sum_{i=1}^{n} \mu_B(x_i)\, tanh\left(\beta\frac{\mu_A(x_i)}{\mu_B(x_i)}\right) + \sum_{i=1}^{n}\big(1-\mu_B(x_i)\big)\, tanh\left(\beta\frac{\big(1-\mu_A(x_i)\big)}{\big(1-\mu_B(x_i)\big)}\right) - \mu_B(x_i) tanh\beta - \big(1-\mu_B(x_i)\big) \quad (19)$$

Again since $x^m sinhx, x^m coshx, x^m tanhx$ are also convex functions for $m \geq 1$, we get the following more general hyperbolic measures of fuzzy directed divergence.

$$D_4(A:B) = \sum_{i=1}^{n} \mu_A{}^m(x_i)\mu_B{}^{1-m}(x_i)\, sinh\left(\beta \frac{\mu_A(x_i)}{\mu_B(x_i)}\right) + \sum_{i=1}^{n} \left(1-\mu_A(x_i)\right)^m \left(1-\mu_B(x_i)\right)^m sinh\left(\beta \frac{\left(1-\mu_A(x_i)\right)}{\left(1-\mu_B(x_i)\right)}\right) - \mu_B(x_i) sinh\beta - \left(1-\mu_B(x_i)\right) sinh\beta \quad (20)$$

$$D_5(A:B) = \sum_{i=1}^{n} \mu_A{}^m(x_i)\mu_B{}^{1-m}(x_i)\, cosh\left(\beta \frac{\mu_A(x_i)}{\mu_B(x_i)}\right) + \sum_{i=1}^{n} \left(1-\mu_A(x_i)\right)^m \left(1-\mu_B(x_i)\right)^m cosh\left(\beta \frac{\left(1-\mu_A(x_i)\right)}{\left(1-\mu_B(x_i)\right)}\right) - \mu_B(x_i) cosh\beta - \left(1-\mu_B(x_i)\right) cosh\beta \quad (21)$$

$$D_6(A:B) = \sum_{i=1}^{n} \mu_A{}^m(x_i)\mu_B{}^{1-m}(x_i)\, tanh\left(\beta \frac{\mu_A(x_i)}{\mu_B(x_i)}\right) + \sum_{i=1}^{n} \left(1-\mu_A(x_i)\right)^m \left(1-\mu_B(x_i)\right)^m tanh\left(\beta \frac{\left(1-\mu_A(x_i)\right)}{\left(1-\mu_B(x_i)\right)}\right) - \mu_B(x_i) tanh\beta - \left(1-\mu_B(x_i)\right) tanh\beta \quad (22)$$

3.2 New Exponential Measures of Fuzzy Directed Divergence

Since $x^m e^{ax}$ is a convex function when $m \geq 1, x > 0$ we get the following measures of fuzzy directed divergence

$$D_7(A:B) = \sum_{i=1}^{n} \mu_A{}^m(x_i)\mu_B{}^{1-m}(x_i)\, e^{a\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)} + \sum_{i=1}^{n} \left(1-\mu_A(x_i)\right)^m \left(1-\mu_B(x_i)\right)^{1-m} e^{a\left(\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)}\right)} - e^a \quad (23)$$

Special case for m=0 and m=1 are

$$D_8(A:B) = \sum_{i=1}^{n} \mu_B{}^m(x_i)\, e^{a\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)} + \sum_{i=1}^{n} \left(1-\mu_B(x_i)\right) e^{a\left(\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)}\right)} - e^a \quad (24)$$

$$D_9(A:B) = \sum_{i=1}^{n} \mu_A{}^m(x_i)\, e^{a\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)} + \sum_{i=1}^{n} \left(1-\mu_A(x_i)\right) e^{a\left(\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)}\right)} - e^a \quad (25)$$

4. Conclusion

In section 2 and 3 by using the convexity of some trigonometric, hyperbolic and exponential function and satisfying the conditions of fuzzy entropy and fuzzy directed divergence we get some new trigonometric, hyperbolic and exponential measures of fuzzy entropy and fuzzy directed divergence.

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