sg-Interior and sg-Closure in Topological spaces
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Abstract: In this paper, we introduce sg-interior, sg-closure and some of its basic properties.

Keywords: sg-open; sg-closed; sg-int(A); sg-cl(A); sg-Hausdorff space.

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1. INTRODUCTION AND PRELIMINARIES

Levine [6] introduced generalized closed sets in topology as a generalization of closed sets. This concept was found to be useful and many results in general topology were improved. Many researchers like Arya et al [2], Balachandran et al [3], Bhattacharya et al [4], Arcockarani et al [1], Gnanambal [5], Malghan [7], Nagaveni [8] and Palaniappan et al [9] have worked on generalized closed sets. In this paper, the notion of sg-interior is defined and some of its basic properties are investigated. Also we introduce the idea of sg-closure in topological spaces using the notions of sg-closed sets and obtain some related results.

Throughout the paper, X and Y denote the topological spaces and respectively and on which no separation axioms are assumed unless otherwise explicitly stated.

Definition 1.1 A subset A of a space X is called

1) A preopen set if A ⊆ int(cl(A)) and a preclosed if cl(int(A)) ⊆ A

2) A regular open set if A = int(cl(A)) and regular closed set if A = cl(int(A))

3) A semi open set if A ⊆ cl(int(A)) and semi closed set if int(cl(A)) ⊆ A

The intersection of all preclosed subsets of X containing A is called pre-closure of A and is denoted by pcl(A)

Definition 1.2: A subset A of a space X is called

1) A g-closed set [6] if cl(A) ⊆ U whenever A ⊆ U and U is open in X

2) semi generalized closed set [4] if scl(A) ⊆ U whenever A ⊆ U and U is semi open in X.

3) generalized preclosed set [7] if clint(A) ⊆ U whenever A ⊆ U and U is open in X.

The complements of the above mentioned closed sets are their respective open sets.

Definition 1.3: Let X be a topological space and let x ∈ X. A subset N of X is said to be sg-neighbourhood of x if there exists a sg-open set G such that x ∈ G ⊂ N.

2. SG–CLOSURE AND INTERIOR IN TOPOLOGICAL SPACE.

Definition 2.1: Let A be a subset of X. A point x ∈ A is said to be sg-interior point of A if x is a sg-interior point of A.

In the paper, X and Y denote the topological spaces and respectively and on which no separation axioms are assumed unless otherwise explicitly stated.

Theorem 2.1: If A be a subset of X. Then sg-int(A) = ∪ { G : G is a sg-open, G ⊂ A }.

Proof: Let A be a subset of X.

X ∈ sg-int(A) ⇔ x is a sg-interior point of A.

⇔ A is a sg-nbd of point x.

⇔ there exists sg-open set G such that x ∈ G ⊂ A.

Hence sg-int(A) = ∪ { G : G is a sg-open, G ⊂ A }.

Theorem 2.2: Let A and B be subsets of X. Then

(i) sg-int(X) = X and sg-int(φ) = φ

(ii) sg-int(A) ⊆ A.

(iii) If B is any sg-open set contained in A, then B ⊂ sg-int(A).

(iv) If A ⊆ B, then sg-int(A) ⊆ sg-int(B).

(v) sg-int(sg-int(A)) = sg-int(A).
Theorem 2.5: If A and B are subsets of X, then sg-int(A ∩ B) ⊆ sg-int(A) ∩ sg-int(B).

Proof: We know that A ∩ B ⊆ A and A ∩ B ⊆ B. We have sg-int(A ∩ B) ⊆ sg-int(A) and sg-int(A ∩ B) ⊆ sg-int(B). This implies that sg-int(A ∩ B) ⊆ sg-int(A) ∩ sg-int(B).

Again let x ∈ sg-int(A) ∩ sg-int(B). Then x ∈ sg-int(A) and x ∈ sg-int(B). Hence x is a sg-int point of each of sets A and B. It follows that A and B is sg-nbhd of x, so their intersection A ∩ B is also a sg-nbhd of x. Hence x ∈ sg-int(A ∩ B). Thus x ∈ sg-int(A ∩ B) implies that x ∈ sg-int(A) ∩ B. Therefore sg-int(A) ∩ sg-int(B) ⊆ sg-int(A ∩ B) ----(1)

From (1) and (2),

We get sg-int(A ∩ B) = sg-int(A) ∩ sg-int(B).

Theorem 2.6: If A is a subset of X, then int(A) ⊆ sg-int(A).

Proof: Let A be a subset of X.

Let x ∈ int(A) ⇒ x ∈ ∪ {G : G is open, G ⊆ A}.

⇒ there exists an open set G such that x ∈ G ⊆ A.

⇒ there exists a sg-open set G such that x ∈ G ⊆ A, as every open set is a sg-open set in X .

⇒ x ∈ ∪ {G : G is sg-open, G ⊆ A}.

⇒ x ∈ sg-int(A).

Thus x ∈ int(A) ⇒ x ∈ sg-int(A). Hence int(A) ⊆ sg-int(A).

Remark 2.1: Containment relation in the above theorem may be proper as seen from the following example.

Example 2.2: Let X = [a,b,c] with topology τ = {X, ø, {b}, {b,c}}. Then sg-O(X) = {X, ø, {a}, {b}, {a,b}, {a,c}, {b,c}}.

Let A = [a,b] and int(A) = {b}. It follows that int(A) ⊆ sg-int(A) and int(A) ≠ sg-int(A).

Theorem 2.7: If A is a subset of X, then g-int(A) ⊆ sg-int(A), where g-int(A) is given by g-int(A) = ∪ {G : G is g-open, G ⊆ A}.

Proof: Let A be a subset of X.

Let x ∈ int(A) ⇒ x ∈ ∪ {G : G is g-open, G ⊆ A}.

⇒ there exists a g-open set G such that x ∈ G ⊆ A.

⇒ there exists a sg-open set G such that x ∈ G ⊆ A, as every g-open set is a sg-open set in X.
\[ \Rightarrow x \in \bigcup \{ G : G \text{ is sg-open, } G \subseteq A \}. \]

\[ x \in \text{sg-int}(A). \]

Hence \( g \text{-int}(A) \subseteq \text{sg-int}(A). \)

**Remark 2.2:** Containment relation in the above theorem may be proper as seen from the following example.

**Example 2.3:** Let \( X = \{a,b,c\} \) with topology \( \tau = \{X, \varnothing, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}. \) Then \( \text{sg-o}(X) = \{X, \varnothing, \{c\}, \{a\}, \{a,b\}, \{a,c\}, \{b,c\}\}. \) & \( g \text{- open} (X) = \{X, \varnothing, \{a\}, \{c\}, \{a,c\}\}. \) Let \( A = \{b,c\}, \) \( \text{sg-int}(A) = \{b,c\} \) & \( g \text{-int}(A) = \{c\}. \) It follows \( g \text{-int}(A) \subseteq \text{sg-int}(A) \) and \( g \text{-int}(A) \neq \text{sg-int}(A). \)

**Definition 2.2:** Let \( A \) be a subset of a space \( X. \) We define the \( \text{sg-} \) closure of \( A \) to be the intersection of all \( \text{sg-} \) closed sets containing \( A. \) In symbols, \( \text{sg-cl}(A) = \bigcap \{ F : A \subseteq F \subseteq \text{sgc}(X) \}. \)

**Theorem 2.8:** If \( A \) and \( B \) are subsets of a space \( X. \) Then

(i) \( \text{sg-cl}(X) = X \) and \( \text{sg-cl}(\varnothing) = \varnothing \)

(ii) \( A \subseteq \text{sg-cl}(A). \)

(iii) If \( B \) is any \( \text{sg-} \) closed set containing \( A, \) then \( \text{sg-cl}(A) \subseteq \text{sg-cl}(B). \)

(iv) If \( A \subseteq B \) then \( \text{sg-cl}(A) \subseteq \text{sg-cl}(B). \)

**Proof:** (i) By the definition of \( \text{sg-} \) closure, \( X \) is the only \( \text{sg-} \) closed set containing \( X. \) Therefore \( \text{sg-cl}(X) = \text{sgcl}(X) = X. \) That is \( \text{sg-cl}(X) = X. \) By the definition of \( \text{sg-} \) closure, \( \text{sg-cl}(\varnothing) = \bigcap \{ \varnothing \} = \varnothing. \) That is \( \text{sg-cl}(\varnothing) = \varnothing. \)

(ii) By the definition of \( \text{sg-} \) closure of \( A, \) it is obvious that \( A \subseteq \text{sg-cl}(A). \)

(iii) Let \( B \) be any \( \text{sg-} \) closed set containing \( A. \) Since \( \text{sg-cl}(A) \) is the intersection of all \( \text{sg-} \) closed sets containing \( A, \) \( \text{sg-cl}(A) \) is contained in every \( \text{sg-} \) closed set containing \( A. \) Hence in particular \( \text{sg-cl}(A) \subseteq B. \)

(iv) Let \( A \) and \( B \) be subsets of \( X \) such that \( A \subseteq B. \) By the definition \( \text{sg-cl}(B) = \bigcap \{ F : B \subseteq F \subseteq \text{sgc}(X) \}. \) If \( B \subseteq F \subseteq \text{sgc}(X), \) then \( \text{sg-cl}(B) \subseteq F. \) Since \( A \subseteq B, \) \( A \subseteq F \subseteq \text{sgc}(X), \) we have \( \text{sg-cl}(A) \subseteq F. \) There fore \( \text{sg-cl}(A) \subseteq \bigcap \{ F : B \subseteq F \subseteq \text{sgc}(X) \} = \text{sg-cl}(B). \)

(i.e) \( \text{sg-cl}(A) \subseteq \text{sg-cl}(A). \)

**Theorem 2.9:** If \( A \subseteq X \) is \( \text{sg-} \) closed, then \( \text{sg-cl}(A) = A. \)

**Proof:** Let \( A \) be \( \text{sg-} \) closed subset of \( X. \) We know that \( A \subseteq \text{sg-cl}(A). \) Also \( A \subseteq A \) and \( A \) is \( \text{sg-} \) closed. By theorem (iii) \( \text{sg-cl}(A) \subseteq A. \) Hence \( \text{sg-cl}(A) = A. \)

**Remarks 2.3:** The converse of the above theorem need not be true as seen from the following example.

**Example 2.4:** Let \( X = \{a,b,c\} \) with topology \( \tau = \{X, \varnothing, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}. \) Then \( \text{sg-cl}(X) = \{X, \varnothing, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}. \) \( \text{sg-cl}(\{b\}) = \{b\}. \) But \( \{b\} \) is not \( \text{sg-} \) closed in \( X. \)

**Theorem 2.10:** Let \( A \) and \( B \) be subsets of a space \( X. \) Then \( \text{sg-cl}(A \cap B) \subseteq \text{sg-cl}(A) \cap \text{sg-cl}(B). \)

**Proof:** Let \( A \) and \( B \) be subsets of \( X. \) Clearly \( A \cap B \subseteq A \) and \( A \cap B \subseteq B. \)

By theorem \( \text{sg-cl}(A \cap B) \subseteq \text{sg-cl}(A) \) and \( \text{sg-cl}(A \cap B) \subseteq \text{sg-cl}(B). \)

Hence \( \text{sg-cl}(A \cap B) \subseteq \text{sg-cl}(A) \cap \text{sg-cl}(B). \)

**Theorem 2.11:** If \( A \) and \( B \) are subsets of a space \( X \) then \( \text{sg-cl}(A \cup B) = \text{sg-cl}(A) \cup \text{sg-cl}(B). \)

**Proof:** Let \( A \) and \( B \) be subsets of \( X. \) Clearly \( A \subseteq A \cup B \) and \( B \subseteq A \cup B. \) We have \( \text{sg-cl}(A \cup B) = \text{sg-cl}(A) \cup \text{sg-cl}(B). \)
If \( A \subset F \in C(X) \), then \( A \subset F \in sg-C(X) \), because every closed set is sg-closed. That is \( sg-cl(A) \subset F \). There fore \( sg-cl(A) \subset \bigcap \{ F \subset X : F \in C(X) \} = cl(A) \).

Hence \( sg-cl(A) \subset cl(A) \).

**Remark 2.4:** Containment relation in the above theorem may be proper as seen from the following example.

**Example 2.5:** Let \( X = \{ a, b, c \} \) with topology \( \tau = \{ X, \varnothing, \{ a \}, \{ b \}, \{ c \}, \{ a, b \}, \{ b, c \} \} \).

Then \( sg-cl(X) = \{ X, \varnothing, \{ a \}, \{ b \}, \{ c \}, \{ a, b \}, \{ b, c \} \} \) and \( g-cl(X) = \{ X, \varnothing, \{ b \}, \{ a, b \}, \{ b, c \} \} \).

Let \( A = \{ b, c \}, \) \( sg-cl(A) = \{ b, c \} \) and \( g-cl(A) = \{ b \} \).

It follows \( g-cl(A) \subset sg-cl(A) \) and \( g-cl(A) \neq sg-cl(A) \).

**Theorem 2.14:** If \( A \) is a subset of \( X \), then \( sg-cl(A) \subset g-cl(A) \), where \( g-cl(A) \) is given by \( g-cl(A) = \bigcap \{ F \subset X : A \subset F \) and \( f \) is a g-closed set in \( X \).

**Proof:** Let \( A \) be a subset of \( X \). By definition of \( g-cl(A) = \bigcap \{ F \subset X : A \subset F \) and \( f \) is a g-closed set in \( X \) \). If \( A \subset F \) and \( F \) is a g-closed subset of \( x \), then \( A \subset F \in sg-cl(X) \), because every g closed is sg-closed subset in \( X \). That is \( sg-cl(A) \subset F \).

Therefore \( sg-cl(A) \subset \bigcap \{ F \subset X : A \subset F \) and \( f \) is a g-closed set in \( X \) = \( g-cl(A) \).

Hence \( sg-cl(A) \subset g-cl(A) \).

**Corrolary 2.1:** Let \( A \) be any subset of \( X \). Then

(i) \( sg-int(A)^c = sg-cl(A)^c \)

(ii) \( sg-int(A) = (sg-cl(A))^c \)

(iii) \( sg-cl(A) = (sg-cl(A))^c \)

**Proof:** Let \( x \in sg-int(A)^c \). Then \( x \not\in sg-int(A) \). That is every sg-open set \( U \) containing \( x \) is such that \( U \not\subset A \).

That is every sg-open set \( U \) containing \( x \) is such that \( U \cap A \neq \varnothing \). By theorem \( x \in sg-int(A)^c \) and there fore \( sg-int(A)^c \subset sg-cl(A)^c \).

Conversely, let \( x \in sg-cl(A)^c \).

Then by theorem, every sg-open set \( U \) containing \( x \) is such that \( U \cap A \neq \varnothing \). That is every sg-open set \( U \) containing \( x \) is such that \( U \not\subset A \).

That is \( x \in sg-int(A)^c \) and \( sg-cl(A)^c \subset (sg-int(A))^c \).

Thus \( sg-int(A)^c = sg-cl(A)^c \).

(i) Follows by taking complements in (i).

(ii) Follows by replacing \( A \) by \( A^c \) in (i).

### 3. Preservation theorems concerning \( g \)-Hausdorff and \( sg \)-Hausdorff spaces

In this section we investigate preservation theorems concerning \( g \)-Hausdorff spaces.

**Defintion 3.1:** A topological space \( X \) is said to be \( g \)-Hausdorff if whenever \( x \) and \( y \) are distinct points of \( X \) there are disjoint \( g \)-open sets \( U \) and \( V \) with \( x \in U \) and \( y \in V \).

It is obvious that every Hausdorff space is \( g \)-Hausdorff space. The following example shows that the converse is not true.

**Example 3.1:** Let \( X = \{ a, b, c \} \) and \( \tau = \{ X, \varnothing, \{ a \} \} \). It is clear that \( X \) is not Hausdorff Space. Since \( \{ a \}, \{ b \} \) and \( \{ c \} \) are all \( g \)-open, it follows that \( H \) is \( g \)-Hausdorff Space.

**Theorem 3.1:** Let \( X \) be a topological space and \( Y \) be Hausdorff. If \( f: X \to Y \) is injective and \( g \)-continuous, then \( x \) is \( g \)-Hausdorff.

**Proof:** Let \( x \) and \( y \) be any two distinct points of \( X \). Then \( f(x) \) and \( f(y) \) are distinct points of \( Y \), because \( f \) is injective. Since \( Y \) is Hausdorff, there are disjoint open sets \( U \) and \( V \) in \( Y \) containing \( f(x) \) and \( f(y) \) respectively. Since \( f \) is \( g \)-continuous and \( U \cap V = \varnothing \), we have \( f(U) \) and \( f(V) \) are disjoint \( g \)-open sets in \( X \) such that \( x \in f(U) \) and \( y \in f(V) \). Hence \( X \) is \( g \)-Hausdorff space.

**Definition 3.2:** A topological space \( X \) is said to be \( g \)-Hausdorff Space if whenever \( x \) and \( y \) are distinct points of \( X \) there are disjoint \( g \)-open sets \( U \) and \( V \) with \( x \in U \) and \( y \in V \).

It is obvious that every \( g \)-Hausdorff space is a \( g \)-Hausdorff space. The following example shows that the converse is not true.

**Example 3.1:** Let \( X = \{ a, b, c \} \) and \( \tau = \{ X, \varnothing, \{ a \} \} \). Since \( \{ a \}, \{ b \} \) and \( \{ c \} \) are all \( g \)-open, then \( X \) is \( g \)-Hausdorff space. Since \( \{ a \}, \{ b \} \) and \( \{ c \} \) are not \( g \)-open in \( X \), it follows that \( \{ a \} \) and \( \{ c \} \) can not be separated by any two disjoint \( g \)-open sets in \( X \).

Hence \( X \) is not \( g \)-Hausdorff Space.

**Theorem 3.2:** Let \( X \) be a topological space \( Y \) be Hausdorff space. If \( f: X \to Y \) is injective and \( g \)-continuous, then \( X \) is \( g \)-Hausdorff Space.

**Proof:** Let \( x \) and \( y \) be any two distinct points of \( X \). Then \( f(x) \) and \( f(y) \) are distinct points of \( Y \), because \( f \) is injective. Since \( Y \) is Hausdorff, there are disjoint open sets \( U \) and \( V \) in \( Y \) containing \( f(x) \) and \( f(y) \) respectively. Since \( f \) is \( g \)-continuous and \( U \cap V = \varnothing \), we have \( f(U) \) and \( f(V) \) are disjoint \( g \)-open sets in \( X \) such that \( x \in f(U) \) and \( y \in f(V) \). Hence \( X \) is \( g \)-Hausdorff space.

**Theorem 3.3:** Let \( X \) be a topological space \( Y \) be Hausdorff space. If \( f: X \to Y \) is injective and \( g \)-irresolute, then \( X \) is \( g \)-Hausdorff space.

**Proof:** Let \( x \) and \( y \) be any two distinct points of \( X \). Then \( f(x) \) and \( f(y) \) are distinct points of \( Y \), because \( f \) is injective. Since \( Y \) is Hausdorff, there are disjoint open sets \( U \) and \( V \) in \( Y \) containing \( f(x) \) and \( f(y) \) respectively. Since \( f \) is \( g \)-irresolute and \( U \cap V = \varnothing \), we have \( f(U) \) and \( f(V) \) are disjoint \( g \)-open sets in \( X \) such that \( x \in f(U) \) and \( y \in f(V) \). Hence \( X \) is \( g \)-Hausdorff space.

### 4. Conclusion

From the definitions of \( g \)-Hausdorff space and \( sg \)-Hausdorff space, we have result.
X is a Hausdorff Space $\implies$ X is a $g$-Hausdorff Space $\implies$ X is a sg- Hausdorff Space.

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