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Two-Sided α-derivations on Left Nearrings

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Abstract:

H.E.Bell and G.Mason[1] proved that if D is a derivation on left near ring N satisfying D(N) ⊆ Z or [D(x),D(y)]=0 for all x,y ϵ N then (N,+) is abelian. In [2], Bell and kappe proved that if d is a derivation of semiprime ring R which is either an endomorphism or anti-endomorphism then d=0. Argafi genaralized this result for a semiprime near ring in [3]. In this paper, we prove that (N,+) is abelian if d(x+y-x-y)=0 and if d+d is additive on I.

Key words:

Near-ring, Derivation, semiprime ring,□(□, 1)-derivation,(1; □)-derivation, two-sided □-derivation

Introduction:

An additive map d:N→N is a derivation if d(xy)=xd(y)+d(x)y for all x,yϵ N or equivalently that d(xy)=d(x)y+x d(y) for all x,yϵ N.

A set N together two binary operations '+' and '.' is called (left) nearring. If

(i) N is a group (not necessarily abelian) under addition.

(ii) Multiplication is associative (so N is a semigroup under multiplication)

(iii) Multiplication distributives over addition on the left for any x,y,z in N, it holds that x.(y+z)=x.y+x.z.

A Nearring N is said to be prime if xNy={0} for n,yϵ N implies x=0. A non-empty subset I of N will be called a semi group ideal if IN⊆I and NI⊆I, if d is a derivation of a semigroup ring R which is either an endomorphism or anti-endomorphism, then d=0.

An additive mapping f:N→N is called a (α, β)-derivation if there exist functions α,β:N→N such that d(xy)=f(x)α(y)+β(x) f(y) for all x,yϵN. An additive mapping d:N→N is called a two –sided α-derivation if d is an (α,1)-derivation as well as (1,α)-derivation. For α=1, a two-sided α-derivation.

Preliminaries:

Lemma 1: Let N be a prime nearring and I a nonzero semigroup ideal of N. If u+v=v+u for all u,v∈ I, then (N,+) is abelian.

Proof: By the hypothesis, we have ux+uy=uy+ux for all u∈I and x,y ∈ N

Then we get u(x+y-x-y)=0 for all u∈ I and x,y∈ N.

It means that I(x+y-x-y)=NI(x-y-x-y)=0.

Since I is a nonzero semigroup ideal we have x+y-x-y=0 for all x,y ∈ N by the primeness of N.

Thus (N,+) is abelian.□

Lemma 2: Let N be a left nearring, d a (α,1)-derivation of N and I a multiplicative semigroup of N which contains 0. If d acts as an anti-homomorphism on I and α(0)=0,then 0x=0 for all x∈ I.

Proof: since x0=0 for all x∈I and d acts as an anti-homomorphism on I it is clear that 0d(x)=0 for all x∈I.

Taking 0x instead of x ,one can obtain d(x)α(0)+0x=0 for all x∈I.

Thus we have 0x=0 for all x∈I.□

Lemma 3: Let N be a nearring and be a multiplicative sub semigroup of N. If d is a two-sided α-derivation of N such that α(xy)=α(x)α(y) for all x,y∈ I then (d(x)α(y)+xd(y))n = d(x)α(y)n+xd(y)n for all n,x,y∈ I. Further-more, if α(I)=I, then(d(x)y+α(x)d(y))n= d(x)yn + α(x)d(y)n for all n,x.

Lemma 4: Let N be a prime nearring and I a nonzero semigroup ideal of N. Let d be a nonzero (α,1)-derivation on N such that α(xy)=α(x)α(y) for all x,y∈ I. If x∈ N and d(I)x={0}, then x=0.

Proof: Assume that d(I)x=0.

Then d(uy)x=0 for all y∈ N,u∈ I.

Hence 0=(d(u)α(y)+ud(y))x=ud(y)x for all y∈ NJ,u∈ I

Since I is a nonzero semigroup ideal and d is non-zero, it is clear that x=0 by the primeness of N.□

Lemma 5: Let N be a prime nearring and I a non-zero semigroup ideal of N and d a nonzero (α,1)-derivation on N. If d(x+y-x-y)=0for all x,yϵ I,then d(z)(x+y-x-y)=0 for all x,y,zϵ I.

Proof: Assume that d(x+y-x-y) =0for all x,yϵ I.

Let us take yz and xz instead of y and x, where zϵ I respectively.

Then

0= d(z(x+y-x-y))

= α(z)d(x+y-x-y)+d(z)(x+y-x-y)

= d(z)(x+y-x-y) for all x,y,z ϵ I.□

Lemma 6: Let N be a nearring and I be a multiplicative sub semigroup of N. Let d be a (α,1)-derivation of N such that α(xy)=α(x)α(y) for all x,y ϵ I and α(I)=I

(i) If d acts as a homomorphism on I, then

x d(y)d(y) = xyd(y) = xd(y)α(y) for all x,yϵ I.

(ii) If d acts as an anti-homomorphism on I, then

xd(y)d(y)= xyd(y) = d(y)xα(y) for all x,yϵ I.

Proof: (i) Let d acts as a homomorphism on I. Then

D(yx)=d(y)α(x)++yd(x)=d(y)d(x) for all x,yϵ I (1)

Substituting xy for x in (1), we have

d(y)α(xy)+y d(xy)=d(y)d(xy)=d(xy)d(y) for all x,yϵ I (2)

By lemma (3), we have

d(y)d(xy)=d(y)d(x)α(y)+d(y)xd(y)=d(yx)α(y)+d(y)xd(y)

using this relation in (2) ,we get

xyd(y)=xd(y)d(y)

Similarly, taking xy instead of y in (1), we obtain

d(yx)=d(xy)α(x)+xyd(x)=d(xy)d(x) for all x,yϵ I (3)

on the other hand

d(xy)d(x)=(d(x)α(y)+xd(y))d(x)=d(x)α(y)d(x)+xd(y)d(x)=d(x)α(y)d(x)+xd(yx)

using this relation in (3), we get

d(xy)α(x)=d(x)d(y)α(x)=d(x)α(x)d(y)

since α(I)=I it is clear that d(x)wd(y)=d(x)wα(y) for all x,y,wϵ I

(ii) Since d acts as an anti-homomorphism on I, we have

d(yx)=d(y)α(x)+yd(x)=d(x)d(y) for all x,yϵ I (4)

taking yx for y in (4) ,we get

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$d(yx)\alpha(x)+yxd(x) = d(x)d(yx)$

$= d(x)(d(y)\alpha(X)+yd(x))$

$= d(x)d(y)\alpha(x)+d(x)yd(x)$

$= d(xy)\alpha(x)+d(x)yd(x)$ for all $x,y\in I$

From this relation we get $d(yx)\alpha(x)=d(xy)\alpha(x)$.

Since $\alpha(I)=I$ we get

$d(x)\alpha(x)y=d(x)yd(x)$ for all $x,y \in I$.

Similarly, taking yx instead of x in (4), one can prove the relation

$xd(y)d(y)=xyd(y)$.□

Main results:

Theorem 1: Let N be a semiprime nearring and I be a subset of N such that $0\in I$ and $IN\subseteq I$. Let d be a two sided α-derivation on N such that $\alpha(I)=I$ and $\alpha(xy)=\alpha(x)\alpha(y)$ for all $x,y \in I$

(i) If d acts as a homomorphism on I ,then $d(I)=\{0\}$

(ii) If d acts as an anti-homomorphism on I and $\alpha(0)=0$,then $d(I)=\{0\}$

Proof: (i) Suppose that d acts as a homomorphism on I. By lemma(6), we have

$xd(y)d(y)=xd(y)\alpha(y)$ for all $x,y \in I$ (5)

by multiplying left side of (5) with d(z), where $z\in I$, and using the hypothesis that d acts as a homomorphism on I together with lemma(3) ,we obtain

$zd(y)xd(y)=0$ for all $x,y,z \in I$

Taking xn instead of x, where $n\in N$, we get

$zd(y)xnd(y)=0$ for all $x,y,z \in I$ and $n\in N$

In particular, $xd(y)xNd(y)=\{0\}$.

By the semiprimeness of N we conclude $xd(y) = 0$.

Since $\alpha(I)=I$, it is clear that

$\alpha(x)d(y)=0$ for all $x,y\in I$ (6)

Substituting yn for y in (6), and right multiplying (6) by d(z),where $z\in I$,we get

$\alpha(x)nd(y)d(z)+d(x)\alpha(n)\alpha(y)d(z)=0$.

Since the second summand is zero by (6) we get

$0 = \alpha(x)nd(y)d(z) = \alpha(x)nd(yz) = \alpha(x)nd(y)\alpha(z)+\alpha(x)nyd(z)$,

that is $xnyd(z)=0$ for all $x,y,z \in I$,$n\in N$.

Since N is semiprime, we have

$yd(z)=0$ for all $y,z\in I$ (7)

Combining (6) and (7) shows that

d(yz)=0 for all y,zϵ I.

In particular, d(xnx)=0 for all xϵ I,nϵ n; and since d acts as a homomorphism on I ,we have

0=d(xn)d(x)=d(x)nd(x)+α(x)d(n)d(x)

Since α (I) = I, the second summand is zero by (7) we have

d(x)=0 for all xϵ I

(ii) Now assume that d acts as an anti-homomorphism on I.

Note that 0a=0 for all aϵ I by lemma (2)

According to lemma (6), we have

xyd(y) = xd(y)d(y) for all x,y ϵ I (8)

d(y)α(y)x = xd(y)d(y) for all x,yϵ I (9)

Replacing x by xd(y) in (8) and using lemma (6), we get

xd(y)yd(y) = d(Y)xd(Y^2)

= d(y)x(d(y)α(y)+yd(y))

= d(y)xd(y)α(y)+d(y)xyd(y)

Hence xd(y)yd(y) = d(y)xd(y)α(y)+d(y)xyd(y) (10)

Substituting xy for n in (8) we have

Xy^2 d(y)=d(y)xy d(y) for all x,yϵ I (11)

Left- multiplying (8) by α(y), we obtain

α(y)xyd (y) = α(y) d(y) nd (y) for all x,yϵ I (12)

Replacing x by y in (8) we get

y^2d (y) =d(y) yd (y)

and right-multiplying this relation by n, we have

Y^2 d (y) x =d(y) y d(y) x for all x,yϵ I (13)

Using (11), (12) and (13) in (10) we obtains

x yd(y)α(y) = 0.

In particular, y n y d (y) α (y) = 0, Where nϵ N.

Hence y d (y) α (y) N y d (y) α (y) ={0}.

By the semiprimeness

Nyd (y) α (y) = o for all n, yϵ I (14)

According to (12), we get α (y) d (y) n d (y) = 0

Using this relation in (9), we have

D (y) α (y) x α (y) = 0 for all x,yϵ I (15)

Replacing n by x n d (y) in 15, we have

D(y)α(y)xd (y)α(y) = d (y) α (y) x n d (y) α (y) x = 0 for all x,y∈ I, n∈ N.

Hence D (y) α (y) x = 0, for all x,y∈ I (16)

Using (16) in(9), we obtain that

d (y) x d (y) = 0,

and so we have

d (y) x n d (y) x=0 for all x, y∈ I ,n∈ N.

Hence xd (y) = 0 for all x, y∈ I (17)

Therefore x d (z) d (yn) x = 0 for all x,y,z∈ I, n∈ N.

Thus 0 = x d (z) (d(y) n+ α (y) d (n)) x = x d (z) d (y) α (y) d (n) x for all x,y,z∈ I , n∈ N.

Since α (I) = I the second summand is zero by (17).

Hence x d (z) d (y) N x = {0} and so

x d (z) d (y) Nx d(z) d(y) = {0}.By the semi primeness of N we get

0 = x d (z) d (y) = xd (yz).

Therefore 0 = x d (y) z +x α (y) d (z) = x α (y) d (z).

In particular 0 = α (y) d (z) n α (y) d (z).

Hence 0 = α (y) d (z).

Recalling (17), we now have 0 = d (xy) for all x,y∈ I.

So d (xxn) = 0 for all x∈ I,n∈ N.

Thus

0 = d(xn) d (x) = (d (x) n + α (x) d (n)) d (x)nd(x) + α(x)d(n)d (x)

= d (x) n d (x) + α (x) d(xn).

Since the second summand is zero, we get d (x) n d (x) = 0.

Therefore d (x) = 0 for all x∈ I.□

Corollary 1: Let N be a semi prime nearring and d a two sided α – derivation of N such that α is onto and α (xy) = α (x) α (y) for all x,y∈ N.

(i) If d acts as a homomorphison on N , then d = 0

(ii) If d acts as an anti homomorphison on N such that α (0) = 0 , then d = 0 .□

Corollary 2: Let N be a prime nearring and I a nonzero subset of N such that 0∈ I and IN⊆I. Let d be a two sided α derivation on N such that α (I) = I and α (xy) = α(x)α(y) for all x,y∈ I.

(i) If d acts as a homomorphison on I, then d = 0.

(ii) If d acts as an anti – homomorphison on I and α (0) = 0, then d = 0.

Proof: By theorem 1 , we have d (x) = 0 for all x∈ I.

Then 0 = d (xn) = d(x) α (n) + x d (n) = x d (n), and so

xmd (n) = 0 for all x∊ I, n,m∊ N.

By the primeness of N we have x = 0 or d (n) = 0 for all x∊ I, n∊ N.

Since I is nonzero, we have d (n) = 0 for all n∊ N.□

Theorem 2: Let N be a prime nearring, I a nonzero semi group ideal of N and d nonzero (α,1)-derivation of N such that α(xy) = α(x) α (y) for all x,y∊I. If d (x+y-x-y) =0 for all x,y∊ I, then (N,+) is abelian.

Proof: Suppose that d (x+y-x-y) = 0 for all x,y∊ I.

Then from lemma (5) we have

d(z) (x+y-n-y) = 0 for all x,y,z∊ I.

Since d≠0, it is clear that by lemma(4)

x+y-x-y=0 for all x,y∊ I.

Hence form by lemma (1) we have

(N,+) is abelian.□

Corallary 3: Let N be a prime nearring, I a nonzero semigroup ideal of N and d a nonzero (α,1)-derivation of N such that α(xy)=α(x)α(y) for all x,y∊ I. If d+d is additive on I, then (N,+) is abelian.

Proof: Assume that d+d is an additive on I, then

(d+d)(x+y) = (d+d)(x)+(d+d)(y)

= d(x)+d(x)+d(y)+d(y) for all x,y∊ I.

On the other hand,

(d+d)(x+y) = d(x+y)+d(x+y)

= d(x)+d(y)+d(x)+d(y) for all x,y∊ I.

The above two expressions for (d+d)(x+) yield

d(x)+d(y)=d(y)+d(x) for all x,y∊ I,

i.e. d(x+y-x-y)=0.

Hence from theorem (2) we have (N,+) is abelian.□

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