The b-chromatic number of mycielskian of cycles

Lisna. P. C, M.S.Sunitha

Abstract—A b-coloring of a graph G is a proper coloring of the vertices of G such that there exist a vertex in each color class joined to at least one vertex in each other color classes. The b-chromatic number of a graph G, denoted by \( \varphi(G) \), is the maximal integer k such that G may have a b-coloring with k colors. The Mycielskian or Mycielski graph \( \mu(H) \) of a graph H with vertex set \( \{v_1, v_2, \ldots, v_n\} \) is a graph G obtained from H by adding \( n + 1 \) new vertices \( u, u_1, u_2, \ldots, u_n \), joining u to each vertex \( u_i (1 \leq i \leq n) \) and joining \( u_i \) to each neighbour of \( v_i \) in H. In this paper we obtained the b-chromatic number of the mycielskian of cycles.

Index Terms—b-chromatic number, b-coloring, b-dominating set, mycielskian, cycle.

1 INTRODUCTION

The concept of b-chromatic number was introduced in 1999 by Irving and Manlove[6], who proved that determining \( \varphi(G) \) is NP-hard in general and polynomial time solvable for trees. The b-chromatic number \( \varphi(G) \) of a graph G is the largest positive integer k such that G admits a proper k-coloring in which every color class has a representative vertex which is adjacent to at least one vertex in each of the other color classes. Such a coloring is called a b-coloring and this representative vertex is known as the b-dominating vertex and the set of b-dominating vertex is known as the b-dominating set [3]. In [1] the b-coloring of cographs and P_4 sparse graphs is discussed. In [3] El Sahili and Kouider M obtained a general formula for the b-chromatic number of regular graphs. In [7] the b-coloring of Kneser graphs is discussed. In [10] Vernold Vivin J and Venkatachalal M obtained the b-chromatic number of corona of two graphs with same number of vertices. In this paper we obtained the b-chromatic number of mycielskian of cycles.

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2 B-CHROMATIC NUMBER OF A GRAPH

The b-chromatic number of a graph G is defined as follows.

Definition 2.1. The b-chromatic number \( \varphi(G) \) of a graph G is the largest positive integer k such that G admits a proper k-coloring in which every color class contains a vertex which is adjacent to at least one vertex in each of the other color classes.

Example 2.2.

Figure 1: b-coloring of a graph with three colors

3 MYCIELSKIAN OF A GRAPH [5]

The mycielskian or mycielski graph of a graph G, denoted by \( \mu(G) \) is defined as follows.

Definition 3.1. The Mycielskian or Mycielski graph \( \mu(H) \) of
a graph \( H \) with vertex set \( \{v_1, v_2, \ldots, v_n\} \) is a graph \( G \) obtained from \( H \) by adding \( n + 1 \) new vertices \( \{u_1, u_2, \ldots, u_n\} \) joining \( u \) to each vertex \( u_i \) for \( 1 \leq i \leq n \). Joining \( u \) to each neighbour of \( v_i \) in \( H \). 

Example 3.2.

\[ \begin{align*}
&\text{Figure 2: Mycielskian of the path } P_5 \\
&\text{4 B-CHROMATIC NUMBER OF MYCIELSKIAN OF CYCLE} \\
&\text{The b-chromatic number of the mycielskian of a cycle is given as follows} \\
&\text{Theorem 4.1. The b-chromatic number of the mycielskian of a cycle is} \\
&\phi(\mu(C_n)) = \begin{cases} 
&\phi(C_n) + 1, \quad n \leq 6 \\
&\phi(C_n) + 2, \quad n \geq 7 
\end{cases} \\
&\text{Proof:} \\
&\text{Let the vertex set of } C_n \text{ be } \{v_1, v_2, \ldots, v_n\} \text{ and that of } \\
&\mu(C_n) \text{ be } \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\} \cup \{u\} \\
&\text{Here } \{u_1, u_2, \ldots, u_n\} \text{ is set of } n \text{ independent vertices in which each} \\
&u_i \text{ is connected to every neighbour of } v_i \text{ and the vertex } u \text{ is connected to every } u_i; \quad 1 \leq i \leq n. \\
&\text{Case 1: } n \leq 6 \\
&\text{Subcase 1: } n = 3 \\
&\text{Here } \phi(C_n) = 3, \text{ so we have to prove that} \\
&\phi(\mu(C_n)) = 4. \text{ On the contrary assume that } \\
&\phi(\mu(C_n)) = 5, \text{ then there will be at least 5 vertices with degree at least 4. But here we have only 3 vertices with degree at least 4. So a b-coloring with 5 colors is not possible.} \\
&\text{A b-coloring with 4 colors can be obtained by assigning color } i \text{ to } v_i \text{ and } u_u; \quad 1 \leq i \leq n. \text{ and color 4 to } u. \\
&\text{Subcase 2: } n = 4 \\
&\text{Here we have exactly 5 vertices, } v_1, v_2, v_3, v_4, u \text{ with degree at least 4. Hence we can check the existence of a b-coloring with 5 colors. Assume that such a coloring is existing, then the b-dominating vertices will be } v_1, v_2, v_3, v_4, \text{ and } u. \text{ Now consider the vertices } v_1 \text{ and } v_3. \text{ The neighbours of } v_1 \text{ are } v_2, v_n, u_2, u_n, \text{ which are same as the neighbours of } v_3. \text{ That is, here the two vertices } v_1 \text{ and } v_3 \text{ are having same neighbours. Hence a b-coloring with 5 colors is not possible. Because if we choose } v_1 \text{ and } v_3 \text{ as the b-dominating vertices then } v_3 \text{ should be adjacent to a vertex which is having the color of } v_1 \text{ and } v_1 \text{ should be adjacent to a vertex which is having the color of } v_3, \text{ but since these vertices have same neighbours we cannot assign the color of } v_1 \text{ and color of } v_3 \text{ to any of these neighbours. Next check the existence of a b-coloring with 4 colors. Here all the vertices are having degree at least 3. Now consider the set } \{v_1, v_2, v_3, v_4\}. \text{ From this set we can choose only two b-dominating vertices because here } v_1 \text{ and } v_3 \text{ are having same neighbours and } v_2 \text{ and } v_4 \text{ are having same neighbours. Hence from this set we can select } v_1, v_2 \text{ or } v_3, v_4 \text{ as the b-dominating vertices. That is we can select any two adjacent } v_i \text{'s as the b-dominating vertices. Next consider the set } \{u_1, u_2, u_3, u_4\}. \text{ Here each vertices are having degree exactly 3. Here any two } u_i \text{'s will have either same neighbours or it will have one common neighbour and two distinct neighbours. The vertices with same neighbours cannot be chosen as b-dominating vertices. So we can choose vertices with one common neighbour and two distinct neighbours as the b-dominating vertices. Thus we can select } u_1, u_2, u_3, u_4 \text{ or } u_2, u_3, u_4, u_1 \text{ as the b-dominating vertices. That is we can select } u_i \text{ and } u_{i+1} \text{, } 1 \leq i \leq 4, j \neq 1 \text{ if } j > 4 \text{ as the b-dominating vertices. Now suppose that } v_i, v_{i+1}, u_j, u_{j+1} \text{ are the b-dominating vertices. Here } u_j \text{ will be adjacent to either } v_i \text{ or } v_{i+1} \text{ but not to both. Similarly } u_{j+1} \text{ will be adjacent to either } v_i \text{ or } v_{i+1} \text{ but not to both. Without loss of generality assume that } u_j \text{ is adjacent to } v_i \text{ and } u_{j+1} \text{ is adjacent to } v_{i+1}. \text{ Assign colors } c_1, c_2, c_3, c_4 \text{ to } v_1, v_{i+2}, u_j, u_{j+1} \text{ respectively. Now } u_j \text{ having color } c_1 \text{ and is adjacent to } v_i \text{ with color } c_1. \text{ To make the vertex } u_j \text{ b-dominating, it should be adjacent to vertices with color } c_1. \text{ Since } v_{i+2} \text{ is adjacent to } v_1 \text{ we cannot assign color } c_2 \text{ to } v_{i+2}. \text{ So assign color } c_2 \text{ to } v_{i+2} \text{ and } c_2 \text{ to } u. \text{ Now consider } u_{j+1}. \text{ The neighbours of } u_{j+1} \text{ are } v_1, v_{i+2} \text{ and } u. \text{ Here } i + 2 = 1 \text{ if } i + 2 > 4. \text{ Since } v_{i+2} \text{ is adjacent to } v_1 \text{ we cannot assign color } c_2 \text{ to } v_{i+2}. \text{ So assign color } c_2 \text{ to } v_{i+2} \text{ and } c_2 \text{ to } u. \text{ Hence } u_{j+1} \text{ will be adjacent to either } v_i \text{ or } v_{i+1}. \text{ Suppose that it is adjacent to } v_i. \text{ As mentioned above, to make the vertex } u_j \text{ b-dominating we have to assign color } c_2 \text{ to } u. \text{ But this will results in a b-coloring with 3 colors. Because here we cannot find a b-dominating vertex to assign color } c_4. \text{ This means that we cannot choose any b-dominating vertex from the set } \{u_1, u_2, u_3, u_4\}. \]
Now select u as a b-dominating vertex. But here also we cannot construct a b-coloring with 4 colors. Because here we have only 3 b-dominating vertices \(v_i, v_{i+1}\) and u. Using these three b-dominating vertices we cannot construct a b-coloring with 4 colors. Hence a b-coloring with 4 colors is not possible here. A b-coloring with 3 colors can be obtained by assigning color 1 to \(v_1\), color 2 to \(v_2, v_3, u_1\) and \(u_2\), and color 3 to \(v_2\) and \(u_2\).

Subcase 3: \(n = 5\)

\(\varphi(C_n) = 3\). So we have to prove that \(\varphi(\mu(C_n)) = 4\). On the contrary assume that \(\varphi(\mu(C_n)) = 5\). Then there should be at least 5 vertices with degree at least 4. Here the degree of each \(u_i\): \(1 \leq i \leq 5\) is 3. But the degree of each \(v_i\): \(1 \leq i \leq 5\) is 4 and the degree of \(u\) is 5. So here we can choose either \(\{v_1, v_2, v_3, v_4, v_5\}\) or any 4 vertices from the set \(\{v_1, v_2, v_3, v_4, v_5\}\) and the vertex u as the b-dominating vertices. Consider the first case. That is select \(\{v_1, v_2, v_3, v_4, v_5\}\) as the b-dominating set. The degree of each of these vertices is 4. So these 5 vertices will become b-dominating only if all the 4 neighbours each \(v_i\) receives distinct colors. Now assign color \(c_1\) to \(v_i\): \(1 \leq i \leq 5\). Now consider the vertex \(v_1\). This vertex is having color \(c_1\) and is adjacent to \(v_2, v_3, u_1\) and \(u_2\). Here \(v_2\) and \(v_3\) are having colors \(c_2\) and \(c_3\) respectively. To make the vertex \(v_1\) b-dominating we have to assign colors \(c_4\) and \(c_1\) to \(u_1\) and \(u_2\). Here \(u_1\) is adjacent to \(v_3\) having color \(c_3\). So we cannot assign color \(c_1\) to \(u_2\). So assign color \(c_2\) to \(u_1\) and \(c_1\) to \(u_2\). Thus \(v_1\) becomes a b-dominating vertex. Next consider the neighbours of \(v_2\). The neighbours of \(v_2\) are \(v_3, v_5, u_3\) and \(u_5\). Here \(v_3\) and \(u_5\) are having color \(c_1\). That is two neighbours of \(v_1\) receives same color. Hence we cannot make the vertex \(v_1\) b-dominating. Hence the vertices \(\{v_1, v_2, v_3, v_4, v_5\}\) will not become a set of b-dominating vertices. Now consider the second case. That is choose any four vertices from the set \(\{v_1, v_2, v_3, v_4, v_5\}\) and \(u\) as the b-dominating vertices. Assign color \(c_1\) to \(u\). Since each \(u_i\): \(1 \leq i \leq 5\) is adjacent to \(u\), we cannot assign color \(c_1\) to any of the \(u_i\)'s. But the four vertices from the set \(\{v_1, v_2, v_3, v_4, v_5\}\) become b-dominating only if they adjacent to a vertex with color \(c_1\).

But from the set five vertices, \(\{v_1, v_2, v_3, v_4, v_5\}\) we have selected only four vertices as the b-dominating vertices. So one vertex will remains here and we can assign color \(c_1\) to this vertex. But still all the four b-dominating vertices will not be adjacent to this vertex. Only two of them will be adjacent to this vertex. Hence in this case also we cannot make a b-coloring with 5 colors. A b-coloring with 4 colors can be obtained by assigning color 1 to \(v_1\) and \(v_4\), color 2 to \(v_2\) and \(u\), color 3 to \(v_3\) and \(v_5\) and color 4 to \(u_1, u_2, u_3, u_4\) and \(u_5\).

Subcase 4: \(n = 6\)

Here also \(\varphi(C_n) = 3\). So we have to prove that \(\varphi(\mu(C_n)) = 4\). On the contrary assume that \(\varphi(\mu(C_n)) = 5\). Then there will be at least 5 vertices with degree at least 4. Here the vertices with degree at least 4 are \(v_1, v_2, v_3, v_4, v_5, v_6\) and \(u\). Note that the degree of \(v_1, v_2, v_3, v_4, v_5, v_6\) is exactly 4 and that of \(u\) is 6. Now we can select any of the 5 vertices from the set \(\{v_1, v_2, v_3, v_4, v_5, v_6, u\}\) as the b-dominating vertices. Suppose that the vertex u is included in the b-dominating set. If we select u as a b-dominating vertex, then the remaining four b-dominating vertices will be from the set \(\{v_1, v_2, ..., v_6\}\). Let the color of u be \(c_1\). Since all the u_i’s are adjacent to u, we cannot assign color \(c_1\) to these vertices. The four vertices from the set \(\{v_1, v_2, ..., v_6\}\) will be b-dominating if it is adjacent to a vertex with color \(c_1\). Note that from this set of six vertices we have selected only four vertices as b-dominating and for the remaining two vertices we can assign color \(c_1\). Hence choose the four b-dominating and the two non b-dominating vertices with color \(c_1\) in such a way that all the b-dominating vertices are adjacent to one of the non b-dominating vertices with color \(c_1\). For example if we choose the b-dominating vertices as \(v_1, v_2, v_3, v_4, v_6\) and the two non b-dominating vertices with color \(v_1\) as \(v_2\) and \(v_5\), then here all the b-dominating vertices will be adjacent to either \(v_2\) or \(v_5\). That is all the b-dominating vertices are adjacent to a vertex with color \(c_1\). Let the color of the b-dominating vertices \(v_1, v_2, v_4, v_5\) and \(v_6\) be \(c_2, c_3, c_1\) and \(c_5\) respectively. Now consider the vertex \(v_1\). \(v_1\) is having color \(c_3\) and is adjacent to \(v_2, v_6, u_3\) and \(u_4\). \(v_3\) and \(v_4\) are having colors \(c_1\) and \(c_5\) respectively. To make the vertex \(v_1\) b-dominating, it should be adjacent to vertices with colors \(c_3\) and \(c_4\). Here \(v_3\) is having color \(c_3\) and is adjacent to \(u_2\). So we cannot assign color \(c_1\) to \(u_2\). So assign color \(c_1\) to \(u_2\) and \(c_3\) to \(u_4\).

Now consider the vertex \(v_3\). \(v_3\) is having color \(c_1\) and is adjacent to \(v_2, v_4, u_2\) and \(u_4\). Here the color of \(u_2\) and \(v_4\) is \(c_1\) and that of \(v_2\) is \(c_1\). To make the vertex \(v_3\) b-dominating, it should be adjacent to vertices with colors \(c_2\) and \(c_5\). But now this vertex has only one uncolored neighbour. So we cannot make this vertex b-dominating. Thus a b-coloring with 5 colors is not possible here. Also we cannot make any b-coloring with 5 colors even if we choose the four b-dominating and the two non b-dominating vertices in any other order. Now suppose that the vertex u not included in the b-dominating set. Thus the five b-dominating vertices will be from the set \(\{v_1, v_2, v_6\}\). As mentioned above here also we cannot make some vertices b-dominating. Hence a b-coloring with 5 colors is not possible here and a b-coloring with 4 colors can be obtained by assigning color 1 to \(v_1, u_3, u_4\) and \(u_5\), color 2 to \(v_2, v_5\) and \(u\), color 3 to \(v_3, v_6\) and \(u_6\) and color 4 to \(v_4, u_1\) and \(u_2\).

Case 2: \(n \geq 7\)
\( \varphi(C_n) = 3 \). So we have to prove that
\( \varphi(\mu(C_n)) = 5 \). On the contrary assume that \( \varphi(\mu(C_n)) = 6 \).
Then there will be at least 6 vertices with degree at least 5. But here we have only one vertex with degree at least 5. So a b-coloring with 6 colors is not possible here. A b-coloring with 5 colors can be obtained by assigning color 1 to \( v_2 \), \( u_4 \), \( u_5 \) and \( u_{n-1} \), color 2 to \( v_5 \), \( v_6 \) and \( v_{n-1} \), color 3 to \( v_4 \), \( v_1 \) and \( u \), color 4 to \( v_5 \), \( u_1 \) and \( u_2 \) and color 5 to \( v_n \), \( u_n \), \( u_3 \) and \( u_4 \). For the remaining \( u \)'s and \( v \)'s assign any of the color from the list \{1, 2, 3, 4, 5\} \{3\} in a proper way. Now this is a b-coloring with 5 colors and here the b-dominating vertices are \( v_2 \), \( v_3 \), \( v_4 \), \( v_5 \) and \( v_n \).

5. CONCLUSION

In this paper we obtained the b-chromatic number of the mycielskian of cycles.

REFERENCES


