The Steady-State Solution of Serial Channel with Feedback, Balking and Reneging Connected with Non-Serial Queuing Processes

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Abstract—In the present paper, we have studied the general queuing model having feedback, balking and reneging in serial queuing processes connected with non serial queuing channels with random order selection for service and such models are of common occurrence in the administrative setup. We have also obtained the mean queue length of the model when queue discipline is first come first order.

Index Terms—Balking, Difference-differential, Exponential service, Feedback, Poisson arrivals, Random selection, Reneging, Steady-State, Waiting space.

1 INTRODUCTION

O’Brien (1954), Jackson (1954) and Hunt (1955) studied the problems of serial queues in the steady state with Poisson assumptions. In these studies, it is assumed that the unit must go through each service channel without leaving the system. Barrer (1955) obtained the steady-state solution of a single channel queuing model having Poisson input, exponential holding time, random selection where impatient customers leave the service facility after a wait of certain time. Finch (1959) studied simple queues with customers at random for service at a number of service stations in series where the arrival from outside was considered at the initial stage. Feedback is permitted either from the terminal server or from each server of the series to the queue waiting for service at that stage by imposing an upper limit on the number of customers in the system at any time. Singh (1984) studied the problem of serial queues introducing the concept of reneging. Singh and Singh (1994) worked on the network of queuing processes with impatient customers. Punam, Singh and Ashok (2011) found the steady-state solution of serial queuing processes where feedback is not permitted.

In our present work, the steady-state solutions are obtained for serial queuing processes with feedback, balking and reneging connected with non serial queuing channels in which

(i) A customer may join any channel from outside and leave the system at any stage after getting service.

(ii) M serial queuing processes with feedback, balking and reneging connected with N non serial queuing channels.

(iii) Feedback is permitted from each channel to its previous channel in serial channels.

(iv) The customer may balk due to long queue at each service channel.

(v) The impatient customer leaves the service facility after wait of certain time.

(vi) The input process in serial channels depends upon queue size and Poisson arrivals are followed.

(vii) Exponential service times are followed.

(viii) The queue discipline is random selection for service

(ix) Waiting space is infinite.

The expressions for marginal probabilities and mean queue length have also been derived whenever the queue discipline is first come first served.

2. FORMULATION OF THE MODEL

The system consists of the serial queues $Q_j (j=1,2,3,....M)$ and non-serial channels $Q_{ij} (i=1,2,3,....N)$ with respective servers $S_j (j=1,2,3,....M)$ and $S_{ij} (i=1,2,3,....N)$. Customers demanding different types of service arrive from outside the system in Poisson stream with parameters $\lambda_j (j=1,2,3,....M)$ and $\lambda_{ij} (i=1,2,3,....N)$ at $Q_j (j=1,2,3,....M)$ and $Q_{ij} (i=1,2,3,....N)$ but the sight of long queue at $Q_j (j=1,2,3,....M)$ may discourage the fresh customer from joining it and may decide not to enter the service channel at $Q_j (j=1,2,3,....M)$. Then the Poisson input rate at $Q_j (j=1,2,3,....M)$ would be $\frac{\lambda_j}{n_j + 1}$ where $n_j$ is the
queue size of $Q_j (j = 1, 2, 3, \ldots, M)$ . Further, the impatient customer joining any service channel $Q_j (j = 1, 2, 3, \ldots, M)$ may leave the queue without getting service after wait of certain time. Service time distributions for servers $S_j (j = 1, 2, 3, \ldots, M)$ and $S_i (i = 1, 2, 3, \ldots, N)$ are mutually independent negative exponential distribution with parameters $\mu_j (j = 1, 2, \ldots, M)$ and $\mu_i (i = 1, 2, \ldots, N)$ respectively. After the completion of service at $S_j$, the customer either leaves the system with probability $p_j$ or joins the next channel with probability $\frac{q_j}{n_{j+1}}$ or join back the previous channel with probability $\frac{r_j}{n_{j-1}+1}$ such that $p_j + \frac{q_j}{n_{j+1}+1} + \frac{r_j}{n_{j-1}+1} = 1 \quad (j = 1, 2, 3, \ldots, M - 1)$ and after the completion of service at $S_M$ the customer either leaves the system with probability $p_M$ or join back the previous channel with probability $\frac{r_M}{n_{M-1}+1}$ or join any queue $Q_i (i = 1, 2, 3, \ldots, N)$ with probability $q_{iM} (i = 1, 2, 3, \ldots, N)$ such that $p_M + \frac{r_M}{n_{M-1}+1} + \sum_q q_{i} = 1$.

It is being mentioned here that $r_j = 0$ for $j = 1$ as there is no previous channel of the first channel.

The applications of such models are of common occurrence. For example, consider the administration of a particular district in a particular state at the level of district head quarter consisting of Block Development officer, Tehsildar, Sub-Divisional Magistrate, District Magistrate etc. These officers correspond to the servers of serial channels. Education Department, Health Department, Irrigation Department etc. connected with the last server of serial queue correspond to non-serial channels. The people meet the officers of the district in connection with their problems. It is also a common practice that officers call the customers (people) for hearing randomly. The senior officer may send any customer to his junior if some information regarding the customer’s problem is lacking. Further District Magistrate may send the customers to different departments such as Education, Health, Irrigation etc if there problems are related to such departments.

3. FORMULATION OF EQUATIONS:

Define: $P(n_1, n_2, n_3, \ldots, n_{M-1}, n_M, m_1, m_2, m_3, \ldots, n_{N-1}, n_N; t) =$ the probability that at time ‘$t$’ there are $n_j$ customers (which may balk, renege or leave the system after being serviced or join the next phase or join back the previous channel) waiting before $S_j (j = 1, 2, 3, \ldots, M - 1, M)$; $m_j$ customers (which may leave the system after being serviced) waiting before the servers $S_i (i = 1, 2, 3, \ldots, N)$.

We define the operators $T_\gamma, T_{\gamma, i}, T_{\gamma, i, j}, T_{\gamma, i, j, k}$ to act upon the vectors $\bar{n} = (n_1, n_2, n_3, \ldots, n_M)$ or $\bar{m} = (m_1, m_2, m_3, \ldots, m_N)$ as follows:

$$T_\gamma(\bar{n}) = (n_1, n_2, n_3, \ldots, n_{i-1}, n_{i+1}, \ldots, n_M)$$

$$T_{\gamma, i}(\bar{n}) = (n_1, n_2, n_3, \ldots, n_{i-1}, n_{i+1}, \ldots, n_M)$$

$$T_{\gamma, i, j}(\bar{n}) = (n_1, n_2, n_3, \ldots, n_{i-1}, n_{i+1}, \ldots, n_M)$$

We write the following Steady–state equations of the queuing system after being serviced) waiting before the servers $S_i (i = 1, 2, 3, \ldots, N)$:

Following the procedure given by Kelly [5], we write the difference - differential equations as

$$\frac{d P(\bar{n}, \bar{m}; t)}{dt} = - \sum_{i=1}^{M} \lambda_i \delta (n_i) \left( \mu_i + C_{m_i} \right) P(\bar{n}, \bar{m}; t) + \sum_{i=1}^{M} \mu_i \delta (m_i) \mu_i \frac{r_i}{n_{i-1}} + \sum_{i=1}^{N} q_{iM} \delta (m_i) \mu_i \frac{r_M}{n_{M-1}} + \sum_{i=1}^{N} \lambda_i \delta (n_i) \left( \mu_i + C_{m_i} \right) P(\bar{n}, \bar{m}; t) + \sum_{i=1}^{M} q_{i} \delta (n_i) \mu_i \frac{r_i}{n_{i-1}}$$

for $n_i \geq 0 \quad (i = 1, 2, 3, \ldots, M)$, $m_j \geq 0 \quad (j = 1, 2, 3, \ldots, N)$;

where $\delta (x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

and $P(\bar{n}, \bar{m}; t) = \bar{0}$ if any of the arguments in negative.

4. STEADY–STATE EQUATIONS

We write the following Steady–state equations of the queuing model by equating the time-derivatives to zero in the equation (1)

$$\sum_{i=1}^{M} \frac{\lambda_i}{n_i} + \sum_{i=1}^{M} \delta (n_i) \left( \mu_i + C_{m_i} \right) + \sum_{i=1}^{N} \delta (m_i) \mu_i$$

where $\bar{0}$ is the zero vector.

$$= \sum_{i=1}^{M} \lambda_i \delta (n_i) \left( \mu_i + C_{m_i} \right) P(\bar{n}, \bar{m}) + \sum_{i=1}^{M} q_{i} \delta (n_i) \mu_i \frac{r_i}{n_{i-1}} + \sum_{i=1}^{M} q_{iM} \delta (m_i) \mu_i \frac{r_M}{n_{M-1}}$$
\begin{align*}
+ \sum_{j=1}^{n} \mu_j q_j P(\tilde{n}, t, \tilde{m})
+ \sum_{j=1}^{n} \lambda_j P(\tilde{n}, t, \tilde{m})
+ \sum_{j=1}^{n} \mu_j P(\tilde{n}, t, \tilde{m})
\end{align*}

for \( n_i \geq 0 \quad (i = 1, 2, 3, \ldots, M) \), \( m_j \geq 0 \quad (j = 1, 2, 3, \ldots, N) \)

5. STEADY-STATE SOLUTIONS

The solutions of the Steady-State equations (2) can be verified to be

\begin{align*}
P(\tilde{n}, \tilde{m}) &= P(\tilde{0}, \tilde{0}) \left( \frac{1}{n_!} \prod_{i=1}^{n} (\mu_i + C_{ii}) \right)^n \\
& \quad \times \left( \frac{\lambda_i + \frac{\mu_i q_i \rho_i}{(n_i + 1)(\mu_i + C_{ii})}}{n_i + 1} \prod_{i=1}^{n} (\mu_i + C_{ii}) \right)^{n_i} \\
& \quad \times \left( \frac{\lambda_{n+1} + \frac{\mu_{n+1} q_{n+1} \rho_{n+1}}{(n_{n+1} + 1)(\mu_{n+1} + C_{n+1})}}{n_{n+1} + 1} \prod_{i=1}^{n} (\mu_i + C_{ii}) \right)^{n_{n+1}}
\end{align*}

\begin{align*}
\rho_i &= \frac{\lambda_i + \frac{\mu_i q_i \rho_i}{(n_i + 1)(\mu_i + C_{ii})}}{n_i + 1} \prod_{i=1}^{n} (\mu_i + C_{ii}) \\
\rho_{n+1} &= \frac{\lambda_{n+1} + \frac{\mu_{n+1} q_{n+1} \rho_{n+1}}{(n_{n+1} + 1)(\mu_{n+1} + C_{n+1})}}{n_{n+1} + 1} \prod_{i=1}^{n} (\mu_i + C_{ii})
\end{align*}

Solving these (4) M-equations for \( \rho_M \) with the help of determinants, we get
\[
\rho_M = \frac{q_M \Delta_M}{\Delta_M - \frac{q_M \mu_M}{(n_{M-1}+1)\left(\mu_{M-1} + C_{M-1}\right)} + \frac{r_M \mu_M}{(n_{M-2}+1)\left(\mu_{M-2} + C_{M-2}\right)}} \Delta_M
\]

Where

\[
\Delta_M = \Delta_{M-1} - \frac{q_{M-1} \mu_{M-1}}{(n_{M-1}+1)\left(\mu_{M-1} + C_{M-1}\right)} + \frac{r_{M-1} \mu_{M-1}}{(n_{M-2}+1)\left(\mu_{M-2} + C_{M-2}\right)} \Delta_{M-2}
\]

With \( \Delta_1 = 1 \)

\[
\Delta_2 = \begin{pmatrix}
1 & -\frac{r_2 \mu_2}{n_2 + 1} \\
\frac{q_2 \mu_1}{n_1 + 1} & \frac{r_2 \mu_2}{\mu_2 + C_{2n_2+1}}
\end{pmatrix}
\]

\[
\Delta_3 = \begin{pmatrix}
1 & -\frac{n_1 + 1}{\mu_1 + c_{n_1+1}} & 0 \\
\frac{q_1 \mu_1}{n_1 + 1} & 1 & -\frac{n_1 + 1}{\mu_1 + c_{n_1+1}} \\
0 & -\frac{n_2 + 1}{\mu_2 + C_{2n_2+1}} & 1
\end{pmatrix}
\]

Since \( \rho_M \) is obtained, we can get \( \rho_{M-1} \) by putting the value of \( \rho_M \) in the last equation of (4), \( \rho_{M-2} \) by putting the values of \( \rho_{M-1} \) and \( \rho_M \) in the last but one equation of (4). Continuing in this way, we shall obtain \( \rho_{M-3}, \rho_{M-4}, \ldots, \rho_3, \rho_2 \) and \( \rho_1 \).

Thus, we write (3) as under
\[ p(\tilde{n}, \tilde{m}) = P(\tilde{0}, \tilde{0}) \left[ \frac{1}{n_1! n_2! n_3!} \prod_{i=1}^{n_1} \left( \frac{\rho_i}{\mu + C_i} \right)^{n_1} \prod_{i=1}^{n_2} \left( \frac{\rho_i}{\mu + C_i} \right)^{n_2} \prod_{i=1}^{n_3} \left( \frac{\rho_i}{\mu + C_i} \right)^{n_3} \right] \]

\[ P(\tilde{n}, \tilde{m}) = P(\tilde{0}, \tilde{0}) \left[ \frac{1}{n_1! n_2! n_3!} \prod_{i=1}^{n_1} \left( \frac{\rho_i}{\mu + C_i} \right)^{n_1} \prod_{i=1}^{n_2} \left( \frac{\rho_i}{\mu + C_i} \right)^{n_2} \prod_{i=1}^{n_3} \left( \frac{\rho_i}{\mu + C_i} \right)^{n_3} \right] \]

\[ n_i = 0 \quad (i = 1, 2, 3, \ldots, M) \quad m_j \geq 0 \quad (j = 1, 2, 3, \ldots, N) \]

We obtain the steady-state marginal probability that there are \(n_i\) customers in state \(i\) from the normalizing conditions.

\[ \sum_{\tilde{n}, \tilde{m}} P(\tilde{n}, \tilde{m}) = 1 \]

and with the restriction that traffic intensity of each service channel of the system is less than unity, \(C_{in_i}\) is the reneging rate at which customer renge after a wait of time \(T_{in}\) whenever there are \(n_i\) customers in the service channel \(Q_i\).

\[ C_{in_i} = \frac{\mu_i n_i}{\mu_i - n_i} \quad (i = 1, 2, 3, \ldots, M) \]

Here it is mentioned that the customers leave the system at constant rate as long as there is a line, provided that the customers are served in the order in which they arrive. Putting \(C_{in_i} = C_i\) \((i = 1, 2, 3, \ldots, M)\) in the steady-state solution (3) then \(\rho_j\) \((j = 1, 2, 3, \ldots, M)\) will change accordingly and the steady-state solution reduces to

\[ P(\tilde{n}, \tilde{m}) = P(\tilde{0}, \tilde{0}) \left[ \frac{1}{n_1! n_2! n_3!} \prod_{i=1}^{n_1} \left( \frac{\rho_i}{\mu + C_i} \right)^{n_1} \prod_{i=1}^{n_2} \left( \frac{\rho_i}{\mu + C_i} \right)^{n_2} \prod_{i=1}^{n_3} \left( \frac{\rho_i}{\mu + C_i} \right)^{n_3} \right] \]

\[ \left( \rho_1 \right)^{n_1} \left( \rho_2 \right)^{n_2} \left( \rho_N \right)^{n_N} \]

Thus \(P(\tilde{n}, \tilde{m})\) is completely determined.

**6. Steady-State Marginal Probabilities**

Let \(P(n_i)\) be the steady-state marginal probability that there are \(n_i\) units in the queue before the first server. This is determined as

\[ P(n_i) = \sum_{n_{i+1}, n_{i+2}, \ldots, n_M} P(\tilde{n}, \tilde{m}) \]

\[ = \sum_{n_{i+1}, n_{i+2}, \ldots, n_M} P(\tilde{0}, \tilde{0}) \left[ \frac{1}{n_1! n_2! n_3!} \prod_{i=1}^{n_1} \left( \frac{\rho_i}{\mu + C_i} \right)^{n_1} \prod_{i=1}^{n_2} \left( \frac{\rho_i}{\mu + C_i} \right)^{n_2} \prod_{i=1}^{n_3} \left( \frac{\rho_i}{\mu + C_i} \right)^{n_3} \right] \]

\[ \left( \rho_1 \right)^{n_1} \left( \rho_2 \right)^{n_2} \left( \rho_N \right)^{n_N} \]

Thus \(P(n_i) = \frac{1}{n_i!} \left( \frac{\rho_i}{\mu + C_i} \right)^{n_i} e^{-\rho_i/\mu_iC_i} \quad n_i > 0\)

Similarly

\[ P(n_j) = \frac{1}{n_j!} \left( \frac{\rho_j}{\mu_j + C_j} \right)^{n_j} e^{-\rho_j/\mu_jC_j} \quad n_j > 0 \]
Further let \( P(m_1), P(m_2), P(m_3), \ldots, P(m_N) \) be the steady-state marginal probabilities that there are \( m_1, m_2, m_3, \ldots, m_N \) customers waiting before server \( S_i (i = 1, 2, 3, \ldots, N) \) respectively.

\[
P(m_i) = \sum_{n_i=0}^{\infty} \sum_{n_{i-1}=0}^{\infty} \cdots \sum_{n_1=0}^{\infty} \sum_{n_0=0}^{\infty} \frac{\nu_m}{n_i!} \frac{\nu_{i-1}}{n_{i-1}!} \frac{\nu_{i-2}}{n_{i-2}!} \cdots \frac{\nu_1}{n_1!} \frac{\nu_0}{n_0!} \left( \frac{\rho_1}{\mu_1 + C_1} \right)^{n_i} \left( \frac{\rho_2}{\mu_2 + C_2} \right)^{n_{i-1}} \cdots \left( \frac{\rho_{i-1}}{\mu_{i-1} + C_{i-1}} \right)^{n_{i-2}} \left( \frac{\rho_i}{\mu_i + C_i} \right)^{n_{i-1}} \left( \frac{\rho_{i+1}}{\mu_{i+1} + C_{i+1}} \right)^{n_i} \left( \frac{\rho_{i+2}}{\mu_{i+2} + C_{i+2}} \right)^{n_{i-1}} \cdots \left( \frac{\rho_N}{\mu_N + C_N} \right)^{n_0}.
\]

\[
= (\rho_1)^{n_i} (1 - \rho_1) \quad m_i > 0
\]

Similarly

\[
P(m_i) = (\rho_i)^{n_i} (1 - \rho_i) \quad m_i > 0
\]

\[
P(m_N) = (\rho_N)^{n_N} (1 - \rho_N) \quad m_N > 0
\]

### 7. Mean Queue Length

Mean queue length before the server \( S_i \) is determined by

\[
L_i = \sum_{n_i=0}^{\infty} n_i P(n_i) = \sum_{n_i=0}^{\infty} n_i \frac{\nu_i}{n_i!} \left( \frac{\rho_i}{\mu_i + C_i} \right)^{n_i} e^{-\left( \frac{\rho_i}{\mu_i + C_i} \right)}
\]

\[
= \frac{\rho_i}{\mu_i + C_i}
\]

Similarly

\[
L_2 = \frac{\rho_2}{\mu_2 + C_2}
\]

\[
L_M = \frac{\rho_M}{\mu_M + C_M}
\]

Mean queue length before the server \( S_{i+1} \) is determined as

\[
L_{i+1} = \frac{\rho_i}{1 - \rho_i}
\]

Similarly

\[
L_{i+1} = \frac{\rho_i}{1 - \rho_i} \quad j = 2, 3, \ldots, N
\]
Numerical Solutions of Mean Queue Length

<table>
<thead>
<tr>
<th>Servers in Series $S_M$</th>
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<th>Departure rate $\mu_{SM}$ before server $S_M$</th>
<th>Marginal mean queue length before the server $S_M$ $\mu_{SM}=\mu_{SM}/(\mu_{SM}+C_{SM})$</th>
<th>Mean Queue Length before the servers $S_{SM}$ before servers $S_{SN}$</th>
<th>Arrival rate $\lambda_{SN}$ before servers $S_{SN}$</th>
<th>Prob. of joining the non-serial servers $q_{SM}$</th>
<th>Service rate $\mu_{SN}$ before servers $S_{SN}$</th>
<th>Mean Queue Length before the servers $S_{SN}$ before servers $S_{IN}$</th>
<th>Sum of Marginal Mean Queue Lengths of Serial and Non-Serial Servers</th>
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Mean queue length of the system = 11.16599866

REFERENCES