Steady flow in pipes of equilateral triangular cross-section in magnetic field

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ABSTRACT: In this paper we have investigate the Steady flow in pipes of equilateral triangular cross-section in magnetic field. We have investigated the velocity, volumetric flow and vortex lines.

KEY WORDS: Steady flow, Equilateral triangular cross section, incompressible fluid, pipes and magnetic field.

NOMENCLATURE

- $u = $Velocity component along x – axis
- $v = $Velocity component along y – axis
- $w (x , y) = $Velocity in x-y plane
- $t = $the time
- $\rho = $The density of fluid
- $P = $the fluid pressure
- $K = $the thermal conductivity of the fluid
- $\mu = $Coefficient of viscosity
- $\nu = $Kinematic viscosity
- $Q = $the volumetric flow
- $\Omega_x = $Vorticity component in x - direction
- $\Omega_y = $Vorticity component in y - direction
- $\Omega_z = $Vorticity component in z - direction

INTRODUCTION

investigated the velocity, volumetric flow and vortex lines.

FORMULATION OF THE PROBLEM

Let \( z \)-axis be taken the direction of flow along the axis of the pipe. Then \( u = 0, v = 0 \) for steady and incompressible fluid the velocity component is independent of \( z \). The equation of continuity.

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{.................(1)}
\]

But \( u = 0, \ v = 0 \)

\[
\frac{\partial w}{\partial z} = 0 \quad \text{.................(2)}
\]

\[\Rightarrow w = \psi(x, y) \quad \text{.................(3)}\]

\( AB = BC = CA = 2a \sqrt{3}, \quad AN = 3a \)

i.e. \( w \) is independent of \( z \)

The Navier-Stokes equations of in the absence of body forces.

\[
\frac{\partial P}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) - \frac{\mu B^2}{\rho} w = 0
\]

\[
\Rightarrow \frac{\partial P}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) - \frac{\mu B^2}{\rho} w = 0 \quad \text{.................(4)}
\]

let \( \frac{\sigma B^2}{\rho \mu} = B^2 \)

It is clear from (3) & (4) \( P \) is independent of \( x \) & \( y \) i.e. \( p \) is the Function of \( z \)

SOLUTION OF THE PROBLEM \( p = p(z) \)

\[
\frac{\partial p}{\partial z} = \frac{dp}{dz} = \text{Constant} = -P
\]

\[\mu \left[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - B^2 w \right] = \frac{dp}{dz} \Rightarrow \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - B^2 w = \frac{-P}{\mu} \quad \text{.................(6)}\]

\[
(D^2 + D^2 - B^2)w = \frac{-P}{\mu} \quad \text{C.F.} = \sum a_n e^{h_n x + h'_n y} \quad \text{Where} \ h_n \text{ &} \ h'_n \text{ are related by} \ h_n^2 + h'_n^2 - B^2 = 0
\]

and \( \mathcal{P.I.} = \frac{1}{D^2 + D^2 - B^2} \left( -\frac{P}{\mu} \right) = \frac{P}{B^2 \mu} \Rightarrow w(x, y) = \sum a_n e^{h_n x + h'_n y} + \frac{1}{B^2 \mu} P \quad \text{Where} \ h_n^2 + h'_n^2 = B^2
\]

Case \(-\) using boundary conditions at \( y = \frac{x+2a}{\sqrt{3}} \)

\[-\frac{P}{B^2 \mu} = \sum_{n=1}^{\infty} a_n e^{h_n x + h'_n \left( \frac{x+2a}{\sqrt{3}} \right)} \quad \text{.................(7)}\]
\[ \text{and at } y = -\frac{x + 2a}{\sqrt{3}} \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n e^{h_n x - h_n'} \left( \frac{x + 2a}{\sqrt{3}} \right) = -\frac{P}{B^2 \mu} \]  
\[ \text{From (7) & (8) } \sum_{n=1}^{\infty} a_n \left[ h_n x + h_n' \left( \frac{x + 2a}{\sqrt{3}} \right) - e \left( h_n x + h_n' \left( \frac{x + 2a}{\sqrt{3}} \right) \right) \right] = 0 \quad \Rightarrow \quad e = h_n x + h_n' \left( \frac{x + 2a}{\sqrt{3}} \right) = e \]  

at \( x = a \)
\[ e^{h_n a} + \frac{h_n a}{\sqrt{3}} + 2ah_n = e^{h_n a - ah_n - 2ah_n} \quad \Rightarrow \quad h_n = 0, \quad h_n^2 + h_n^2 = B^2 \quad : h_n = B \quad \text{& } h_n' = 0 \]
\[ \sum_{n=1}^{\infty} a_n e^{b_n x} + \frac{P}{B^2 \mu} = 0 \quad \Rightarrow \quad \frac{P}{B^2 \mu} = e^{b_n} \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n = -\frac{P}{B^2 \mu} e^{-b_n} \Rightarrow w_i(x, y) = \frac{P}{B^2 \mu} \left[ 1 - e^{b(x-a)} \right] \]

Case - II \( w(x, y) = 0 \) at \((-2a, 0)\) \& \( w(x, y) = 0 \) at \((a, a\sqrt{3})\)
\[ -\frac{P}{B^2 \mu} = \sum_{n=1}^{\infty} a_n e^{-2ah_n} \]  
\[ -\frac{P}{B^2 \mu} = \sum_{n=1}^{\infty} a_n e^{ah_n + a\sqrt{3} h_n'} \]  
\[ a_n + a\sqrt{3} h_n' = -2ah_n \quad \Rightarrow \quad a\sqrt{3} h_n' = -3ah_n \quad \Rightarrow \quad h_n' = -\sqrt{3} h_n \]
\[ h_n^2 + h_n^2 = B^2 \quad \Rightarrow \quad h_n = \pm \frac{B}{2} \quad \text{& } h_n' = \pm \frac{\sqrt{3} B}{2} \quad , \quad \sum_{n=1}^{\infty} a_n = -\frac{P}{B^2 \mu} e^{B_n} \]
\[ w_2(x, y) = \frac{P}{B^2 \mu} \left[ 1 - e^{\frac{B(x+y+2a)}{2}} \right] \]

Case – III \( w(x, y) = 0 \) at \((-2a, 0)\) \& \( w(x, y) = 0 \) at \((a, -a\sqrt{3})\)
\[ -\frac{P}{B^2 \mu} = \sum_{n=1}^{\infty} a_n e^{-2ah_n} \]  
\[ -\frac{P}{B^2 \mu} = \sum_{n=1}^{\infty} a_n e^{ah_n - a\sqrt{3} h_n'} \]  

On solving: \( h_n = \frac{B}{2} \), \( h_n' = \frac{\sqrt{3} B}{2} \) \& \( \sum_{n=1}^{\infty} a_n = -\frac{P}{B^2 \mu} e^{B_n} \Rightarrow w_3(x, y) = \frac{P}{B^2 \mu} \left[ 1 - e^{\frac{B(x+y+2a)}{2}} \right] \]
\[ w(x, y) = \frac{P}{B^2 \mu} \left[ 1 - e^{\frac{B(x+y+2a)}{2}} + e^{\frac{\sqrt{3} B y}{2}} \right] - e^{B(x-a)} \]
\[ = \frac{P}{B^2 \mu} \left[ 1 - e^{\frac{B(x+y+2a)}{2}} + e^{\frac{\sqrt{3} B y}{2}} \right] - e^{B(x-a)} \]
\[ w(x, y) = \frac{P}{B^2 \mu} \left[ 1 - 2 \cosh \frac{\sqrt{3} By}{2} e^{-\frac{B(x+2a)}{2}} - e^{B(x-a)} \right] \] ........ (13)

The volumetric Flow

\[ Q = \int \int w(x, y) \, dx \, dy = \int_{x=-a}^{a} \int_{y=-\frac{x+a}{2}}^{\frac{x+a}{2}} \frac{P}{B^2 \mu} \left( 1 - 2 \cosh \frac{\sqrt{3} By}{2} e^{-\frac{B(x+2a)}{2}} - e^{B(x-a)} \right) \, dx \, dy \]

\[ = \frac{2P}{B^2 \mu} \int_{-2a}^{a} \int_{0}^{\frac{x+2a}{2B}} \left( 1 - 2 \cosh \frac{\sqrt{3} By}{2} e^{-\frac{B(x+2a)}{2}} - e^{B(x-a)} \right) \, dy \, dx \]

\[ = \frac{2P}{B^2 \mu} \int_{-2a}^{a} \left\{ \frac{x+2a}{2B} - \frac{4}{\sqrt{3B}} \sinh \frac{B(x+2a)}{2} e^{-\frac{B(x+2a)}{2}} - \frac{(x+2a)}{2B} e^{B(x-a)} \right\} \, dx \]

\[ = \frac{1}{2} \int_{-2a}^{a} e^{B(x+2a)/2} - 1 \, dx = \frac{1}{2} \left[ e^{B(x+2a)/2} - a \right]_{-2a}^{a} = \frac{1}{2} \left[ e^{3aB}/B - a - B - 2a \right] = \frac{1}{2} \left[ e^{3aB}/B - B - 3a \right] = \frac{1}{2} \left[ e^{3aB} - 1 - 3aB \right] \]

Let \[ I_1 = \int_{-2a}^{a} e^{B(x+2a)/2} \sinh \frac{B(x+2a)}{2} \, dx = \frac{1}{2} \left( e^{B(x+2a)/2} - \frac{B(x+2a)}{2} \right) \]

Let \[ I_2 = \int_{-2a}^{a} \left( x+2a \right) e^{B(x-a)} \, dx = \left[ \frac{(x+2a)}{2B} e^{B(x-a)} \right]_{-2a}^{a} = \frac{3a}{2B} e^{3aB} - \frac{1}{2} \left( e^{3aB} - 1 - 3aB \right) \]

Let \[ I_3 = \int_{-2a}^{a} \left( x+2a \right) \sqrt{3} \, dx = \frac{\left( x+2a \right)^2}{2\sqrt{3}} \left[ -2a \right] = \frac{3a^2}{2\sqrt{3}} + \frac{3a}{2\sqrt{3}} = \frac{3a}{2\sqrt{3}} \]

\[ Q = \frac{2P}{B^2 \mu} \left[ I_3 - 4/\sqrt{3B} - I_1 - I_2 \right] = \frac{2P}{B^2 \mu} \left[ \frac{3a^2}{2} - 4/\sqrt{3B} - \frac{3a}{2B} e^{3aB} - 1 - 3aB \right] - 1/\sqrt{3B} \left( 3aB - 1 + e^{-3aB} \right) \]

\[ Q = \frac{2P}{B^2 \mu} \left\{ \frac{9a^2}{2\sqrt{3} B^2} + \frac{3}{2\sqrt{3} B^2} e^{3aB} + \frac{2}{\sqrt{3} B^2} + 6a \frac{3a}{\sqrt{3} B^2} - \frac{3a}{\sqrt{3} B^2} + \frac{3a}{\sqrt{3} B^2} \right\} \]

\[ = \frac{2P}{B^2 \mu} \left\{ \frac{9a^2}{2\sqrt{3} B^2} + \frac{3}{2\sqrt{3} B^2} e^{3aB} - \frac{3a}{\sqrt{3} B^2} e^{-3aB} \right\} \]
\[ Q = \frac{2 P}{\sqrt{3} \mu B^2} \left\{ \frac{9a^2}{2} + \frac{3}{B} \left( a + \frac{1}{B} \right) - \frac{1}{B^2} \left( 2e^{3aB} - e^{-3aB} \right) \right\} \]  \hspace{2cm} \text{.........(14)}

Since \( w(x, y) = \frac{P}{\mu B^2} \left[ 1 - 2 \cosh \frac{\sqrt{3} B y}{2} e^{-\frac{B(x+2a)}{2}} - e^{B(x-a)} \right] \)

Let \( \bar{q} = u \hat{i} + v \hat{j} + w \hat{k} = \frac{P}{\mu B^2} \left[ 1 - 2 \cosh \frac{\sqrt{3} B y}{2} e^{-\frac{B(x+2a)}{2}} - e^{B(x-a)} \right] \hat{k} \)

Let \( \Omega_x, \Omega_y, \Omega_z \) are vorticity components

\[ \Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{P}{\mu B^2} \left[ -\sqrt{3} \sinh \frac{\sqrt{3} B y}{2} e^{-\frac{B(x+2a)}{2}} \right] = -\frac{\sqrt{3} P}{\mu B} e^{-\frac{B(x+2a)}{2}} \sinh \frac{\sqrt{3} B y}{2} \]

\[ \Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = -\frac{P}{\mu B^2} \left[ -B \cosh \frac{\sqrt{3} B y}{2} e^{-\frac{B(x+2a)}{2}} - B e^{B(x-a)} \right] = \frac{P}{\mu B} \left[ \cosh \frac{\sqrt{3} B y}{2} e^{-\frac{B(x+2a)}{2}} + e^{B(x-a)} \right] \]  \& \( \Omega_z = 0 \)

Equation of vortex line

\[ \frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z} \]

\[ \frac{dx}{\sqrt{3} e^{-\frac{B(x+2a)}{2}} \sinh \frac{\sqrt{3} B y}{2}} = \frac{dy}{\frac{P}{\mu B} \left[ \cosh \frac{\sqrt{3} B y}{2} e^{-\frac{B(x+2a)}{2}} + e^{B(x-a)} \right]} = \frac{dz}{0} \]

Taking 1st Two

\[ \int \left( \cosh \frac{\sqrt{3} B y}{2} \frac{\sinh \frac{\sqrt{3} B y}{2}}{e^{-\frac{B(x+2a)}{2}}} \right) \left( \sinh \frac{\sqrt{3} B y}{2} \right) dy = C_1 \]

\[ \int \cosh \frac{\sqrt{3} B y}{2} \left( \frac{e^{-\frac{B(2x-2y+2a)}{2}}}{\sqrt{3} B} \right) dy = \sqrt{3} \left( \frac{2}{\sqrt{3} B} \cosh \frac{\sqrt{3} B y}{2} \right) = C_1 \]

\[ \int \cosh \frac{B(x+2a)}{2} \left( \frac{e^{-\frac{B(x-4a)}{2}}}{\sqrt{3} B} \right) dy = C_1 \]

\[ \frac{2}{B} \sinh \frac{B(x+2a)}{2} + \frac{2}{B} e^{-\frac{B(x+2a)}{2}} + \frac{2}{B} C_1 \cosh \frac{\sqrt{3} B y}{2} = C_1 \Rightarrow \sinh \frac{B(x+2a)}{2} + e^{-\frac{B(x+2a)}{2}} + \cosh \frac{\sqrt{3} B y}{2} = \frac{C_1}{B} = A \]
The first vortex line
\[ e^{-\frac{B(x-4a)}{2}} + \text{Sinh} \frac{\sqrt{3} B y}{2} + \text{Cosh} \frac{\sqrt{3} B y}{2} = \sqrt{5} \] .................\( \sqrt{15} \)

taking last two \( dz = 0 \) ⇒ the second vortex line \( z = B \) ................. \( \text{(16)} \)

Table for Velocity: Let \( P = 2, \ \mu = .5 \) are fixed and \( B = \sqrt{\frac{\sigma B_0^2}{\rho \mu}} \), \((x, y)\) are change

<table>
<thead>
<tr>
<th>( \sqrt{\frac{\sigma B_0^2}{\rho \mu}} )</th>
<th>((x, y))</th>
<th>(-9, \frac{1}{6\sqrt{3}})</th>
<th>(-12, \frac{1}{2\sqrt{3}})</th>
<th>(-15, \frac{1}{\sqrt{3}})</th>
<th>(-18, \frac{2}{\sqrt{3}})</th>
<th>(-21, \frac{3}{\sqrt{3}})</th>
<th>(-24, \frac{4}{\sqrt{3}})</th>
<th>(-27, \frac{5}{\sqrt{3}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( w(x, y) )</td>
<td>2.209</td>
<td>3.589</td>
<td>3.8997</td>
<td>3.969</td>
<td>3.9896</td>
<td>3.996</td>
<td>3.999</td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>( w(x, y) )</td>
<td>.8315</td>
<td>8.795</td>
<td>12.519</td>
<td>14.203</td>
<td>15.025</td>
<td>15.45</td>
<td>15.683</td>
</tr>
<tr>
<td>( \frac{1}{3} )</td>
<td>( w(x, y) )</td>
<td>-8.346</td>
<td>9.178</td>
<td>19.622</td>
<td>25.677</td>
<td>29.324</td>
<td>31.58</td>
<td>33.024</td>
</tr>
<tr>
<td>( \frac{1}{4} )</td>
<td>( w(x, y) )</td>
<td>-25.164</td>
<td>1.914</td>
<td>21.41</td>
<td>34.21</td>
<td>42.82</td>
<td>48.71</td>
<td>52.82</td>
</tr>
<tr>
<td>( \frac{1}{6} )</td>
<td>( w(x, y) )</td>
<td>-99.8</td>
<td>-42.65</td>
<td>.316</td>
<td>32.23</td>
<td>56.26</td>
<td>74.54</td>
<td>88.58</td>
</tr>
</tbody>
</table>

CONCLUSION AND DISCUSSION

In this paper, we have investigated the velocity \( w(x, y) \) by the table-1 of equation (13) between velocity and point \((x, y)\), it is clear that the velocity \( w(x, y) \) of increases in the interval \( \left(-9, \frac{1}{6\sqrt{3}}\right) \) at the different values of \( \sqrt{\frac{\sigma B_0^2}{\rho \mu}} \). Again the velocity \( w(x, y) \) increases correspondingly in the interval \( \left(-9, \frac{1}{6\sqrt{3}}\right) \) when \( \sqrt{\frac{\sigma B_0^2}{\rho \mu}} \) decreases from 1 to \( \frac{1}{6} \). Negative sign of velocity \( w(x, y) \) shows that the direction of flow is in opposite to the direction of motion of fluid. Also we have investigated the volumetric flow, vortex lines respectively by the equations (14), (15) & (16).

REFERENCES


