Some Properties of the Lattice of Convex Edge Set of a Connected Directed Graph

Asha Saraswathi B. And Lavanya S.
Department Of Mathematics, P.A.College Of Engineering
Mangalore, D.K -574153
E-Mail Id: ashasaraswathib@gmail.com, lav_abh@yahoo.co.in

ABSTRACT

Let G be a connected directed graph and E(G) be the directed edge set of G. A subset C of E(G) is said to be convex if for any $e_i, e_j \in C$, there is a directed path containing $e_i, e_j$ and the edge set of every $e_i - e_j$ geodesic is contained in C. Let $\text{Con}(G)$ be the set of all convex edge sets of G together with empty set ordered by set inclusion relation. Then $\text{Con}(G)$ forms a lattice if and only if G has an Euler trail. In this paper cardinality of the lattice $\text{Con}(G)$ is discussed. Also some of the properties of the lattice $\text{Con}(G)$ are studied.

Index term: Lattices, Chains, Irreducibility, Connected digraphs, Convex edge sets, Paths, Cycles

MSC: 06B99, 05C20, 05C38

1. Introduction

Motivated by the studies on the lattice of convex sets of a connected graph [8], the set of convex edge sets of connected digraphs together with empty set is considered in [1] and it is found that this set forms a lattice with respect to the partial order set inclusion if and only if digraph contains an Euler trail. In this paper we studied properties of these lattices when the digraph G is directed path and directed cycle. Also irreducibility criteria and conditions under which $\text{Con}(G)$ becomes lower semimodular is discussed. It is proved that if $|E(G)| \geq 3$, $\text{Con}(G)$ satisfies lower covering condition if and only if G is a directed cycle $C_3$.

For terminologies and notations used in this paper we refer to [3] and [4].

2. Preliminaries

Let G be a finite connected digraph. E(G) be the edge set of G. A set $C \subseteq E(G)$ is said to be convex in G if for every two edges $e_i, e_j \in C$, there is a directed path containing $e_i, e_j$ and the edge set of every $e_i - e_j$ geodesic (i.e. shortest directed path containing $e_i$ and $e_j$) is contained in C. In a digraph G, a walk in which no edge is repeated is a (directed) trail. A closed walk in which no edge is repeated is a (directed) cycle. A trail containing all the edges of G is Euler trail and a circuit containing all the edges of G is Euler circuit. An element ‘a’ of a lattice L is join irreducible if $a = b \lor c$ implies that both $b$ and $c$ cover $a \land b$.

For a finite connected digraph G, let the set of all convex edge sets in G together with empty set be denoted by $\text{Con}(G)$. Define a binary relation $\leq$ on $\text{Con}(G)$ by, for $A, B \in \text{Con}(G)$, $A \leq B$ if and only if G has an Euler trail. Then clearly $\leq$ is a partial order on $\text{Con}(G)$. Moreover $\langle \text{Con}(G), \leq \rangle$ forms a lattice where for $A, B \in \text{Con}(G)$, $A \land B = \text{Min} \{A, B\}$ and $A \lor B = \langle A \cup B \rangle$ is the smallest convex edge set containing $A \cup B$.

For example, the lattice given in Fig 2.2 represents the lattice $\langle \text{Con}(G), \leq \rangle$ of the connected digraph G given in Fig 2.1.
Hereafter we consider digraph $G$ containing an Euler trail and use $\text{Con}(G)$ to represent the lattice $<\text{Con}(G), \leq>$. 

### 3. On the Lattice $\text{Con}(G)$

**Remark 3.1:** $\text{Con}(G)$ is a chain if and only if $G$ is a directed graph with single edge.

**Remark 3.2:** If $G$ is a directed graph with two edges, then $\text{Con}(G)$ will be as shown in Fig 3.1 which is a Boolean algebra.

![Fig 3.1](image_url)

**Theorem 3.3:** If $G$ is a directed cycle with $n$ edges, then $|\text{Con}(G)| = \left(\left\lceil \frac{n}{2} \right\rceil \times n \right) + 2$

(Where $\left\lceil \frac{n}{2} \right\rceil = \text{smallest integer} \geq \frac{n}{2}$)

**Proof:** Let $G$ be the directed cycle. There are $n$ convex sets with single element, $n$ convex sets with two elements, and so on, finally $n$ convex sets with $\left\lceil \frac{n}{2} \right\rceil$ elements. Hence there are $\left\lceil \frac{n}{2} \right\rceil \times n$ such convex sets. Therefore $|\text{Con}(G)| = \left(\left\lceil \frac{n}{2} \right\rceil \times n \right) + 2$, including $\emptyset$ and $E(G)$.

**Theorem 3.4:** If $G$ is a directed path with $n$ edges, then $|\text{Con}(G)| = \frac{n(n+1)}{2} + 1$

**Proof:** There are $n$ convex sets with single edge, $n-1$ convex sets with two edges, $n-2$ convex sets with threeedges and so on, finally one convex set with $n$ edges. Including empty set, $|\text{Con}(G)| = n + (n-1) + (n-2) + \ldots + 1 + 1 = \frac{n(n+1)}{2} + 1$

**Theorem 3.5:** An element $A \in \text{Con}(G)$ is doubly irreducible if and only if $A = \{e_i\}$ where $e_i = \{u,v\}$ is a pendant edge with indegree of $u$ is $0$ or $1$ and outdegree of $v$ is $0$ or $2$ respectively OR $\{u,v\}$ and $\{v,u\}$ is a directed cycle with indegree of $u=1$ and outdegree of $v=1$.

**Proof:** Let $A \in \text{Con}(G)$ be doubly irreducible. If $A$ contains more than one element say $A = \{e_1, e_2, \ldots e_n\}$, then $A = \bigvee_{i=1}^{n} \{e_i\}$ and therefore $A = \{e_i\}$ for some $i$, since $A$ is join irreducible.

If $e_i = \{u, v\}$ is not a pendant edge, then indegree of $u$ is one or more and outdegree of $v$ is one or more, then there will be a directed path $e_i e_k e_j$. Then $\{e_i\} = \{e_j, e_k\} \setminus \{e_i\}$, contradiction to $\{e_i\}$ is meet irreducible. Let $e_i = \{u, v\}$ be a pendant edge with indegree of $u = 2$ or more, then there will be edges $e_j$ and $e_k$ such that $e_j = \{u, u\}$, $e_k = \{u, u\}$ and $\{e_j\} = \{e_i, e_k\} \setminus \{e_j\}$, contradiction to $\{e_j\}$ is meet irreducible. Similarly if $e_i = \{u, v\}$ is a pendant edge with outdegree of $v = 2$ or more, then we get a contradiction to $\{e_i\}$ is meet irreducible. Thus if indegree of $u=2$ or more OR outdegree of $v=2$ or more, then $\{e_i\}$ becomes meet reducible.

Conversely $A = \{e_i\}$ is join irreducible. If $A$ is meet reducible say $A = B \wedge C = B \cap C$ for some $B, C \in \text{Con}(G)$ such that $A \neq B, A \neq C$. Then $\{e_i\} \in B \cap C$. Consider $\{e_i\} \in B, \{e_k\} \in C$ where $e_j \neq e_k, e_k \neq e_i$. Let $e_i, e_j, e_k, \ldots e_i$ be the shortest path connecting $e_i, e_j$ in $B$. Also let $e_i, e_j, e_k, \ldots e_k$ be the shortest path connecting $e_i, e_k$ in $C$. If $f_1 = g_1$, then $f_1 \in B \cap C$ contradiction to $B \cap C = \{e_i\}$. Also if $f_1 \neq g_1$, then outdegree of $v > 1$ contradiction to the fact that outdegree of $v$ is at most 1.

**Theorem 3.6:** Let $G$ be a directed graph with $|E(G)| \geq 3. \text{Con}(G)$ satisfies lower covering condition if and only if $G$ is a directed cycle $C_3$.

**Proof:** If $G$ is $C_3$, then $\text{Con}(G)$ is distributive [1] and hence $\text{Con}(G)$ satisfies lower covering condition. Conversely, let $\text{Con}(G)$ satisfies lower covering condition. If $G$ is not $C_3$, then $G$ contains a trail (which is not a circuit) say $e_i e_j e_k$. Clearly $\emptyset = \{e_j\} \leq \{e_k\} \wedge \{e_k\}$. But $\{e_j\} \leq \{e_i\} \vee \{e_k\}$. Which implies $\{e_k\} \leq \{e_i\} \vee \{e_k\}$ contradiction to $\text{Con}(G)$ satisfies lower covering condition. Hence $G$ must be $C_3$.

**Theorem 3.7:** $\text{Con}(G)$ is lower semimodular (LSM) in the following cases.

1) $G$ is a directed cycle $C_3$

2) $G$ is of the form given in Fig 2.1

3) $G$ is a directed path or directed path containing two element cycles at its end vertices.

4) $G$ is a directed path containing three element cycles at its end vertices.

**Proof:** If $G$ is a directed cycle $C_3$, then $\text{Con}(G)$ is modular [1]. Every modular lattice is LSM. If $G$ is of the form given in Fig 2.1, then $\text{Con}(G)$ will be as shown in Fig 2.2. Clearly it is LSM. Let $G$ be as given in case 3. Let $e_1 e_2 \ldots e_n$ be the Euler Trail. All possible convex edge sets are as follows. Emptyset, \{e_1\}, \{e_2\}, \ldots \{e_n\}, \{e_1, e_2\}, \ldots \{e_{n-1}, e_n\}, \{e_1, e_2, e_3\}, \ldots \{e_{n-2}, e_{n-1}, e_n\}, \{e_1, e_2, e_3, e_4\}, \ldots \{e_n\} Continuing like this $\{e_1, e_2, \ldots e_n\}$ is the maximum element. If $A \vee B$ covers both $A$ and $B$, then $A, B, A \vee B$ are of the form $\{e_1, e_{i+1}, \ldots e_k\}$, $\{e_{i+1}, e_{i+2}, \ldots e_{k+1}\}$ and $\{e_i, e_{i+1}, \ldots e_{k+1}\}$ respectively. Clearly $A \wedge B = \ldots$
\{e_{i+1}, e_{i+2}, \ldots e_k\} and it is covered by both A and B. Thus Con(G) is LSM.

Similarly, if G is a directed path containing 3 element cycles at its end vertices, then Con(G) will be LSM.

**Theorem 3.8:** If G contains a cycle of length \( \geq 4 \), then Con(G) is not LSM.

**Proof:** Let G contains a directed cycle \( C_n \). Say \( e_1e_2\ldots e_n \) with \( n \geq 4 \).

Case 1: When \( n \) is even

Taking \( A = \{e_1, e_2, \ldots e_{\lceil \frac{n}{2} \rceil}\} \) and \( B = \{e_{\lceil \frac{n}{2} \rceil + 1}, e_{\lceil \frac{n}{2} \rceil + 2}, \ldots, e_n\} \), we get \( A \lor B = \{e_1, e_2, \ldots e_n\} \) and \( A \land B = \emptyset \). Clearly \( A \lor B \) covers both A and B. But \( A \land B \) is not covered by A and B. Because \( \emptyset \) is always covered by singleton sets. Here both A and B contain two or more elements (since \( n \geq 4 \)).

Case 2: When \( n \) is odd

Taking \( A = \{e_1, e_2, \ldots e_{\lceil \frac{n}{2} \rceil}\} \) and \( B = \{e_{\lceil \frac{n}{2} \rceil + 1}, e_{\lceil \frac{n}{2} \rceil + 2}, \ldots, e_n\} \), we get \( A \lor B = \{e_1, e_2, \ldots e_n\} \) and \( A \land B = \{e_{\lceil \frac{n}{2} \rceil + 1}\} \). Clearly \( A \lor B \) covers both A and B. But \( A \land B \) is not covered by A and B. Because singleton sets are always covered by two element sets. Here both A and B contain three or more elements (since \( n \geq 5 \)).

**Remark 3.9:** If G is any of the forms given in Theorem 3.7, then Con(G) is LSM and hence they satisfy Jordan Dedekind chain condition. Infact if G is a directed cycle \( C_n \), then it can be observed that Con(G) satisfies Jordan Dedekind chain condition. As there are \( n \) convex sets with single element, \( \lceil \frac{n}{2} \rceil \) convex sets with two elements and so on, finally \( n \) convex sets with \( \lceil \frac{n}{2} \rceil \) elements. Empty set is covered by single element set. Single element sets are covered by two element sets. Two element sets are covered by three element sets continuing like this, \( \lceil \frac{n}{2} \rceil \) element sets are covered by \( E(G) \).

Therefore all maximal chains connecting any two elements of Con(G) are of the same length.

**References:**


