Abstract

Recently Lahiri and Banerjee [1] have introduced the concept of Ritt-order of an entire Dirichlet Series and proved sum and product theorems. They as obtained Ritt order for derivatives. In this paper, we introduced the concept of L-Ritt order and discuss it for sum, products and derivatives of functions.

Keywords: Entire dirichlet series, Ritt order, relative L – Ritt order, property (A).

1. Introduction, Definition and Lemmas

For entire functions $g_1$ and $g_2$ let $G_1(r) = \max \{g_1(z) : |z| = r\}$ and $G_2(r) = \max \{g_2(z) : |z| = r\}$.

If $g_1$ is non constant then $G_1(r)$ is strictly increasing and a continuous function of $r$ and its inverse $G_1^{-1} : (g_1(0), \infty) \to (0, \infty)$ exits and $\lim_{R \to \infty} G_1^{-1}(R) = \infty \ldots (1.1)$

Bernal [5] introduced the definition of relative order of $g_1$ with respect to $g_2$ denoted by $\rho_{g_2}(g_1)$ as follows
\[ \rho_{g_1}(g_1) = \inf\{\mu > 0 : G_1(r) < G_2(r^\mu) \text{ for all } r > r_0(\mu) > 0\} . \tag{1.2} \]

Let \( f(s) \) be an entire function of the complex variable \( s = \sigma + it \) defined by everywhere absolutely convergent Dirichlet series \( \sum_{n=1}^{\infty} a_n e^{\lambda_n} \). (1.3)

where \( 0 < \lambda_n < \lambda_{n+1}(n \geq 1) \), \( \lambda_n \to \infty \) as \( n \to \infty \) and \( a_n s \) are complex constants.

If \( \sigma_c \) and \( \sigma_a \) denote respectively the abscissa of convergence and absolute convergence of (1.3) then in this case clearly \( \sigma_c = \sigma_a = \infty \).

Let \( F(\sigma) = \max\{f(\sigma + it)\} \). (1.4)

Then the Ritt order \([16]\) of \( f(s) \) denoted by \( \rho(f) \) is given by

\[ \rho(f) = \lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log \log F(\sigma)}{\sigma} = \lim_{\sigma \to \infty} \frac{\log[2]}{\sigma} F(\sigma). \tag{1.5} \]

In other words \( \rho(f) = \inf\{\mu > 0 : \log F(\sigma) < \exp(\sigma \mu) \text{ for all } \sigma > R(\mu)\} . \tag{1.6} \)

Similarly the lower Ritt order of \( f(s) \) denoted by \( \lambda(f) \) may be defined.

In the paper we prove sum results on the related to relative L-Ritt order of an entire Dirichlet series. where \( L = L(\sigma) \) is a positive continuous function increasing slowly i.e. \( L(a\sigma) \approx L(\sigma) \) as \( \sigma \to \infty \) for every constants \( a \). In the paper we do not explain the standard definitions and notations in the theory of entire functions as those are available in [6]. The following definitions are well known.

**Definition 1.** The relative Ritt order of \( f(s) \) with respect to an entire function \( g(s) \) is defined by

\[ \rho_g(f) = \inf\{\mu > 0 : \log F(\sigma) < G(\sigma \mu) \text{ for all } \sigma \} \tag{1.7} \]

where \( G(r) = \max\{|g(s)| : |s| = r\} \). Clearly \( \rho_g(f) = \rho(f) \) if \( g(s) = e^s \). The following analogous definition from [5] will be needed.

**Definition 2.** A nonconstant entire function \( g(s) \) is said to have the property (A) if for any \( \delta > 1 \) and positive \( \sigma, [G(\sigma)]^\delta \leq G(\sigma^s) \) holds where \( G(\sigma) = \max\{|g(s)| : |s| = \sigma\} \).
Definition 3. The L-Ritt order \( \rho^L(f) \) and the L-Ritt lower order \( \lambda^L(f) \) of \( f(s) \) are defined as follows respectively

\[
\rho^L(f) = \limsup_{\sigma \to \infty} \frac{\log^{[2]} F(\sigma)}{\sigma L(\sigma)} \quad (1.8) \quad \lambda^L(f) = \liminf_{\sigma \to \infty} \frac{\log^{[2]} F(\sigma)}{\sigma L(\sigma)} \quad (1.9)
\]

Where \( \log^{[k]} x = \log(\log^{[k-1]} x) \) for \( k=1,2,3, \ldots \) and \( \log^{[0]} x = x \). Similarly one can define the relative L-Ritt and relative lower L-Ritt order of \( f(s) \).

Definition 4. The relative L-Ritt order \( \rho^L(g,f) \) and the relative lower L-Ritt order \( \lambda^L(g,f) \) of \( f(s) \) with respect to entire \( g(s) \) are respectively defined as

\[
\rho^L(g,f) = \limsup_{\sigma \to \infty} \frac{G^{-1} \log F(\sigma)}{\sigma L(\sigma)} \quad (1.10) \quad \lambda^L(g,f) = \liminf_{\sigma \to \infty} \frac{G^{-1} \log F(\sigma)}{\sigma L(\sigma)} \quad (1.11)
\]

Bernal [5] has proved the following.

**Lemma 1 [5].** If \( \alpha > 1, \ 0 < \beta < \alpha \) then \( G(\alpha \sigma) > \beta G(\sigma) \) for all large \( \sigma \).

**Lemma 2 [5].** If \( g \) is transcendental with \( g(0) = 0 \) then for all large \( \sigma \) and \( 0 < \delta < 1 \).

\[
G(\sigma^\delta) < \tilde{G}(\sigma) < G(2\sigma) \text{ where } \tilde{G}(\sigma) = \max \left\{ \| g'(z) \| : \| z \| = \sigma \right\}
\]

After Bernal, several papers on relative order of entire functions have appeared in the literature where growing interest of researcher on this topic has been noticed (see for example [2],[3],[4],[12],[13],[14],[15],[16]). During the past decades, several authors (see for example [17],[18],[20]) made close investigation on the properties of entire Dirichlet series related to Ritt order.

2. Main Results: Following the sections 1, have proved the following theorem.

**Theorem 1.** (a) \( \rho^L(g,f) = \limsup_{\sigma \to \infty} \frac{G^{-1} \log F(\sigma)}{\sigma L(\sigma)} \).

(b) If \( F_1(\sigma) \leq F_2(\sigma) \) for all large \( \sigma \), then \( \rho^L(g,f_1) \leq \rho^L(g,f_2) \).

**Proof:** (a) If \( \varepsilon > 0 \) is arbitrary then from the definition.
\[ \rho^L_g(f) + \varepsilon > \frac{G^{-1} \log F(\sigma)}{\sigma L(\sigma)} \quad \text{for all large } \sigma \]  

(2.1)

and there exist a sequence of value \( \sigma = \sigma_n \) tending to infinity.

\[ \frac{G^{-1} \log F(\sigma_n)}{\sigma_n L(\sigma_n)} > \rho^L_g(f) - \varepsilon \]  

(2.2) From (2.1) and (2.2)

\[ \limsup_{\sigma \to \infty} \frac{G^{-1} \log F(\sigma)}{\sigma L(\sigma)} = \rho^L_g(f). \]

**Proof:** (b) For arbitrary \( \varepsilon > 0 \) and for all large \( \sigma \), we can write from (a).

\[ F_2(\sigma) < \exp\left[ G(\sigma L(\sigma) \left( \rho^L_g(f_1) + \varepsilon \right) \right] \]

Since \( F_1(\sigma) \leq F_2(\sigma) \) for all large \( \sigma \), we obtain

\[ \rho^L_g(f_1) = \lim_{\sigma \to \infty} \frac{G^{-1} \log F(\sigma)}{\sigma L(\sigma)} \leq \rho^L_g(f_2) + \varepsilon \]

Since \( \varepsilon > 0 \) is arbitrary \( \rho^L_g(f_1) \leq \rho^L_g(f_2) \).

### 2.1 Sum and Product Theorems

In this section, we assume that \( f_1, f_2 \) etc. are entire functions of \( s \) defined by everywhere absolutely convergent ordinary Dirichlet series \( \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \sum_{n=1}^{\infty} \frac{b_n}{n^s} \) etc. The product of two such series is considered by Dirichlet product method, which is also everywhere absolutely convergent (see [9], pp 66).

**Theorem 2.** Let \( g(s) \) be an entire function having the property (A). Then

(i) \( \rho^L_g(f_1 \pm f_2) \leq \max\{\rho^L_g(f_1), \rho^L_g(f_2)\} \) Sign of equality holds when \( \rho^L_g(f_1) \neq \rho^L_g(f_2) \)

and (ii) \( \rho^L_g(f_1 f_2) \leq \max\{\rho^L_g(f_1), \rho^L_g(f_2)\} \).

**Proof:** (i) We may suppose that \( \rho^L_g(f_1) \) and \( \rho^L_g(f_2) \) both are finite, because in the contrary case the inequality follows immediately. We prove (i) for addition only, because the proof for
subtraction is analogous.

Let \( f = f_1 + f_2 \) , \( \rho = \rho_g^L(f) \rho^L_i = \rho_g^L(f_i) \) \( i = 1,2 \) and \( \rho^L_i(f_1) \leq \rho^L_i(f_2) \).

For arbitrary \( \varepsilon > 0 \) and for all large \( \sigma \), we have from Theorem 1(a)

\[
F_i(\sigma) < \exp\left[ G\left( \alpha L(\sigma) \left( \rho_g^L(f_1) + \varepsilon \right) \right) \right]
\]

\[
\leq \exp\left[ G\left( \alpha L(\sigma) \left( \rho_g^L(f_2) + \varepsilon \right) \right) \right]
\]

and \( F_2(\sigma) < \exp\left[ G\left( \alpha L(\sigma) \left( \rho_g^L(f_2) + \varepsilon \right) \right) \right] \)

So for all large \( \sigma \) \( F(\sigma) \leq F_i(\sigma) + F_2(\sigma) \)

\[
\leq 2 \exp\left[ G\left( \alpha L(\sigma) \left( \rho_g^L(f_2) + \varepsilon \right) \right) \right]
\]

\[
< \exp\left[ G\left( \alpha L(\sigma) \left( \rho_g^L(f_2) + \varepsilon \right) \right) \right]^{2}, \text{ since for all } x, 2 \exp(x) < \exp(x^2)
\]

\[
\leq \exp\left[ G\left( \alpha L(\sigma) \left( \rho_g^L(f_2) + \varepsilon \right) \right) \right]^{\delta} \text{ for every } \delta > 1, \text{ by property (A). Therefore}
\]

\[
\frac{- \log F(\sigma)}{\alpha L(\sigma)} < \left( \rho_g^L(f_2) + \varepsilon \right)^{\delta} \frac{\sigma}{\alpha} \left( L(\sigma) \right)^{-1} \text{ for all large } \sigma
\]

Taking first \( \delta \to 1+0 \) and then limit superior as \( \sigma \to \infty \) and nothing that \( \varepsilon > 0 \) is arbitrary, we obtain \( \rho_g^L(f) < \rho_g^L(f_2) \). This proves the first part of (i).

For the second part of (i), let \( \rho_g^L(f_1) < \rho_g^L(f_2) \).

and suppose that \( \rho_g^L(f_1) < \mu < \lambda < \rho_g^L(f_2) \).

Then for all large \( \sigma \) \( F_i(\sigma) < \exp\left[ G\left( \alpha L(\sigma) \mu \right) \right] \)

(2.3)

and there exist an increasing sequence \( \{ \sigma_n \}, \sigma_n \to \infty \)

\[
F_2(\sigma_n) > \exp\left[ G\left( \sigma_n L(\sigma_n) \lambda \right) \right] \text{ for } n = 1,2,3,... \quad (2.4)
\]

Using Lemma 1, by setting \( \alpha = \frac{\lambda}{\mu}, r = \sigma \mu, \beta = 1+\varepsilon, 0 < \varepsilon < 1 \) such that \( 1 < \beta < \alpha \), we obtain

\[
G\left( \frac{\lambda}{\mu} \sigma \mu \right) > (1+\varepsilon)G(\sigma \mu)
\]

i.e. \( G(\lambda \sigma) > (1+\varepsilon)G(\sigma \mu) \)
Therefore using (2.3) and (2.4) and the fact that \( G(\sigma) > \frac{\log 2}{\epsilon} \) for all large \( \sigma \), we obtain

\[
F_2(\sigma_n) > \exp[G(\sigma_n, L(\sigma_n), \lambda)]
\]

\[> \exp[(1 + \epsilon)G(\sigma_n, L(\sigma_n), \mu)] \]

\[> 2 \exp[G(\sigma_n, L(\sigma_n), \mu)] \]

\[> 2F_1(\sigma_n), \text{ for all large } n. \text{ (2.5)} \]

Now

\[
F(\sigma_n) \geq F_2(\sigma_n) - F_1(\sigma_n)
\]

\[> F_2(\sigma_n) - \frac{1}{2} F_2(\sigma_n), \text{ using (2.5)} \]

\[= \frac{1}{2} F_2(\sigma_n) \]

\[> \frac{1}{2} \exp[G(\sigma_n, L(\sigma_n), \lambda)], \text{ from (2.4)} \]

\[> \exp[(1 - \epsilon)G(\sigma_n, L(\sigma_n), \lambda)], \text{ for all large } n. \]

Let \( \rho^L_g(f_1) < \lambda_1 < \lambda < \rho^L_g(f_2) \), and \( 0 < \epsilon < \frac{\lambda - \lambda_1}{\lambda} \) (which is clearly permissible).

Using Lemma 1, by setting \( \alpha = \frac{\lambda}{\lambda_1}, \beta = \frac{1}{1 - \epsilon}, r = \sigma\lambda_1 \), we have, because \( 0 < \beta < \alpha \)

\[
G\left(\frac{\lambda}{\lambda_1} \sigma \lambda_1 \right) > \frac{1}{1 - \epsilon} G(\sigma \lambda_1),
\]

i.e. \( (1 - \epsilon)G(\lambda \sigma) > G(\sigma \lambda_1) \).

Hence for all large \( n \), \( F(\sigma_n) > \exp[G(\sigma_n, L(\sigma_n), \lambda_1)] \),

i.e. \( \frac{G^{-1} \log F(\sigma_n)}{\sigma_n L(\sigma_n)} > \lambda_1 \) for all large \( n. \)

This gives \( \rho^L_g(f) \geq \lambda_1 \). Since \( \lambda \) & \( \lambda_1 \) both are arbitrary in the interval \( (\rho^L_g(f_1), \rho^L_g(f_2)) \),

We have \( \rho^L_g(f) \geq \max\{\rho^L_g(f_1), \rho^L_g(f_2)\} \),
i.e. \( \rho_g^L(f_1 + f_2) \geq \max \{ \rho_g^L(f_1), \rho_g^L(f_2) \} \).

This in conjunction with the first part of (i) gives

\[ \rho_g^L(f_1 + f_2) = \max \{ \rho_g^L(f_1), \rho_g^L(f_2) \} \]

which proves (i) completely.

(ii) Let \( f = f_1 f_2 \) and the notations \( \rho_g^L(f), \rho_g^L(f_1) \) and \( \rho_g^L(f_2) \) have the analogous meanings as in (i). If \( \rho_g^L(f_1) \leq \rho_g^L(f_2) \) then for arbitrary \( \varepsilon > 0 \) for all large \( \sigma \)

\[
F(\sigma) \leq F_1(\sigma) F_2(\sigma) \\
< \exp[G(\sigma L(\sigma) (\rho_g^L(f_1) + \varepsilon))] \exp[G(\sigma L(\sigma) (\rho_g^L(f_2) + \varepsilon))] \\
\leq \exp[2G(\sigma L(\sigma) (\rho_g^L(f_2) + \varepsilon))] \\
\leq \exp[2G(\sigma L(\sigma) (\rho_g^L(f_2) + \varepsilon))^\delta] 
\]

for every \( \delta > 1 \), by property (A).

The above gives \( \frac{G^{-1} \log F(\sigma)}{\sigma L(\sigma)} \leq (\rho_g^L(f_2) + \varepsilon)^\delta (L(\sigma))^{\delta - 1} \) for all large \( \sigma \). Letting \( \delta \rightarrow 1 + 0 \) and then considering the fact that \( \varepsilon > 0 \) is arbitrary, we obtain \( \rho_g^L(f) \leq \rho_g^L(f_2) \) which proves the theorem.

2.2 Relative L-Ritt order of the derivative

**Theorem 3.** Let \( f(s) \) be an entire function defined by the Dirichlet series (1) having finite L-Ritt order \( \rho_g^L(f) \) and \( f'(s) \) be its derivative. Then \( \rho_g^L(f) = \rho_g^L(f') \) where \( g(s) \) is a transcendental entire function.

**Proof:** It is known([17], p139) that for all large value of \( \sigma \) and arbitrary \( \varepsilon > 0 \)

\[
F(\sigma) - \varepsilon < (\sigma L(\sigma) - \sigma_0 L(\sigma_0)) F'(\sigma) + \left| f'(s_0) \right| (3.1)
\]

where \( s_0 = \sigma_0 + i t_0 \) is a fixed complex number and \( F'(\sigma) = \lim_{\varepsilon \rightarrow \infty} \left| f'(\sigma + i \varepsilon) \right| \).

The inequality (3.1) implies \( F(\sigma) < (\sigma L(\sigma)) F'(\sigma) + A + \varepsilon \),
where $A$ is a constant. Taking logarithm, we see that for all large value of $\sigma$

$$\log F(\sigma) < \log\left[\alpha L(\sigma)F^-(\sigma)\right] + B_{\sigma}$$

Where $B_{\sigma} \to \infty$ as $\sigma \to \infty$

$$< \log F^-(\sigma) + \log(\alpha L(\sigma)) + B_{\sigma}$$

$$< \log F^-(\sigma) + \alpha L(\sigma)\left(\rho^L_\sigma\left(f^-'\right) + \varepsilon\right) + B_{\sigma}$$

$$< \log F^-(\sigma) + \alpha L(\sigma)\left(\rho^L_\sigma\left(f^-'\right) + 2\varepsilon\right)$$

$$< G[\alpha L(\sigma)\left(\rho^L_\sigma\left(f^-'\right) + \varepsilon\right) + \alpha L(\sigma)\left(\rho^L_\sigma\left(f^-'\right) + 2\varepsilon\right)$$

$$< G[\alpha L(\sigma)\left(\rho^L_\sigma\left(f^-'\right) + 2\varepsilon\right)](3.2)$$

because $G[\alpha L(\sigma)\left(\rho^L_\sigma\left(f^-'\right) + \varepsilon\right) + \alpha L(\sigma)\left(\rho^L_\sigma\left(f^-'\right) + 2\varepsilon\right)] < 1$ for all large $\sigma$ on using

([5], (d), p213) and ([6], p165).

From (3.2) $\rho^L_\sigma\left(f^-'\right) = \lim_{\sigma \to \infty} \frac{G^{-1}\log F(\sigma)}{\alpha L(\sigma)} \leq \rho^L_\sigma\left(f^-'\right) + 2\varepsilon$.

Since $\varepsilon > 0$ is arbitrary, $\rho^L_\sigma\left(f^-'\right) = \rho^L_\sigma\left(f^-'\right)$

To obtain the reverse inequality, we use the following inequality from ([17], p139)So

$$F^-'(\sigma) - \varepsilon \leq \frac{1}{\delta} F(\sigma + \delta)(3.3)$$

where $\varepsilon > 0$ is arbitrary and $\delta > 0$ is fixed.

So $\log F^-'(\sigma) \leq \log\left(\frac{1}{\delta} F(\sigma + \delta) + \varepsilon\right)$

$$= \log F(\sigma + \delta) + \log\left(\frac{1}{\delta} + \frac{\varepsilon}{F(\sigma + \delta)}\right)$$

$$\leq G\left[(\sigma + \delta)L(\sigma + \delta)(\rho^L_\sigma(f) + \varepsilon)\right] + \log\left(\frac{1}{\delta} + \frac{\varepsilon}{F(\sigma + \delta)}\right)$$

$$\leq G\left[(\sigma + \delta)L(\sigma + \delta)(\rho^L_\sigma(f) + 2\varepsilon)\right]$$

for all large $\sigma$. 

Therefore \( \rho_g^L(f^\prime) = \limsup_{\sigma \to \infty} \frac{G^{-1}\log F'(\sigma)}{\alpha L(\sigma)} \leq \rho_g^L(f) + 2\varepsilon. \)

Since \( \varepsilon > 0 \) is arbitrary, \( \rho_g^L(f^\prime) \leq \rho_g^L(f) \) which proves the theorem.

If we assume \( g(0) = 0 \), a simpler proof of the following theorem may be provided which relates the L-Ritt order of \( f \) relative to \( g \) and to its derivative \( g^\prime \).

**Theorem 4.** Let \( f(s) \) be an entire function defined by the Dirichlet series (1) and \( g(s) \) be an entire transcendental function with \( g(0) \), then

\[
\frac{1}{2} \rho_g^L(f) \leq \rho_g^L(f) \leq \rho_g^L(f).
\]

**Proof:** Since \( g(s) \) is transcendental with \( g(0) = 0 \), we have by Lemma 2 for all large \( \sigma \) and \( 0 < \delta < 1 \)

\[
G(\sigma^\delta) < \tilde{G}(\sigma) < G(2\sigma),
\]

where \( \tilde{G}(\sigma) = \max \{ |g^\prime(s)| : |s| = \sigma \} \). By computations it follows that

\[
\frac{1}{2} G^{-1}(\sigma) < G^{-1}(\sigma) < \left( G^{-1}(\sigma) \right)^\frac{1}{\delta},
\]

for all large \( \sigma \). Therefore we can write for all large \( \sigma \)

\[
\frac{1}{2} G^{-1}\left[ \log(F(\sigma)) \right]_{\delta} < \frac{G^{-1}\left[ \log(F(\sigma)) \right]}{\alpha L(\sigma)} \leq \left( G^{-1}\left[ \log(F(\sigma)) \right] \right)^\frac{1}{\delta},
\]

since \( \log F(\sigma) \) is increasing and tending to infinity as \( \sigma \to \infty \) (see [8], [9]). Letting \( \delta \to 1 - 0 \), we obtain for all large \( \sigma \)

\[
\frac{1}{2} G^{-1}\left[ \log(F(\sigma)) \right]_{\delta} < \frac{G^{-1}\left[ \log(F(\sigma)) \right]}{\alpha L(\sigma)} \leq \frac{G^{-1}\log F(\sigma)}{\alpha L(\sigma)},
\]

and this gives

\[
\frac{1}{2} \rho_g^L(f) \leq \rho_g^L(f) \leq \rho_g^L(f).
\]

which proves the theorem.
References


