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Relation between Fourier, Laplace and Z-transforms

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Abstract : This paper makes an attempt of consolidated and comparative study of Fourier Transform, Laplace Transform and z-transform. It also shows sequential mathematical flow of interlinking of the three transforms.

Key words : Fourier series, Fourier Integral, Fourier Transform, Laplace Transform, Z Transform.

INTRODUCTION

We tried to obtain a good answer for the Fourier and Laplace and Z- transforms relationship. Many of the explanations just mention that the relationship is that s=a+iw, so the Fourier transform becomes a special case of the laplace transform.

In the calculus we know that certain functions have power series representations of the form

$f(x) = \sum_{n=0}^{\infty} c_n x^n$ where $c_n = \frac{f^n(0)}{n!}$

These type of power series are useful for numerical calculations in addition to various other uses. If the function f is periodic with period p it can be written as sum of sine and cosine functions as

$f(x) = \sum_{n=1}^{\infty} a_n cos\frac{2n\pi x}{p} + \sum_{n=1}^{\infty} b_n sin\frac{2n\pi x}{p}$

let $\frac{2\pi}{p} = \omega$

$f(x) = \sum_{n=1}^{\infty} a_n \cos(n\omega x) + \sum_{n=1}^{\infty} b_n \sin(n\omega x)$

$\cos(n\omega x)\, f(x) = \cos(n\omega x) \sum_{n=1}^{\infty} a_n \cos(n\omega x) + \cos(n\omega x) \sum_{n=1}^{\infty} b_n \sin(n\omega x$

Integrating on both sides over c to c+p the RHS terms become zero except for the case when n=m

$\int_c^{c+p} \cos(m\omega x)\, f(x) dx = \int_c^{c+p} \frac{a_m}{2} dx = \frac{a_m p}{2}$

$a_m = \frac{2}{p}\int_c^{c+p} \cos(m\omega x)\, f(x)\, dx$ Similarly

$b_m = \frac{2}{p}\int_c^{c+p} \sin(m\omega x)\, f(x) dx$ And for n=0

$a_0 = \frac{2}{p}\int_c^{c+p} f(x) dx$

Therefore Fourier series is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega x) + \sum_{n=1}^{\infty} b_n \sin(n\omega x)$

Periodic signal may be written as

$f(x) = a\cos(\omega x) + b\sin(\omega x)$

$a = r cos\emptyset \qquad b = r sin\emptyset$

$f(x) = \sqrt{a^2+b^2}\cos(\omega x - \emptyset)$

Where $\sqrt{a^2+b^2}$ is amplitude and $\cos(\omega x - \emptyset)$ is phase

Now $e^{i\emptyset} = cos\emptyset + sin\emptyset$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega x) + \sum_{n=1}^{\infty} b_n \sin(n\omega x)$

$= c_0 + \sum_{n=1}^{\infty} c_n e^{in\omega x} + \sum_{n=1}^{\infty} d_n e^{-in\omega x}$

Where $c_n = \frac{a_n - ib_n}{2} \quad d_n = \frac{a_n + ib_n}{2}$

$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega x}$ and $c_n = \frac{1}{p}\int_c^{c+p} f(x) e^{-in\omega x} dx$

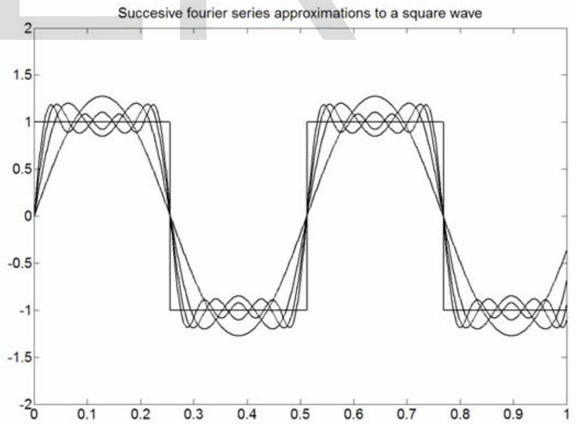
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{p}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{p})$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(t)\, cos\frac{n\pi t}{p}\, dt$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(t)\, sin\frac{n\pi t}{p}\, dt$$

$$f(x) = \frac{1}{2p}\int_{-p}^{p} f(t)dt + \sum_{n=1}^{\infty}\frac{1}{p}\int_{-p}^{p} f(t)\cos(\frac{n\pi t}{p})\cos(\frac{n\pi x}{p})dt + \sum_{n=1}^{\infty}\frac{1}{p}\int_{-p}^{p} f(t)\sin(\frac{n\pi t}{p})\sin(\frac{n\pi x}{p})dt$$

……………………………………………………………Eqn(1)



The above diagram explains the function of Fourier series.

Relation to Fourier -Transform

Taking limit as $p \to \infty$ in Eqn(1), as $f(x)$ is absolutely integrable $\int_{-\infty}^{\infty} |f(t)| dt < \infty$ we have

$$\lim_{p\to\infty} \frac{1}{p}\int_{-p}^{p} f(t) dt = 0$$

Let $\delta s = \pi/p$, $n.\delta s$ = s

Taking limit as $\delta s \to 0 \; as \; p \to \infty$

We get,

$$f(x) = \frac{1}{\pi}\int_{-\frac{\delta s}{\pi}}^{\frac{\delta s}{\pi}} f(t) \sum_{n=1}^{\infty} \cos[n\delta s(t-x)]\delta s\, dt$$

$$\sum \cos[n\delta s(t-x)] = \int_0^{\infty} \cos[s(t-x)]ds$$

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t) \int_0^{\infty} \cos[s(t-x)]\, ds\, dt$$

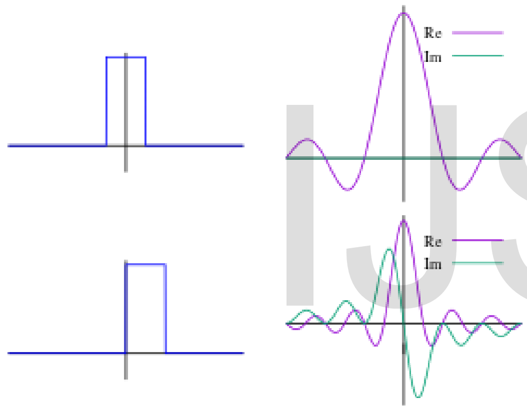
$$f(x) = \frac{1}{\pi}\int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos[s(t-x)]\, dt\, ds$$

$$f(x) = \frac{1}{2\pi}\int_0^{\infty} \int_{-\infty}^{\infty} f(t)\ [e^{is(t-x)} + e^{-is(t-x)}\ ]\, dt\, ds$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\ [e^{is(t-x)}\ ]\, dt\, ds$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} \int_{-\infty}^{\infty} f(t)\ [e^{ist}\ ]\, dt\, ds$$

This is exponential form of Fourier Integral Theorem.
F(s) = $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\ [e^{ist}\ ]\, dt$ is the Fourier Transform of f(t).
The location of the constant $\frac{1}{2\pi}$ is arbitrarily selected.
One may write as
F(s) = $\int_{-\infty}^{\infty} f(t)\ [e^{ist}\ ]\, dt$
The following diagram shows the output of time signal in Fourier Transform

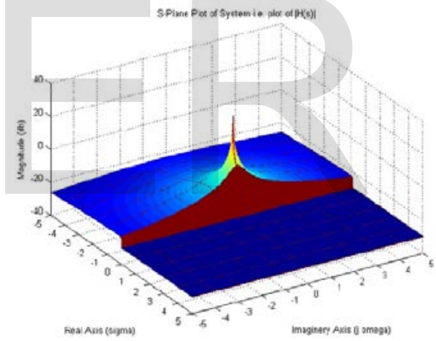


Relation to Laplace -Transform

Suppose the function and its derivative f' both are piecewise continuous functions for all t>=0 , then fourier transform of f (t) may not exist.
Then these functions are best handled by Laplace Transformation as it doesn't require f to be absolutely integrable.
For that we assume a related function as
g(t) = $e^{-kt} f(t)h(t)$, where k is positive real constant
h(t) Heaviside unit function.
Hence f satisfy the condition $\int_0^{\infty} e^{-kt} f(t)\ dt < \infty$
As g satisfies the conditions of Fourier integral theorem we can write
g(t) = $\frac{1}{2\pi}\int_0^{\infty} e^{-ist} \int_{-\infty}^{\infty} g(x) e^{isx}\ dx\ ds$
f(t)h(t) = $\frac{e^{ct}}{2\pi}\int_0^{\infty} e^{-ist} \int_{-\infty}^{\infty} f(x) e^{-(c-is)x}\, dx\, ds$
If we put p= -c then it takes the form
f(t)h(t) = $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt} \int_0^{\infty} f(x)e^{-px}\, dx\, dp$
Again as above these include pair of formulae, one of which is L(p) =$\int_0^{\infty} f(t)e^{-pt}dt$
Which we call as Laplace Transform.
Or

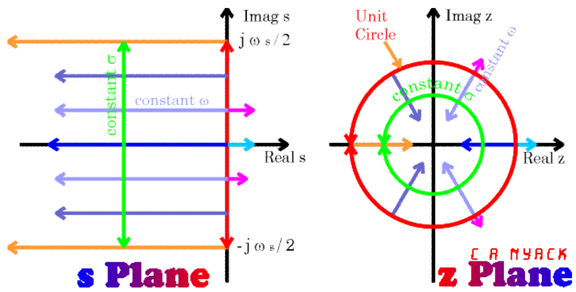
We can also proceed as follows
F(s) = $\int_{-\infty}^{\infty} f(t)\ [e^{ist}\ ]\, dt$ where s is real
This shows Fourier transform is the Laplace Transform when s is purely imaginary.
F(s) = L{x(t)} where s= iω
F(σ+iω)=$\int_{-\infty}^{\infty} f(t)e^{-(\sigma+i\omega)t}$
= $\int_{-\infty}^{\infty} f(t)\, e^{-(\sigma+i\omega)t}$dt
= $\int_{-\infty}^{\infty} [f(t)\, e^{-\sigma t}]e^{-i\omega t}\, dt$
= $\int_{-\infty}^{\infty} [f_1(t)]e^{-i\omega t}\, dt$
$F_{1(t)} = F[f_1(t)]$
This shows The Laplace transform is the Fourier transform of the transformed signal $f_1(t)$ = f(t)e^{-st}. Depending on whether *s* is positive/negative this represents a growing/negative signal

The following figure explains graphically the relation of the Fourier and Laplace Transform
Fourier Transform can be thought of as Laplace transform evaluated on the iw (imaginary) axis, neglecting the real part of complex frequency 's'. As shown in the figure below, the 3D graph represents the laplace transform and the 2D portion at real part of complex frequency 's' represents the fourier transform.



Relation to Z-Transform

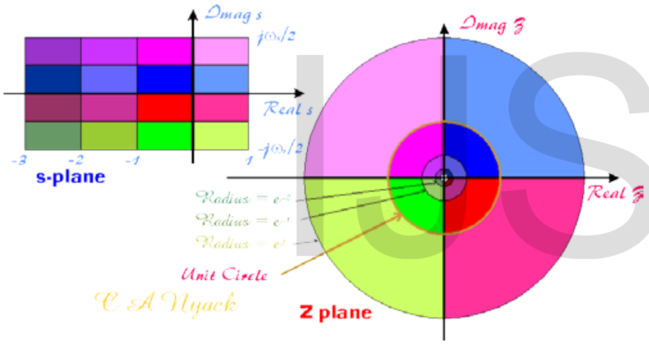
The Laplace Transform of a sampled signal can be written as:- $X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$
If the following substitution is made in the Laplace Transform $z = e^{sT}$
The definition of the z transform results.$X(z) = \sum_{k=0}^{\infty} x(k)z^{-k}$
The relation between s and z can also be written:- $s = \frac{1}{T} lnz$



The mapping of the s plane to the z plane is illustrated by the above diagram and the following 2 relations. Lines of any given colour in the s plane maps to lines of the same colour in the z plane.

$$s = \sigma + j\omega$$
$$z = e^{\sigma T} e^{j\omega T}$$

The above relations show the following:-
The imaginary axis of the s plane between minus half the sampling and plus half the sampling frequency maps onto the unit circle in the z plane.
The portion of the s plane to the left of the red line maps to the interior of the unit circle in the z plane.
The portion of the s plane to the right of the red line maps to the exterior of the unit circle in the z plane.
The green line(line of constant sigma) maps to a circle inside the unit circle in the z plane.
Lines of constant frequency in the s plane maps to radial lines in the z plane.
The origin of the s plane maps to z = 1 in the z plane.
The negative real axis in the s plane maps to the unit interval 0 to 1 in the z plane.
The s plane can be divided into horizontal strips of width equal to the sampling frequency. Each strip maps onto a different Riemann surface of the z "plane".
Mapping of different areas of the s plane onto the Z plane is shown below.



Summary

Transforms are used because the time-domain mathematical models of systems are generally complex differential equations. Transforming these complex differential equations into simpler algebraic expressions makes them much easier to solve. Once the solution to the algebraic expression is found, the inverse transform will give you the time-domain response.
Laplace is used for stability studies and Fourier is used for sinusoidal responses of systems. Fourier is used primarily for **steady state signal analysis**, while Laplace is used for **transient signal analysis**. Laplace is good at looking for the response to pulses, step functions, delta functions, while Fourier is good for continuous signals.
The systems are stable if the real part of s is negative, that is to say there is a transient that will vanish in time, in those cases, it is enough to use Fourier. Laplace can determine the full response of a system, be it stable or unstable, including transient Whereas Z- transforms is used if the input signal is discrete.

CONCLUSION

We can conclude from this there is close relationship between Fourier, Laplace and Z Transforms. One transform can be inherited from another by changing the format of the variable ‘s’ or ‘z’

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