Abstract: This paper makes an attempt of consolidated and comparative study of Fourier Transform, Laplace Transform and z-transform. It also shows sequential mathematical flow of interlinking of the three transforms.

Key words: Fourier series, Fourier Integral, Fourier Transform, Laplace Transform, Z Transform.

INTRODUCTION

We tried to obtain a good answer for the Fourier and Laplace and Z-transforms relationship. Many of the explanations just mention that the relationship is that S = a + iw, so the Fourier transform becomes a special case of the laplace transform.

In the calculus we know that certain functions have power series representations of the form

\[ f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{where} \quad c_n = \frac{f^{(n)}(0)}{n!} \]

These type of power series are useful for numerical calculations in addition to various other uses. If the function f is periodic with period p it can be written as sum of sine and cosine functions as

\[ f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{p}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{p}\right) \]

Integrating on both sides over c to c+p the RHS terms become zero except for the case when n=m

\[ \int_{c}^{c+p} \cos(n\pi x) f(x) dx = \int_{c}^{c+p} \sin(n\pi x) f(x) dx = 0 \]

\[ a_n = \frac{1}{p} \int_{c}^{c+p} \cos(n\pi x) f(x) dx \]

\[ b_n = \frac{1}{p} \int_{c}^{c+p} \sin(n\pi x) f(x) dx \]

And for n=0

\[ a_0 = \frac{1}{p} \int_{c}^{c+p} f(x) dx \]

Therefore Fourier series is

\[ f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{p}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{p}\right) \]

Periodic signal may be written as

\[ f(x) = a \cos(\omega x) + b \sin(\omega x) \]

\[ a = r \cos \theta \quad b = r \sin \theta \]

\[ f(x) = \sqrt{a^2 + b^2} \cos(\omega x - \phi) \]

Where \( \sqrt{a^2 + b^2} \) is amplitude and \( \cos(\omega x - \phi) \) is phase

Now \( e^{j\theta} = \cos \theta + j \sin \theta \)

\[ f(x) = \sum_{n=1}^{\infty} a_n \cos(n\omega x) + \sum_{n=1}^{\infty} b_n \sin(n\omega x) \]

\[ c_n = \frac{1}{p} \int_{c}^{c+p} f(x) e^{-j \frac{2\pi n x}{p}} dx \]

\[ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{j \frac{2\pi n x}{p}} \quad \text{and} \quad c_n = \frac{1}{p} \int_{c}^{c+p} f(x) e^{-j \frac{2\pi n x}{p}} dx \]

\[ f(x) = \sum_{n=1}^{\infty} \frac{a_n}{j} + \sum_{n=1}^{\infty} b_n j \cos(n \pi x/p) + \sum_{n=1}^{\infty} b_n j \sin(n \pi x/p) \]

\[ f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right) \]

\[ a_n = \frac{1}{p} \int_{c}^{c+p} f(t) \cos\left(\frac{n\pi t}{p}\right) dt \]

\[ b_n = \frac{1}{p} \int_{c}^{c+p} f(t) \sin\left(\frac{n\pi t}{p}\right) dt \]

The above diagram explains the function of Fourier series.

Relation to Fourier -Transform

Taking limit as \( p \to \infty \) in Eqn(1), as \( f(x) \) is absolutely integrable \( \int_{-\infty}^{\infty} |f(t)| dt < \infty \) we have

\[ \lim_{p \to \infty} \int_{-p}^{p} f(t) dt = 0 \]

Let \( \delta s = \pi/p \quad n. \delta s = s \)

Taking limit as \( \delta s \to 0 \) as \( p \to \infty \)

We get,

\[ f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sum_{n=1}^{\infty} \cos(n \delta s (t-x)) \delta s \ dt \]
This is exponential form of Fourier Integral Theorem.

\[ F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} \, dt \]

The location of the constant \( \frac{1}{\sqrt{2\pi}} \) is arbitrarily selected.

One may write as

\[ F(s) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} \, dt \]

The following diagram shows the output of time signal in Fourier Transform

\[ \text{Relation to Laplace -Transform} \]

Suppose the function and its derivative \( f' \) both are piecewise continuous functions for all \( t \geq 0 \), then Fourier transform of \( f(t) \) may not exist.

Then these functions are best handled by Laplace Transformation as it doesn’t require \( f \) to be absolutely integrable.

For that we assume a related function as

\[ g(t) = e^{-kt} f(t) h(t) \]

where \( k \) is positive real constant \( h(t) \) Heaviside unit function.

Hence \( f \) satisfy the condition \( \int_{0}^{\infty} e^{-kt} f(t) \, dt < \infty \)

As \( g \) satisfies the conditions of Fourier integral theorem we can write

\[ g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} \, d\omega \]

\[ f(t) h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} \, d\omega \]

If we put \( p = -c \) then it takes the form

\[ f(t) h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(i\omega) e^{-c-i\omega t} \, d\omega \]

Again as above these include pair of formulae, one of which is \( L(p) = \int_{0}^{\infty} f(t) e^{-pt} \, dt \)

Which we call as Laplace Transform.

We can also proceed as follows

\[ F(s) = \int_{0}^{\infty} f(t) \left[ e^{-st} \right] \, dt \]

This shows Fourier transform is the Laplace Transform when \( s \) is purely imaginary.

\[ F(s) = \mathcal{L}[x(t)] \text{ where } s = i\omega \]

\[ F(\sigma+i\omega) = \int_{0}^{\infty} f(t) e^{-(\sigma+i\omega)t} \, dt \]

\[ = \int_{0}^{\infty} \left[ f(t) e^{-\sigma t} \right] e^{-i\omega t} \, dt \]

\[ = \int_{0}^{\infty} \left[ f(t) e^{-\sigma t} \right] e^{-i\omega t} \, dt \]

\[ F_{z}(c) = F[f_{c}(t)] \]

This shows The Laplace transform is the Fourier transform of the transformed signal \( f_{c}(t) = f(t)e^{-ct} \). Depending on whether \( s \) is positive/negative this represents a growing/negative signal.

The following figure explains graphically the relation of the Fourier and Laplace Transform

Fourier Transform can be thought of as Laplace transform evaluated on the iw (imaginary) axis, neglecting the real part of complex frequency ‘s’. As shown in the figure below, the 3D graph represents the laplace transform and the 2D portion at real part of complex frequency ‘s’ represents the fourier transform.

\[ \text{Relation to Z-Transform} \]

The Laplace Transform of a sampled signal can be written as:

\[ X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k} \]

If the following substitution is made in the Laplace Transform \( z = e^{\frac{\pi}{T}} \)

The definition of the z transform results \( X(z) = \sum_{k=0}^{\infty} x(k)z^{-k} \)

The relation between \( s \) and \( z \) can also be written:-

\[ s = \frac{1}{T} \ln z \]

The mapping of the s plane to the z plane is illustrated by the above diagram and the following 2 relations. Lines of any
given color in the s plane maps to lines of the same color in
the z plane.

\[ s = \sigma + j\omega \]
\[ z = e^{\sigma T} e^{j\omega T} \]

The above relations show the following:-
The imaginary axis of the s plane between minus half the
sampling and plus half the sampling frequency maps onto
the unit circle in the z plane.
The portion of the s plane to the left of the red line maps to
the interior of the unit circle in the z plane.
The portion of the s plane to the right of the red line maps to
the exterior of the unit circle in the z plane.
The green line (line of constant sigma) maps to a circle inside
the unit circle in the z plane.
Lines of constant frequency in the s plane maps to radial lines
in the z plane.
The origin of the s plane maps to \( z = 1 \) in the z plane.
The negative real axis in the s plane maps to the unit interval
0 to 1 in the z plane.
The s plane can be divided into horizontal strips of width
equal to the sampling frequency. Each strip maps onto a
different Riemann surface of the z "plane”.
Mapping of different areas of the s plane onto the Z plane is
shown below.

CONCLUSION

We can conclude from this there is close relationship
between Fourier, Laplace and Z Transforms. One transform
can be inherited from another by changing the format of the
variable ‘s’ or ‘z’

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Summary

Transforms are used because the time-domain mathematical
two systems are generally complex differential
equations. Transforming these complex differential
equations into simpler algebraic expressions makes them
much easier to solve. Once the solution to the algebraic
expression is found, the inverse transform will give you the
time-domain response.
Laplace is used for stability studies and Fourier is used for
sinusoidal responses of systems. Fourier is used primarily for
steady state signal analysis, while Laplace is used for
transient signal analysis. Laplace is good at looking for the
response to pulses, step functions, delta functions, while
Fourier is good for continuous signals.
The systems are stable if the real part of s is negative, that is
to say there is a transient that will vanish in time, in those
cases, it is enough to use Fourier. Laplace can determine the
full response of a system, be it stable or unstable, including
transient Whereas Z- transforms is used if the input signal is
discrete.