QUASI-P NORMAL AND QUASI-n-P NORMAL COMPOSITION, WEIGHTED COMPOSITION AND COMPOSITE MULTIPLICATION OPERATORS

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Abstract — In this paper we characterized quasi-P normal, quasi-n-P normal composition, weighted composition, composite multiplication operators

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1 INTRODUCTION
An operator T is said to be Self adjoint operator, If T satisfies $T^* = T$. An operator T is said to be normal, If T satisfies $TT^* = T^*T$. An operator T is said to be n-power normal, If T satisfies $T^n T^n = T^n T^n$. An operator T is said to be binormal, If $TT^*$ and $T^*T$ commute (i.e) $(TT^*)T^* = T^*T(TT^*)$. An operator T is said to be quasi-normal, If $T$ and $T^*$ commute. A operator T is said to be quasi-n-normal, If $T$ and $T^n$ commute. An operator T is said to be quasi-P normal, If $T$ and $T^*$ commute. An operator T is said to be quasi-n-P normal, If $T$ and $T^n$ commute.

SOME PROPERTIES OF QUASI-P NORMAL AND QUASI-n-P NORMAL OPERATORS

THEOREM-1: If $T \in B(H)$ is isometry, Then T is Quasi-p normal.

PROOF: Let T is isometry, we have $T^*T = I$

Now

$$(T + T^*)(TT^*) = (T + T^*)I = (T + T^*)$$

$$(TT^*)(T + T^*) = I(T + T^*) = (T + T^*)$$

(1)

(2)

From (1) and (2) are same.
Hence T is quasi-p normal.

THEOREM-2: Every quasi-normal operator is quasi-p normal.

PROOF: Let T is quasi-normal operator, Then

$$T(T^*T) = (T^*T)T$$

Taking adjoint on the both side of (3) we get,

$$(T(T^*T))^* = (T^*T)^*T^*$$

$$T^*T^* = T^*T$$

(3)

(4)

THEOREM-3: If T is a quasi-n-p normal and $\mu$ is any scalar which is real. Then $\mu T$ is also a quasi-n-p normal operator.

PROOF: Let T is quasi-n-p normal operator, Then

$$(T + T^*)(T^n T^n) = (T^n T^n)(T + T^*)$$

If $\mu$ is any scalar which is real, Then

$$(\mu T + (\mu T)^*)((\mu T)^*+(\mu T)) = (\mu T + (\mu T)^*)(\mu T^*)^*$$

$$(\mu T^*)^*(\mu T) + (\mu T)^* = (\mu T^*)^*(\mu T) + (\mu T)^*$$

$$= \mu^2 + \mu^2 T^* + T$$

(5)

(6)

From (5) and (6) are same. Hence T is quasi-n-p normal.

THEOREM-4: If T is a self-adjoint operator then T is a quasi-n-p normal operator.

PROOF: Let T is a self-adjoint operator

$$T^* = T$$

Now,

$$(T + T^*)(T^n T^n) = (T + T)(T^n T^n)$$

$$= 2T + 2T^n$$

(7)

(8)

From (9) and (10), Hence T is quasi-n-p normal.

THEOREM-5: Let T be a quasi-n-p normal operator on a Hilbert space H. Let S be a self-adjoint operator for which T and S commute, Then ST is also a quasi-n-p normal operator.
THEOREM-6: Let $T \in B(H)$ be a quasi-n-p normal operator which is unitary equivalent to $S$ if and only if $TU = UT$ and $T^*U = UT^*$. Then $S$ is a quasi-n-p normal.

THEOREM-7: If $T$ is a quasi-n-normal operator which is n-power normal also, then $T$ is quasi-n-p normal operator.

THEOREM-8: Let $T_1$ and $T_2$ be two quasi-n-p normal operators which each is the adjoint of the other, then $T_1 T_2$ is a quasi-n-p normal operator.

THEOREM-9: If $T$ be a self adjoint operator on a Hilbert space $H$ and $S$ be any operator on $H$, then $S^*TS$ is a quasi-n-p normal operator on $H$.

THEOREM-10: If $T$ is a quasi-n-p normal, then $T^*$ is a quasi-n-p normal operator.

PROOF: Let $T$ be a quasi-n-p normal.

\[ (T + T^*)(T^*T^n) = (T^nT)(T + T^*) \] \tag{11}

Substituting $T^*$ for $T$ in (11), we have

\[ (T^* + T)(T^*T^n) = (T^nT)(T^* + T) \] \tag{12}

Hence $T^*$ is quasi-n-p normal.

THEOREM-11: Let $T$ be a quasi-n-p normal operator. Which is a unitary operator also, then $T^{-1}$ is a quasi-n-p normal.

THEOREM-12: Let $T \in B(H)$, $A = (T^*T^n) + (T + T^*)$ and $B = (T^*T^n) - (T + T^*)$, then $T$ is quasi-n-p normal operator if and only if $A$ commutes with $B$.

THEOREM-13: Let $T \in B(H)$, $X = (T^*T^n)(T + T^*)$, $A = (T^*T^n) + (T + T^*)$ and $B = (T^*T^n) - (T + T^*)$, then $T$ is quasi-n-p normal operator if and only if $X$ commutes with $A$ and $B$.

QUASI-P NORMAL AND QUASI-n-P NORMAL COMPOSITION OPERATORS

Let $C$ be the composition operator on $L^2(\mu)$. Then the adjoint $C^*$ is given by $C^*f = h.E[f] o T^{-1}$ for $f \in L^2(\mu)$.

Lemma-14: Let $P$ be the projection of $L^2(X, \Sigma, \mu)$ onto $\overline{R(C)}$. Then

\[ (i) \ C^*f = hf \text{ and } CC^*f = (h \circ T)Pf, \text{ for all } f \in L^2(\mu). \]
\[ (ii) \ \overline{R(C)} = \{ f \in L^2(\mu) : f \text{ is } T^{-1}(\Sigma) \text{ measurable} \}. \]
\[ (iii) \text{ If } f \text{ is } T^{-1}(\Sigma) \text{ measurable, } g \text{ and } fg \text{ belong to } L^2(\mu), \text{ then } P(fg) = fP(g). (f \text{ need not be in } L^2(\mu)). \]
\[ (iv) \ (C^*C)f = h_kf \text{ for } k \in N. \]
\[ (v) \ (CC^*)f = (h \circ T)_\alpha P(f). \]
\[ (vi) \ E \text{ is the identity operator on } L^2(\mu) \text{ if and only if } T^{-1}(\Sigma) = \Sigma. \]

The following theorem characterizes the quasi-p normal and quasi-n-p normal composition operators.

THEOREM-15: Every quasinormal composition operator is quasi-p normal operator.

PROOF: Let $C$ be quasinormal composition operator, then

\[ C(C^*C) = (C^*C)C \] \tag{13}

Taking adjoint on both sides, we get

\[ (C(C^*C))^* = (C^*C)C^* \]
\[ C^*C^* = C^*C \]
\[ (C + C^*)(C^*C) = (C^*C)(C + C^*) \]
\[ = C^*C + C^*C^* \]
\[ = (C^*C)(C + C^*) \]

Hence $C$ is quasi-p normal composition operator.

THEOREM-16: A composition operator $C$ on $L^2(\mu)$ is quasi-p normal if and only if $C^*$ is quasi-p normal.

THEOREM-17: A composition operator $C$ on $L^2(\mu)$ is quasi-p normal if and only if $(C + C^*)$ commutes with $M_\mu$. Where $M_\mu$ is the multiplication operator induced by $f_\mu = (\frac{f}{\int f})$.

THEOREM-18: Let $C$ be the quasi-p normal operator (if and only if $(h_\mu T). (f o T) + h. E[h] o T^{-1} . E[f] o T^{-1} = h . (f o T) + h^2 E[f] o T^{-1}$.\]

PROOF: Let $C$ be quasi-p normal operator, then

\[ (C + C^*)(C^*C) = (C^*C)(C + C^*) \]
\[ = C^*C + C^*C^* \]

Consider,

\[ (C^*C)f = C^*(h.f) \]
\[ C^*(C^*f) = C^* (h.f) \]
\[ = h.E[h] o T^{-1} . E[f] o T^{-1} \]
\[ (C^*C)f = C^*C(f o T) \]
\[ = h . (f o T) \]
\[ (C^*C)f = (C^*C)(h.E[f] o T^{-1}) \]
\[ = h^2 E[f] o T^{-1} \]

Hence $C$ is quasi-p normal if and only if $(f o T) + h . E[h] o T^{-1} . E[f] o T^{-1} = h . (f o T) + h^2 E[f] o T^{-1}$.

THEOREM-19: Let $C \in B(L^2(\lambda))$, then $C^*$ is quasi-p normal operator if and only if $h.E[h]. E[f] o T^{-1} + (h o T^2). E[f] o T = (h o T). E[h]. E[f] o T^{-1} + (h o T^2). E[f] o T$.

THEOREM-20: If $C$ is quasi-n-normal and n-power normal operator, then $C$ is quasi-n-p normal composition operator.

THEOREM-21: Let $C$ in $L^2(\mu)$ be quasi-n-p normal composition operator. Then $C^*$ is quasi-n-p normal composition operator.

THEOREM-22: If $C$ is quasi-n-p normal composition operator on $L^2(\mu)$. Then $\alpha C$ is quasi-n-p normal composition operator for every real number $\alpha$. 

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THEOREM-23: A composition operator $C$ on $L^2(\mu)$ is quasi-n-p normal if and only if $(C + C^*)$ commutes with $h.E[f]oT^{n-1}$.

THEOREM-24: Let $C$ be quasi-n-p normal if and only if $(h.oT^*).E[f]^oT^{n-1} + h.E[h]oT^{-1}.E[f]^oT^{n-2} = h.E[f]^oT^n$. 

THEOREM-25: Let $C \in B(L^2(\mu))$. Then $C$ is the quasi-p normal operator if and only if $(h_n.oT^2).E[f]oT^{n-2} - (h_n.oT^2).E[f]oT^{n-2} = (h_n.oT^2).E[f]oT^{n} + (h_n.oT^2).E[f]oT^{n}$.

QUASI-P NORMAL AND QUASI-n-P NORMAL WEIGHTED COMPOSITION OPERATORS

Let $W$ be the weighted composition operator on $L^2(\mu)$. Let $W^*$ be its adjoint which is given by $W^* f = h.E(u.f) o T^{-1}$ for $f \in L^2(\mu)$. For a positive integer $n$, $W^n f = u_n. f \circ T^{-n}$. For $f \in L^2(\mu)$, $W^n f = u_n. f \circ T^{-n}$. 

Proposition-26: For $u \geq 0$:

(i) $W^* W f = h.E[(u^2)] o T^{-1} \cdot f$.

(ii) $W W^* f = u(h \circ T)E(u.f)$.

THEOREM-27: Let $W$ be a weighted composition operator. Then $W$ is quasi-p normal operator if and only if $h.E[u^2]oT^2 - h.E[u^2]oT^{n-2} = (h.oT^{n-1}).E[f]oT^{-1}(h_n.oT^2) + (h.oT^{n-1}).E[f]oT^{-1}(h_n.oT^2)$.

PROOF: Let $W$ be a quasi-p normal operator. Then 


Consider


Hence $W$ is quasi-p normal operator if and only if $u(h.oT^2).E[f]oT^{-2} + h.E[u]oT^{n-1}E[u^2]oT^{-1} + h.E[u^2]oT^n - (h.oT^2).E[f]oT^{-2} = 0$.

THEOREM-28: Let $W$ be a weighted composition operator. Then $W$ is quasi-p normal operator if and only if $h.E[u^2]oT^n - h.E[u]oT^{n-1}E[u^2]oT^{-1} + (h.oT^2).E[f]oT^2 - (h.oT^2).E[f]oT^{-2} = 0$.

THEOREM-29: Let $W$ be a weighted composition operator. Then $W$ is quasi-p normal operator if and only if $(u.(h.oT^2).E[u^2]oT^n - h.E[u]oT^{n-1}E[u^2]oT^{-1} + (h.oT^2).E[f]oT^2 - (h.oT^2).E[f]oT^{-2}) = 0$.

THEOREM-30: Let $W$ be a weighted composition operator. Then $W$ is quasi-p normal operator if and only if $h.E[u]oT^{n-1}E[u^2]oT^{-1} + h.E[u]oT^{n-1}E[u^2]oT^{-1} = 0$.

QUASI-P NORMAL AND QUASI-n-P NORMAL COMPOSITE MULTIPLICATION OPERATORS

A composite multiplication operator is a linear transformation acting on a set of complex valued measurable functions $f$ of the form $M_{u,T}(f) = C_{u,T}(f) = (u.f) o T$. Where $u$ is a complex valued measurable function. In case, $u = 1$ almost everywhere $M_{u,T}$ becomes a composition operator. The adjoint of $M_{u,T}$ is given by $M_{u,T}^* f = u(h \circ T)E(u.f)$.

THEOREM-31: Let $M_{u,T}$ on $L^2(\mu)$ be a composite multiplication operator. Then for each, $\lambda \geq 0$, $M_{u,T}^\lambda$ is a quasi-p normal operator if and only if $h.E[(u^2)]oT^{n-1}E[(u^2)]oT^{-1} + h.E[(u^2)]oT^n - (h.oT^2).E[f]oT^{-2} = 0$.

THEOREM-32: Let $M_{u,T}$ on $L^2(\mu)$ be a composite multiplication operator. Then for each, $\lambda \geq 0$, $M_{u,T}^\lambda$ is a quasi-p normal operator if and only if $h.E[(u^2)]oT^{n-1}E[(u^2)]oT^{-1} + h.E[(u^2)]oT^n - (h.oT^2).E[f]oT^{-2} = 0$.

THEOREM-33: Let $M_{u,T}$ on $L^2(\mu)$ be a composite multiplication operator. Then for each, $\lambda \geq 0$, $M_{u,T}^\lambda$ is a quasi-p normal operator if and only if $h.E[(u^2)]oT^{n-1}E[(u^2)]oT^{-1} + h.E[(u^2)]oT^n - (h.oT^2).E[f]oT^{-2} = 0$.

THEOREM-34: Let $M_{u,T}$ on $L^2(\mu)$ be a composite multiplication operator. Then for each, $\lambda \geq 0$, $M_{u,T}^\lambda$ is a quasi-p normal operator if and only if $h.E[(u^2)]oT^{n-1}E[(u^2)]oT^{-1} + h.E[(u^2)]oT^n - (h.oT^2).E[f]oT^{-2} = 0$. Where $u$ is a complex valued measurable function. In case, $u = 1$ almost everywhere $M_{u,T}$ becomes a composition operator. The adjoint of $M_{u,T}$ is given by $M_{u,T}^* f = u(h \circ T)E(u.f)$. 


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