PROXIMITY STRUCTURES AND GRILL

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Abstract— The concept of proximity has already studied with meticulous care from different aspects and via various approaches. In this paper, we have continued the investigation of proximity in terms of a different concept via grill, where the deliberations in the article include certain characterizations. The associated discussions and results were carried out with grills as a prime supporting tool. The basic properties of the induced proximity have discussed here in some details. We have also shown various relations between the proximity induced via grill and the original proximity under some suitable conditions applied on the grill under consideration.

Index Terms—Grill, Ef-proximity space, Basic proximity

Subject Classification: 54A10, 54E05

1 INTRODUCTION

In principal, a proximity space is a non-empty set X with a binary relation \( \delta \) between subsets with the intuitive meaning that \( A \delta B \) holds, when "A is near B" in some sense. The proximity relation satisfies axioms which are identical with some of the typical axioms of the connection relation. Each proximity space determines a natural topology with useful properties, and the theory possesses deep results, rich machinery and tools; the main work on proximity spaces is the book by Naimpally and Warrack [4]. The idea of grill was initiated by Choquet [1], thereafter it has been observed in connection with many mathematical investigations such as the theories of proximity spaces, compactifications etc., Grills are extremely useful and convenient for many situations as a tool like filters. A grill G on a topological space X is defined as a collection of non-empty subsets of X such that:

(i) \( A \in G \) and \( A \subseteq B \subseteq X \rightarrow B \in G \).
(ii) \( A, B \subseteq X \) and \( A \cup B \subseteq G \rightarrow A \in G \) or \( B \in G \).

The current work aims to establish a generalized proximity using grill and Ef-proximity on a set. In section 2, all preliminaries and theorems of proximity structures and grills which will be needed in the sequel have been briefly mentioned. In section 3, \( \varphi_G(\delta) \) and \( \psi_C(\delta) \) operators on \( P(X) \) w.r.t. grill G and Ef-proximity \( \delta \), are defined and various properties of them are investigated. Also, the new generated proximity will be introduced. Furthermore, the relation between the new generated proximities and the new generated topologies using a given proximity or topology and grills will be discussed.

2. PRELIMINARIES

This section is devoted to recall the known results concerning grill and proximity spaces. Many of the results below are translations of facts about grill and proximity spaces of which proofs are available in [1, 2, 4, 5]. Since space is at a premium, we will usually cite this source for those proofs which are known and not immediately obvious.

Proposition 2.1 Arbitrary union of grills is a grill, but the intersection of two grills is not generally a grill.

Definition 2.2 Let X be a set, then the principal grill generated by a non-empty subset \( A \) of \( X \) is \( [A]=\{B \subseteq X : A \delta B \subseteq \emptyset\} \).

Definition 2.3 A basic proximity structure on a non-empty set \( X \) is a binary relation \( \delta \) on \( P(X) \) such that:

(i) \( A \delta B \rightarrow B \delta A \).
(ii) \( (A \cup B) \delta C \rightarrow A \delta C \) or \( B \delta C \).
(iii) \( A \delta B \rightarrow A \delta \emptyset \) and \( B \delta \emptyset \).
(iv) \( A \cap B \delta \emptyset \rightarrow A \delta B \).

A basic proximity \( \delta \) is said to be Ef-proximity, if it satisfies (where \( \delta^- \) means negation of \( \delta \)):

\( A \delta^- B \rightarrow \exists E \subseteq X \text{ s.t. } A \delta^- E \) and \( (X-E) \delta^- B \).

A\( ^c \) represents the complement of a set A.

Proposition 2.4 Let \( (X, \delta) \) be an Ef-proximity space and \( A \subseteq X \), then:

(i) \( \text{Cl}_\delta(A)=\{x \in X : U \cap A \neq \emptyset \ \forall \delta^- \text{nbd.U of } x\} \} \) is a Kuratowski closure operator and the collection \( \tau_\delta=[A \subseteq X : \text{Cl}_\delta(A)=\{A\}] \) is a topology on \( X \) which is a completely regular if, \( (X, \delta) \) is an Ef-proximity space.
(ii) \( A \) is a \( \delta \)-open set w.r.t. \( \tau_\delta \), if \( (x \delta^- A^- \ \forall x \in A) \).
(iii) \( A \) is a \( \delta \)-closed set w.r.t. \( \tau_\delta \), if \( (x \delta A \rightarrow x \in A) \).

Note that \( \delta \) and \( \tau_\delta \) are called compatible.

Proposition 2.5 For every subset \( A \) and \( B \) of an Ef-proximity space \( (X, \delta) \), then:

(iii) If \( A \delta B \), \( A \subseteq C \) and \( B \subseteq D \), then \( C \delta D \).
Definition 2.6 If \( \delta_1 \) and \( \delta_2 \) are two proximities on a set \( X \), then \( \delta_1 \) is called finer than \( \delta_2 \) (in symbols \( \delta_2 < \delta_1 \)) if, \( A \delta_1 B \) implies \( A \delta_2 B \).

Theorem 2.7 (i) Let \( \delta_1 \) and \( \delta_2 \) be Ef-proximities on a set \( X \), then \( \delta_1 < \delta_2 \) implies \( \tau_{\delta_1} \subseteq \tau_{\delta_2} \).
(ii) Each completely regular topological structure \( \tau \) on a set \( X \) generates a unique compatible proximity structure \( \delta \) given by

\[
A \delta B \Leftrightarrow \exists U \subseteq X \text{s.t. } U \text{-nbds of } x \subseteq \phi(A) \text{ and } U \text{-nbds of } x \subseteq \phi(B).
\]

Definition 2.8 A subset \( B \) of any proximity space \( (X, \delta) \) is called a \( \delta \)-neighborhood of a set \( A \) (in symbols \( A << B \)), if \( A \delta^* (X - B) \). The family \( \{B \subseteq X : A << B\} \) is called the \( \delta \)-nbd-system of \( A \).

Definition 2.9 Let \( (X, \delta) \) be a proximity space and \( x \in X \). Let \( G = \{U \subseteq X : U \text{-nbd of } x\} \) is a grill on \( X \) generated by \( \delta \).

Example 3.2 Let \( (X, \delta, G) \) be a Ef-proximity space with a grill \( G \) on \( X \).

3. OPERATOR \( \phi_G(\delta) \) and GENERATED PROXIMITY

Definition 3.1 Let \( (X, \delta) \) be a proximity space and \( G \) be a grill on \( X \). An operator \( \phi_G : P(X) \rightarrow P(X) \) (where \( P(X) \) stands for the power set of \( X \), denoted by \( \phi_G(A, \delta) \) for \( A \in P(X) \)) of \( \phi_G(A, \delta) \), called the operator associated with the grill \( G \) and the proximity \( \delta \), and is defined by \( \phi_G(A) = \{x \in X : U \text{-nbd of } x \} \) is a grill on \( X \) generated by \( \delta \).

Example 3.2 Let \( (X, \delta, G) \) be an Ef-proximity space with a grill \( G \) and \( A \subseteq X \). Then:
(i) \( G = \phi(A) \text{ if } A \subseteq P(X) \).
(ii) \( G = \phi(A) \text{ if } A = [A] \).

Proposition 3.3 Let \( (X, \delta, G) \) be an Ef-proximity space with a grill \( G \) and \( A \subseteq X \) be \( \delta \)-closed, then \( \phi_G(A) \subseteq A \).

Proof Let \( A \subseteq X \) be a \( \delta \)-closed set and \( x \in A \), then \( A \subseteq \delta \)-nbd. of \( x \) and \( (A \cap \delta A) = \emptyset \). Consequently, \( x \notin \phi_G(A) \) and so \( \phi_G(A) \subseteq A \).

Theorem 3.4 Let \( (X, \delta) \) be an Ef-proximity space and let \( G \) and \( H \) be two grills on \( X \). For a subset \( A \) of \( X \), the following statements hold:
(i) \( G \subseteq H \rightarrow \phi_G(A) \subseteq \phi_H(A) \).
(ii) \( \phi_{G,H}(A) = \phi_G(A) \cup \phi_H(A) \).

Proof (i) Obvious,
(ii) The inclusion \( \phi_G(A) \cup \phi_H(A) \subseteq \phi_{G,H}(A) \) follows directly from (i). Let \( x \notin \phi_G(A) \cup \phi_H(A) \), then \( x \notin \phi_G(A) \) and \( x \notin \phi_H(A) \).

Hence, \( \exists U, U_2 \text{-nbds of } x \) s.t. \( U \cap \neg \neg A \subseteq G \) and \( U \cap \neg \neg A \subseteq H \). Let \( U = U_1 \cup U_2 \text{-nbds of } x \) s.t. \( U \cap \neg \neg A \subseteq G \) and \( U \cap \neg \neg A \subseteq H \). This follows that, \( (U \cap \neg \neg A) \cap H \cap \emptyset \). Let \( x \notin \phi_{G,H}(A) \) i.e. \( \phi_{G,H}(A) \subseteq \phi_G(A) \cup \phi_H(A) \).

Next theorems discuss some of properties of the \( \phi_G(\delta) \)-operator.

Theorem 3.5 Let \( (X, \delta, G) \) be an Ef-proximity space with a grill \( G \). Then, for any \( A, B \subseteq X \) the following hold:
(i) \( \phi_G(A) \subseteq \phi_G(B) \text{ if } A \subseteq B \).
(ii) \( \phi_G(A \cup B) = \phi_G(A) \cup \phi_G(B) \).
(iii) \( \phi_G(A \cap B) \subseteq \phi_G(A) \cap \phi_G(B) \).
(iv) \( \phi_G(A) \subseteq \phi_G(B) \).
(v) \( \phi_G(A) = \emptyset \text{ if } A \subseteq G \).
(vi) \( \phi_G(A) \cap \phi_G(B) \subseteq \phi_G(A \cap B) \).

Proof (i) Obvious from the definition of \( \phi_G(\delta) \).
(ii) In view of (i), it suffices to show that \( \phi_G(A \cup B) \subseteq \phi_G(A) \cap \phi_G(B) \). Suppose \( x \notin \phi_G(A) \cup \phi_G(B) \). Then, there are \( \delta \)-nbds. \( U \) and \( V \) of \( x \) such that \( A \subseteq U \) and \( B \subseteq V \) \( \emptyset \). Hence, \( A \subseteq (U \cap V) \subseteq (A \cup B) \subseteq \emptyset \). Now \( U \subseteq V \cap U \subseteq \emptyset \). Consequently, \( x \notin \phi_G(A \cup B) \).

(vi) Obvious from the definition of the operator \( \phi_G \).
(vii) Let \( A, B \subseteq X \) and \( A \subseteq (A-B) \cap (B-A) \). Then, (ii) and (iii) implies that \( \phi_G(A) \subseteq \phi_G(A-B) \cap \phi_G(B-A) \). Consequently, \( \phi_G(A) \subseteq \phi_G(A-B) \).

Theorem 3.6 Let \( (X, \delta, G) \) be an Ef-proximity space with a grill \( G \). Then, \( \forall A \subseteq X \) the following hold:
(i) \( \phi_G(A \cup B) = \phi_G(A) \cup \phi_G(B) \).
(ii) \( \phi_G(A) \cap \phi_G(B) \subseteq \phi_G(A \cup B) \).

Proof (i) Obvious by using Theorem 3.5.
(ii) Let \( E = (A-B) \cap (B-A) \subseteq G \). Then, \( E = (B-A) \cup (B-A) \subseteq \emptyset \). By using (ii) of Theorem 3.5 and (i) of this theorem, we have \( \phi_G(A) = \phi_G(E) \cap \phi_G(B) \). Which implies \( \phi_G(A) = \phi_G(E) \cap \phi_G(B) \).

Theorem 3.7 Let \( (X, \delta, G) \) be an Ef-proximity space with a grill \( G \). Then, \( \forall A \subseteq X \) the following hold:
(i) \( x \delta \phi_G(A) \rightarrow x \delta A \).
(ii) \( x \in \phi_G(A) \rightarrow x \delta (A-E) \).

Proof (i) Let \( x \in \phi_G(A) \), then by using Theorem 3.5, \( x \delta \phi_G(A) \subseteq \emptyset \). Since \( \delta \) is an Ef-proximity on \( X \), then \( x \delta A \).

(ii) Let \( x \in \phi_G(A) \), then \( x \delta \phi_G(A) \subseteq \emptyset \). By using of Theorem 3.6, \( x \delta \phi_G(A) \subseteq \emptyset \). Consequently, by using (i) of this Theorem, \( x \delta (A-E) \subseteq \emptyset \). Conversely, assume \( x \in \phi_G(A) \), then \( x \delta (A-E) \subseteq \emptyset \). Put \( E = U \cup A \), then \( (A-E) \subseteq (U-A) \subseteq \emptyset \). And so \( x \delta (A-E) \subseteq \emptyset \). Consequently, \( x \in \phi_G(A) \) if \( x \delta (A-E) \subseteq \emptyset \).
Proposition 3.8 Let \((X, \delta, G)\) be an Ef-proximity space with a grill \(G\) and \(A \subseteq X\). If \(U\) is a \(\delta\)-nbd of \(x\), then \(U \cap \varphi_C(A) = U \cap \varphi_C(U \cap A)\).

Proof In view of (i) of Theorem 3.5. and \(U \cap A \subseteq A\), then \(\varphi_C(U \cap A) \subseteq \varphi_C(A)\) and so \(U \cap \varphi_C(U \cap A) \subseteq U \cap \varphi_C(A)\). Let \(x \notin \varphi_C(U \cap A)\) and \(U\) is a \(\delta\)-nbd. of \(x\), then \(\exists \delta\)-nbd. \(V\) of \(x\) s.t. \((V \cap U \cap A) \notin G\). Since \(U\) is a \(\delta\)-nbd. of \(x\), then \(W = V \cap U\) is a \(\delta\)-nbd. of \(x\) and \((W \cap A) \notin G\). Consequently, \(x \notin \varphi_C(A)\) and so \(U \cap \varphi_C(U \cap A) \subseteq U \cap \varphi_C(U \cap A)\). Then, \(U \cap \varphi_C(A) = U \cap \varphi_C(U \cap A)\).

Theorem 3.9 Let \((X, \delta, G)\) be a proximity space with a grill \(G\) and \(A \subseteq X\), then \(\varphi_C(\varphi_C(A)) \subseteq \varphi_C(A) = \text{Cl}_G(\varphi_C(A))\).

Proof We shall show that \(\text{Cl}_G(\varphi_C(A)) \subseteq \varphi_C(A)\). Let \(x \notin \varphi_C(A)\), then \(\exists U \delta\)-nbd. of \(x\) such that \(U \cap A \notin G\). By using (v) of Theorem 3.5., then \(\varphi_C(U \cap A) = \varphi_G\). By using Proposition 3.8., then \(U \cap \varphi_C(A) = \varphi_G\) and so \(\varphi_C(A) \subseteq X - U\). Consequently, \(x \notin \varphi_C(A)^*\) which implies that \(x \notin \text{Cl}_G(\varphi_C(A))\). Then, \(\text{Cl}_G(\varphi_C(A)) \subseteq \varphi_C(A)\) and \(\varphi_C(A) = \text{Cl}_G(\varphi_C(A))\) i.e. \(\varphi_C(A)\) is closed w.r.to \(\delta\). By using (iv) of Theorem 3.5. \(\varphi_C(\varphi_C(A)) \subseteq \text{Cl}_G(\varphi_C(A))\) and so \(\varphi_C(\varphi_C(A)) \subseteq \varphi_C(A)\).

Proposition 3.10 Let \((X, \delta, G)\) be a proximity space with a grill \(G\) and \(x \in X\). Then:

(i) \(\varphi_G(x) = x\)
(ii) \(A \subseteq \varphi_G(A) \forall \delta\)-nbd. \(A\) of \(x\).

Proof

(i) Obvious

(ii) Let \(A \subseteq \varphi_G(A) \forall \delta\)-nbd. of \(x\), then \(A \subseteq \text{Cl}_G(A)\). By using Proposition 3.8. and (i) of this proposition \(A \cap \varphi_G(A) = A \cap \varphi_G(A \cap X)\) and hence \(A = A \cap \varphi_G(A)\). Consequently, \(A \subseteq \varphi_G(A)\).

Corollary 3.11 Let \((X, \delta, G)\) be a proximity space with a grill \(G\), then for a non-empty subset \(A\) of \(X\) the following statements are hold:

(i) \(\varphi_G(A)^* \subseteq \varphi_G(A)\).

(ii) If \(A\) is \(\delta\)-open, then \(x \in \varphi_G(A)\) iff \(x \in A\).

Proof Obvious in view of Theorems 3.5., 3.7. and Proposition 3.10.

Proposition 3.12 Let \(\delta_1, \delta_2\) be two proximities on a non-empty set \(X\) and \(\delta_2\) is finer than \(\delta_1\). Then, for any grill \(G\) on \(X\) and any subset \(A\) of \(X\), \(\varphi_G(A, \delta_2) \subseteq \varphi_G(A, \delta_1)\).

Proof Let \(A \subseteq X\) and \(x \notin \varphi_G(A, \delta_1)\), then \(\exists \delta_1\)-nbd. \(U\) of \(x\) s.t. \(U \cap A \notin G\). By hypothesis \(\delta_2 < \delta_1\). \(U\) is a \(\delta_2\)-nbd. of \(x\) and \(U \cap A \notin G\), and so \(x \notin \varphi_G(A, \delta_2)\). Consequently, \(\varphi_G(A, \delta_2) \subseteq \varphi_G(A, \delta_1)\).

Theorem 3.13 Let \(G\) be a grill on a space \(X\). We define a map \(\psi_G : P(X) \rightarrow P(X)\), by \(\psi_G(A) = A \cup \varphi_G(A, \tau, A)\), (for \(A \subseteq P(X)\)), is a Kuratowski closure operator. Moreover, \(\psi_G(A) \subseteq \text{Cl}_G(A)\).

Proof From the definition of \(\psi_G\), statements (ii) and (v) of Theorem 3.5. and Theorem 3.9.; \(\psi_G(A)\) is a Kuratowski closure operator and \(\psi_G(A) \subseteq \text{Cl}_G(A)\).

For a given Ef-Proximity space with grill \((X, \delta, G)\), a new proximity structure is generated by using the closure operator \(\psi_G\) as we see in the following theorem.

Theorem 3.14 Let \(A\) and \(B\) be subsets of a space \((X, \delta, G)\). Then, the relation \(\delta^*\) generated by the closure operator \(\psi_G\) which is defined by \(A \delta^* B\) iff \(\psi_G(A) \cap \psi_G(B) \neq \emptyset\) is a basic proximity on \(X\), which is finer then \(\delta\).

Proof \(\delta^*\) is a basic proximity on \(X\) follows directly from the definition of \(\delta^*\). Moreover, \(A \delta^* B \iff \psi_G(A) \cap \psi_G(B) \neq \emptyset \rightarrow \text{Cl}_G(A) \cap \text{Cl}_G(B) \neq \emptyset \rightarrow \text{Cl}_G(A) \delta \text{Cl}_G(B) \rightarrow A \delta B\). Hence, \(\delta < \delta^*\).

Theorem 3.15 Let \((X, \delta)\) be an Ef-proximity space and let \(G\) and \(H\) be grills on \(X\) with \(G \subseteq H\), then \(\delta^* H < \delta^* G\).

Proof Let \(A, B \subseteq X\) and \(A \delta^* B\), then \(A \delta G \cap B \delta G\). Since \(G \subseteq H\) it follows that \(H \subseteq \psi_G(A) \cap H \subseteq \psi_G(B) \neq \emptyset\). Consequently, \(A \delta^* H \) and so \(\delta^* H < \delta^* G\).

The proof of the following theorems is obvious and omitted.

Theorem 3.16 Let \((X, \delta)\) be an Ef-proximity space with grills \(G\) and \(H\) on \(X\). Then, \(\delta^* H \subseteq \delta^* G \cup \delta^* G\).

Theorem 3.17 Let \(\delta^*\) be a proximity relation on \(X\) generated by an Ef-proximity relation \(\delta\) and a grill \(G\) on \(X\), then:

(i) \(\delta^*\) is a discrete proximity, if \(G = P(X) - \{\emptyset\}\).

(ii) If \(G = [A]\), then \(A \delta^* B \iff \text{Cl}_G(A) \cap \text{Cl}_G(B) \neq \emptyset \forall A, B \subseteq X\).

Some induced proximity and topological structures and some relations among these structures by using the operator \(\varphi_G\).

Note that; for a topological space with grill \((X, \tau, G)\) the induced topology \(\tau^*\) is the topology generated by the closure operator \(\psi_G(A) = A \cup \varphi_G(\tau, A)\), where \(\varphi_G(\tau, A) = \{x \in X : U \cap A \in \tau \ \forall U \in \text{Cl}_G(A)\}\). Moreover, \(\varphi_G(\tau, A) = \varphi_G(\tau, A)\).

The relation between \(\delta^*\) and \(\delta^*\) have been investigated.

Lemma 3.18 Let \((X, \tau, G)\) be a completely regular topological space with grill \(G\) and \(\delta\) is a compatible proximity structure. Then, \(\forall A \subseteq X, \varphi_G(\tau, A) = \varphi_G(\delta, A)\) and the equality holds if \((X, \tau)\) is a compact space.

Proof Let \(x \notin \varphi_G(\delta, A)\), then there is \(\delta\)-nbd. \(U\) of \(x\) s.t. \(U \cap A \notin G\). Since every \(\delta\)-nbd. is a \(\tau\)-nbd. Then, there is \(\tau\)-open nbd. \(V\) of \(x\) s.t. \(V \subseteq U\) and \(U \cap A \notin G\). Hence, \(x \notin \varphi_G(\tau, A)\) and so \(\varphi_G(\tau, A) \subseteq \varphi_G(\delta, A)\). Let \((X, \tau)\) be a compact space and let \(x \notin \varphi_G(\delta, A)\), then there is \(\text{int}\)-open nbd. \(U\) of \(x\) s.t. \(U \cap A \notin G\). Since \(\tau\) is a compact completely regular topology, then \(U\) is a \(\delta\)-nbd.
of $x$. Consequently, $x \notin A^*(\delta, G)$ and so, $\varphi_C(\delta, A) \subseteq \varphi_C(\tau, A)$. This follows that $\varphi_C(\tau, A) = \varphi_C(\delta, A)$.

**Theorem 3.19** Let $(X, \tau, G)$ be a completely regular space with a grill, then the induced topology $\tau^*$ is compatible with a proximity coarser than $(\delta)^*$ and if the space $(X, \tau, G)$ is also compact, then $\tau^*$ is compatible with $(\delta)^*$.

**Proof** Obvious, by using Lemma 3.18.

**Theorem 3.20** If $(X, \tau, G)$ is a compact completely regular space with a grill $G$ and $\tau^*$ is a compact topology on $X$. Then, $\delta^* < (\delta)^*$.

**Proof** Since $\tau^*$ is compact, $\varphi_C(\delta, A) = \varphi_C(\tau, A) = \varphi_C(\tau^*, A) = \varphi_C(\delta^*, A)$. Consequently, $(\delta)^* = (\delta_\tau)^*$. Also, since $\delta^* < (\delta)^*$, then $\delta^* < (\delta_\tau)^*$.

**Theorem 3.21** Let $(X, \delta, G)$ be an Ef-proximity space with a grill $G$ and let $A \subseteq X$, then $\varphi_R G R (\tau R \delta R, A) \subseteq \varphi_R G R (\delta R, A)$.

**Proof** Let $x \notin \varphi_R G R (\delta R, A)$, then there is a $\delta$-nbd. $U$ of $x$ s.t. $U \cap A \notin \mathcal{G}$, which implies there is a $\tau_\delta$-open nbd $V$ of $x$ s.t. $V \subseteq U$ and $U \cap A \notin \mathcal{G}$. Hence, $x \notin \varphi_R G R (\tau_\delta, A)$.

**Theorem 3.22** $\tau^* \subseteq (\tau_\delta)^*$.

**Proof** Let $A$ be a $\tau^*$-closed, then $\forall x \notin A$, $x \notin A$. i.e $\psi([x]) \cap \psi(A) = \emptyset \ \forall x \in A$ which implies that $x \notin \varphi_C(\delta, A)$. Consequently, by using Theorem 3.21. $x \notin \varphi_C(\tau_\delta, A) \ \forall x \in A$. This follows that $\varphi_C(\tau_\delta, A) \subseteq A$, then $A$ is a $(\tau_\delta)^*$-closed.

**References**


