Operation Approaches on $\alpha-(\gamma,\beta)$-Open (closed) Mappings and $\gamma$ generalized $\alpha$-open sets

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Abstract

In this paper the concept of $\alpha-(\gamma,\beta)$-open (closed) mappings have been introduced and studied. Further $\gamma$-$g\alpha$-open (closed) sets, $\gamma$-$\alpha T_b$ space, $\gamma$-$\alpha T_d$ space and $\gamma$-$T_{g\alpha}$ space have been introduced and some of their basic properties are studied.

Key words: ($\alpha$-$\gamma,\beta$)-continuous mappings, $\alpha$-($\gamma,\beta$)-continuous mappings, $\gamma$-$g\alpha$-open (closed) sets, $\gamma$-$\alpha T_b$ space, $\gamma$-$\alpha T_d$ space, $\gamma$-$T_{g\alpha}$ space.

1 Introduction

O.Njastad [6] introduced $\alpha$-open sets in a topological space and studied some of its properties. Kasahara [3] defined the concept of an operation on topological spaces and introduced $\alpha$-closed graphs of an operation. Ogata [7] called the operation $\alpha$ as $\gamma$ operation and introduced the notion of $\tau_\gamma$ which is the collection of all $\gamma$-open sets in a topological space $(X, \tau)$. Further he introduced the concept of $\gamma$-$T_i$ spaces ($i = 0, \frac{1}{2}, 1, 2$) and characterized $\gamma$-$T_i$ spaces using the notion of $\gamma$-closed set or $\gamma$-open sets. G.Sai Sundara Krishnan and N.Kalaivani [9] introduced $\alpha$-$\gamma$-open sets in topological spaces and studied some of their basic properties.

In his paper paper in section 3 we studied some properties of $\alpha$-($\gamma,\beta$)-continuous mappings, $\alpha$-($\gamma,\beta$)-homeomorphism and we introduced the notion of $\alpha$-$\beta$-$T_i$ ($i = \frac{1}{2}, 1, 2$) spaces.

In sections 4 and 5 we introduced the concept of $\alpha$-($\gamma,\beta$)-open and $\alpha$-($\gamma,\beta$)-closed mappings and characterized the mappings with $\alpha$-$\gamma$-interior and $\alpha$-$\gamma$-closure operators and investigated

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their basic properties.

In section 6 γ-gα-open sets, γ-gα- closed sets ,γ-α T_b, γ-α T_d and γ-T_gα space have been introduced and some of their properties are discussed.

2 Preliminaries

In this section we recall some of the basic Definitions and Remarks.

Definition 2.1 [6] Let (X,τ) be a topological space and A be a subset of X. Then A is said to be α-open set if A ⊆ int(cl(int(A))) and α-closed set if cl(int(cl(A))) ⊇ A.

Definition 2.2 [9] Let (X,τ) be a topological space and γ be an operation on τ. Then a subset A of X is said to be a α-γ-open set if and only if A ⊆ τγ − int(τγ − cl(τγ − int(A)))

Definition 2.3 [9] Let (X,τ) be a topological space and γ be an operation on τ. Then a subset A of X is said to be a α-γ-closed if and only if X − A is α-γ-open.

Remark 2.4 [9] Let (X,τ) be a topological space and γ be an operation on τ and A be a subset of X. Then A is α-γ-closed if and only if A ⊇ τγ − cl(τγ − cl(τγ − cl(A)))

Definition 2.5 [9] Let (X,τ) be a topological space and γ be an operation on τ and A be a subset of X. Then τα−γ-interior of A is the union of all α-γ-open sets contained in A and it is denoted by τα−γ-int(A) = ∪ {U : U is a α-γ-open set and U ⊆ A}

Definition 2.6 [9] Let (X,τ) be a topological space and γ be an operation on τ. Let A be a subset of X. Then τα−γ-closure of A is the intersection of all α-γ-closed sets containing A and it is denoted by τα−γ-cl(A) = ∩ {F : F is a α-γ-closed set and A ⊆ F}

Remark 2.7 [9] Let (X,τ) be a topological space and γ be an operation on τ. A subset A of X is said to be α-γ-generalized closed (written as α-γ g-closed set) if τα−γ-cl(A) ⊆ U whenever A ⊆ U and U is α-γ-open set in (X,τ).

Definition 2.8 [7] A mapping f : X → Y is said to be (γ,β)-continuous if for each x of X and each open set V containing f(x) there exists an open set U such that x ∈ U and f(Uγ) ⊆ Vβ.

Definition 2.9 [7] Let (X,τ) be a topological space and γ be an operation on τ. A subset A of X is said to be γ-generalized closed (written as γ-g.closed set) if clγ ⊆ U whenever A ⊆ U and U is γ-open in (X,τ).

Definition 2.10 [2] A mapping f : (X,τ) → (Y,σ) is said to be α-(γ,β)-continuous if and only if for any α-β-open set U of Y, f−1(U) is α-γ-open in X.
3 Some properties of \(\alpha-(\gamma,\beta)\)-continuous mapping and 
\(\alpha-\beta-T_i\) spaces

Let \((X, \tau)\) and \((Y, \sigma)\) be two topological spaces. \(\gamma: \tau \to P(X)\) and \(\beta: \sigma \to P(Y)\) be operations on \(\tau\) and \(\sigma\) respectively.

**Definition 3.1** Let \((X, \tau)\) be a topological space and \(\gamma: \tau \to P(X)\) be an operation on \(\tau\). Then a subset \(A\) of \(X\) is said to be a \(\alpha-\gamma\) neighbourhood of a point \(x \in X\) if there exists a \(\alpha-\gamma\)-open set \(U\) such that \(x \in U \subseteq A\).

**Theorem 3.2** A mapping \(f: (X, \tau) \to (Y, \sigma)\) is \(\alpha-(\gamma,\beta)\)-continuous if and only if for each \(x \in X\), the inverse of every \(\alpha-\beta\)-neighbourhood of \(f(x)\) is \(\alpha-\gamma\)-neighbourhood of \(x\).

**Proof:** Let \(x \in X\) and \(B\) be a \(\alpha-\beta\)-neighbourhood of \(f(x)\). By Definition 3.1 there exists a \(V \in \sigma_{\alpha-\beta}(Y)\) such that \(f(x) \in V \subseteq B\). This implies that \(x \in f^{-1}(V) \subseteq f^{-1}(B)\). Since \(f\) is \(\alpha-(\gamma,\beta)\)-continuous, \(f^{-1}(V) \in \tau_{\alpha-\gamma}(X)\). Hence \(f^{-1}(B)\) is a \(\alpha-\gamma\)-neighbourhood of \(x\).

Conversely, Let \(B \in \sigma_{\alpha-\beta}\). Put \(A = f^{-1}(B)\). Let \(x \in A\). Then \(f(x) \in B\). \(B\) is a \(\alpha-\beta\)-neighbourhood of \(f(x)\). So by hypothesis, \(A = f^{-1}(B)\) is a \(\alpha-\gamma\)-neighbourhood of \(x\). Hence by Definition 3.1 there exists \(A_x \in \tau_{\alpha-\gamma}\) such that \(x \in A_x \subseteq A\). This implies that \(A = \bigcup_{x \in A} A_x\). By Theorem 3.4 \([9]\) \(A\) is \(\alpha-\gamma\)-open in \(X\). Therefore \(f\) is \(\alpha-(\gamma,\beta)\)-continuous.

**Theorem 3.3** A mapping \(f: (X, \tau) \to (Y, \sigma)\) is \(\alpha-(\gamma,\beta)\)-continuous if and only if for each point \(x \in X\) and each \(\alpha-\beta\)-neighbourhood \(B\) of \(f(x)\), there is a \(\alpha-\gamma\)-neighbourhood \(A\) of \(x\) such that \(f(A) \subseteq B\).

**Proof:** Let \(x \in X\) and \(B\) be a \(\alpha-\beta\)-neighbourhood of \(f(x)\). Then there exists \(O_{f(x)} \in \sigma_{\alpha-\beta}\) such that \(f(x) \in O_{f(x)} \subseteq B\). It follows that \(x \in f^{-1}(O_{f(x)}) \subseteq f^{-1}(B)\). By hypothesis, \(f^{-1}(O_{f(x)}) \subseteq \tau_{\alpha-\gamma}\). Let \(A = f^{-1}(B)\). Then it follows that \(A\) is \(\alpha-\gamma\)-neighbourhood of \(x\) and \(f(A) = f(f^{-1}(B)) \subseteq B\).

Conversely, let \(U \in \sigma_{\alpha-\beta}\). Take \(W = f^{-1}(U)\). Let \(x \in W\). Then \(f(x) \in U\). Thus \(U\) is a \(\alpha-\beta\)-neighbourhood of \(f(x)\). By hypothesis, there exists a \(\alpha-\gamma\) neighbourhood \(V_x\) of \(x\) such that \(f(V_x) \subseteq U\). Thus it follows that \(x \in V_x \subseteq f^{-1}(f(V_x)) \subseteq f^{-1}(U) = W\). Since \(V_x\) is a \(\alpha-\gamma\)-neighbourhood of \(x\), which implies that there exists a \(W_x \in \tau_{\alpha-\gamma}\) such that \(x \in W_x \subseteq W\). This implies that \(W = \bigcup_{x \in W} W_x\). By Theorem 3.4 \([9]\) \(W\) is \(\alpha-\gamma\)-open in \(X\). Thus \(f\) is \(\alpha-(\gamma,\beta)\)-continuous.

**Theorem 3.4** Let \(f: (X, \tau) \to (Y, \sigma)\) be a mapping. Then the following statements are equivalent:

(i) \(f\) is \(\alpha-(\gamma,\beta)\)-continuous.
(ii) \(f[\tau_{\alpha-\gamma} - cl(A)] \subseteq \sigma_{\alpha-\beta} - cl[f(A)]\) holds for every subset \(A\) of \((X, \tau)\).
(iii) For every \(\alpha-\beta\)-closed set \(V\) of \((Y, \sigma)\), \(f^{-1}(V)\) is \(\alpha-\gamma\)-closed in \((X, \tau)\).
Proof:
(i) $\rightarrow$ (ii). Let $y \in f(\tau_{\alpha-\gamma} - cl(A))$ and $V$ be any $\alpha$-$\beta$-open set containing $y$. Using Theorem 3.3, then there exists a point $x \in X$ and a $\alpha$-$\gamma$-open set $U$ such that $x \in U$ with $f(x) = y$ and $f(U) \subseteq V$. Since $x \in \tau_{\alpha-\gamma} - cl(A)$, we have $U \cap A \neq \emptyset$ and hence $\phi \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A)$. This implies that $y \in \sigma_{\alpha-\beta} - cl(f(A))$. Therefore, we have $f(\tau_{\alpha-\gamma} - cl(A)) \subseteq \sigma_{\alpha-\beta} - cl(f(A))$.

(ii) $\rightarrow$ (iii). Let $V$ be a $\alpha$-$\beta$-closed set in $Y$. Then $\sigma_{\alpha-\beta} - cl(V) = V$. By (ii) $f(\tau_{\alpha-\gamma} - cl(f^{-1}(V))) \subseteq \sigma_{\alpha-\beta} - cl(f(f^{-1}(V))) \subseteq \sigma_{\alpha-\beta} - cl(V) = V$ holds. Therefore $\tau_{\alpha-\gamma} - cl(f^{-1}(V)) \subseteq f^{-1}(V)$ and thus $f^{-1}(V) = \tau_{\alpha-\gamma} - cl(f^{-1}(V))$. Hence $f^{-1}(V)$ is $\alpha$-$\gamma$-closed in $X$.

(iii) $\rightarrow$ (i). Let $B$ be any $\alpha$-$\beta$-open set in $Y$. Consider $V = Y - B$. Then $V$ is $\alpha$-$\beta$-closed in $Y$. By (iii) $f^{-1}(V)$ is $\alpha$-$\gamma$-closed in $X$. Hence $f^{-1}(B) = f^{-1}(Y - B) = X - f^{-1}(V)$ is $\alpha$-$\gamma$-open in $X$. Hence $f$ is $\alpha$-$(\gamma, \beta)$- continuous.

Theorem 3.5 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a $\alpha$-$(\gamma, \beta)$- continuous mapping and injective. If $Y$ is $\alpha$-$\beta$-$T_2$ (resp.$\alpha$-$\beta$-$T_1$), then $X$ is $\alpha$-$\gamma$-$T_2$ (resp.$\alpha$-$\gamma$-$T_1$).

Proof: Suppose $Y$ is $\alpha$-$\beta$-$T_2$. Let $x$ and $y$ be two distinct points of $X$. Then, there exists two $\alpha$-$\beta$-open sets $U$ and $V$ such that $f(x) \in U$, $f(y) \in V$ and $U \cap V = \emptyset$. Since $f$ is $\alpha$-$(\gamma, \beta)$- continuous, for $U$ and $V$, there exist two $\alpha$-$\gamma$-open sets $W$ and $S$ such that $x \in W$ and $y \in S$, $f(W) \subseteq U$ and $f(S) \subseteq V$, implies that $W \cap S \neq \emptyset$. Hence $X$ is $\alpha$-$\gamma$-$T_2$. In a similar way we can prove that $X$ is $\alpha$-$\gamma$-$T_1$ whenever $Y$ is $\alpha$-$\beta$-$T_1$.

Theorem 3.6 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \delta)$ be two mappings.

(i) If $f$ is $\alpha$-$(\gamma, \beta)$- continuous and $g$ is $\beta$-$(\delta)$- continuous, then $g \circ f$ is $\alpha$-$(\gamma, \delta)$- continuous;

(ii) If $f$ is $\alpha$-$(\gamma, \beta)$- continuous and $g$ is $\alpha$-$(\delta)$- continuous, then $g \circ f$ is $\alpha$-$(\gamma, \delta)$- continuous;

(iii) If $f$ is $\alpha$-$(\gamma, \beta)$- continuous and $g$ is $\alpha$-$(\beta, \delta)$- continuous, then $g \circ f$ is $\alpha$-$(\gamma, \delta)$- continuous;

Proof: Follows from the Definitions 2.20[7], 4.1[2] and 6.1[2].

4 $\alpha$-$(\gamma, \beta)$-open mappings

In this section we introduce the concept of $\alpha$-$(\gamma, \beta)$-open mappings and study some of its basic properties.

Let $(X, \tau)$ and $(Y, \sigma)$ be two topological spaces. $\gamma : \tau \rightarrow P(X)$ and $\beta : \sigma \rightarrow P(Y)$ be operations on $\tau$ and $\sigma$ respectively.

Definition 4.1 A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\alpha$-$(\gamma, \beta)$-open if and only
if for each \( A \in \tau_{\alpha-\gamma}, f(A) \in \sigma_{\alpha-\beta} \).

**Example 4.2** Let \( X = \{a, b, c\}, Y = \{1, 2, 3\}, \tau = \{\varphi, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\} \) and \( \sigma = \{\varphi, Y, \{2\}, \{1, 3\}\} \). Define operations \( \gamma : \tau \to P(X) \) and \( \beta : \sigma \to P(Y) \) by \( A^\gamma = cl(A) \) for every \( A \in \tau \) and \( B^\beta = cl(B) \) for every \( B \in \sigma \).

Define \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = 1, f(b) = 3 \) and \( f(c) = 2 \). The image of every \( \alpha-\gamma \)-open set is \( \alpha-\beta \)-open under \( f \). Hence \( f \) is \( \alpha-(\gamma, \beta) \)-open.

**Remark 4.3** Every \( \alpha-(\gamma, \beta) \)-open mapping is \( (\gamma, \alpha-\beta) \)-open. But the converse need not be true.

Let \( X = \{a, b, c\}, Y = \{1, 2, 3\}, \tau = \{\varphi, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\} \) and \( \sigma = \{\varphi, Y, \{1\}, \{3\}, \{1, 2\}, \{1, 3\}\} \). Define operations \( \gamma : \tau \to P(X) \) and \( \beta : \sigma \to P(Y) \) by

\[
A^\gamma = \begin{cases} \emptyset & \text{if } A = \{a\} \\ A \cup \{c\} & \text{if } A \neq \{a\} \end{cases}
\]

\[
B^\beta = \begin{cases} cl(B) & \text{if } b \in B \\ B & \text{if } b \notin B \end{cases}
\]

Define \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = 1, f(b) = 1 \) and \( f(c) = 2 \). The image of every \( \gamma \)-open set in \( X \) is \( \alpha-\beta \)-open in \( Y \) under \( f \). Hence \( f \) is \( (\gamma, \alpha-\beta) \)-open. But the image of every \( \alpha-\gamma \)-open set is not \( \alpha-\beta \)-open. Hence \( f \) is not \( \alpha-(\gamma, \beta) \)-open.

**Remark 4.4** If \( f : (X, \tau) \to (Y, \sigma) \) is said to be \( \alpha-(\gamma, \beta) \)-open and \( g : (Y, \sigma) \to (Z, \delta) \) is \( \alpha-(\beta, \delta) \)-open, then the composition \( g \circ f : (X, \tau) \to (Z, \delta) \) is \( \alpha-(\gamma, \delta) \)-open mapping.

**Theorem 4.5** A mapping \( f : (X, \tau) \to (Y, \sigma) \) is \( \alpha-(\gamma, \beta) \)-open if and only if for each \( x \in X \), and for every \( A \in \tau_{\alpha-\gamma} \) such that \( x \in A \), there exists \( B \in \sigma_{\alpha-\beta} \) such that \( f(x) \in B \) and \( B \subseteq f(A) \).

**Proof:** Let \( A \) be a \( \alpha-\gamma \)-open set of \( x \in X \). Then \( f(x) \in f(A) \). Therefore \( f(A) \) is a \( \alpha-\beta \)-open neighbourhood of \( f(x) \) in \( Y \). Then by Theorem 3.3 there exists a \( \alpha-\gamma \)-open neighbourhood \( B \in \sigma_{\alpha-\beta} \) such that \( f(x) \in B \subseteq f(A) \).

Conversely, Let \( A \in \tau_{\alpha-\gamma} \) such that \( x \in A \). Then by assumption, there exists \( B \in \sigma_{\alpha-\beta} \) such that \( f(x) \in B \subseteq f(A) \). Therefore \( f(A) \) is a \( \alpha-\beta \)-neighbourhood of \( f(x) \) in \( Y \) and this implies that \( f(A) = \bigcup_{f(x) \in f(A)} B \). Then by Theorem 3.4 [8] \( f(A) \) is \( \alpha-\beta \)-open in \( Y \). Hence \( f \) is \( \alpha-(\gamma, \beta) \)-open.

**Theorem 4.6** A mapping \( f : (X, \tau) \to (Y, \sigma) \) is \( \alpha-(\gamma, \beta) \)-open if and only if for each \( x \in X \), and for every \( \alpha-\gamma \)-neighbourhood \( U \) of \( x \in X \) there exists a \( \alpha-\beta \)-neighbourhood \( V \) of \( f(x) \in Y \) such that \( V \subseteq f(U) \).
Proof: Let $U$ be a $\alpha$-$\gamma$-neighbourhood of $x \in X$. Then by Definition 3.1 there exists a $\alpha$-$\gamma$-open set $W$ such that $x \in W \subseteq U$. This implies that $f(x) \in f(W) \subseteq f(U)$. Since $f$ is a $\alpha$-($\gamma$, $\beta$)-open mapping, we have $f(W)$ is $\alpha$-$\beta$-open. Hence $V = f(W)$ is a $\alpha$-$\beta$-neighbourhood of $f(x)$ and $V \subseteq f(U)$.

Conversely, Let $U \in \tau_{a-\gamma}$ and $x \in U$. Then $U$ is a $\alpha$-$\gamma$-neighbourhood of $x$. So by hypothesis, there exists a $\alpha$-$\beta$-neighbourhood $V$ of $f(x)$ such that $f(x) \in V \subseteq f(U)$. That is, $f(U)$ is a $\alpha$-$\beta$-neighbourhood of $f(x)$. Thus $f(U)$ is a $\alpha$-$\beta$-neighbourhood of each of its points. Therefore $f(U)$ is $\alpha$-$\beta$-open. Hence $f$ is $\alpha$-($\gamma$, $\beta$)-open.

**Theorem 4.7** A mapping $f : (X, \tau) \to (Y, \sigma)$ is $\alpha$-($\gamma$, $\beta$)-open if and only if $f(\tau_{a-\gamma} - \text{int}(A)) \subseteq \sigma_{a-\beta} - \text{int}(f(A))$, for all $A \subseteq X$.

**Proof:** Let $x \in \tau_{a-\gamma} - \text{int}(A)$. Then there exists $U \in \tau_{a-\gamma}$ such that $x \in U \subseteq A$. So $f(x) \in f(U) \subseteq f(A)$. Since $f$ is $\alpha$-($\gamma$, $\beta$)-open, $f(U)$ is $\alpha$-$\beta$-open in $Y$. Hence $f(x) \in \sigma_{a-\beta} - \text{int}(f(A))$. Thus $f(\tau_{a-\gamma} - \text{int}(A)) \subseteq \sigma_{a-\beta} - \text{int}(f(A))$.

Conversely, Let $U \in \tau_{a-\gamma}$. Then by hypothesis, $f(U) = f(\tau_{a-\gamma} - \text{int}(U)) \subseteq \sigma_{a-\beta} - \text{int}(f(U)) \subseteq f(U)$ or $f(U) \subseteq \sigma_{a-\beta} - \text{int}(f(U)) \subseteq f(U)$. This implies that $f(U)$ is $\alpha$-$\beta$-open. So $f$ is $\alpha$-($\gamma$, $\beta$)-open.

**Theorem 4.8** A mapping $f : (X, \tau) \to (Y, \sigma)$ is $\alpha$-($\gamma$, $\beta$)-open if and only if $\tau_{a-\gamma} - \text{int}(f^{-1}(B)) \subseteq f^{-1}(\sigma_{a-\beta} - \text{int}(B))$, for all $B \subseteq Y$.

**Proof:** Let $B$ be any subset of $Y$. Clearly, $\tau_{a-\gamma} - \text{int}(f^{-1}(B))$ is $\alpha$-$\gamma$-open in $X$. Also $f(\tau_{a-\gamma} - \text{int}(f^{-1}(B))) \subseteq f(f^{-1}(B)) \subseteq B$. Since $f$ is $\alpha$-($\gamma$, $\beta$)-open and by Theorem 4.7, we have $f(\tau_{a-\gamma} - \text{int}(f^{-1}(B))) \subseteq \sigma_{a-\beta} - \text{int}(B)$. Hence $\tau_{a-\gamma} - \text{int}(f^{-1}(B)) \subseteq f^{-1}(f(\tau_{a-\gamma} - \text{int}(f^{-1}(B)))) \subseteq \sigma_{a-\beta} - \text{int}(B)$. This implies that $\tau_{a-\gamma} - \text{int}(f^{-1}(B)) \subseteq f^{-1}(\sigma_{a-\beta} - \text{int}(B))$ for all $B \subseteq Y$.

Conversely, Let $A \subseteq X$. By hypothesis, we obtain $\tau_{a-\gamma} - \text{int}(A) \subseteq \tau_{a-\gamma} - \text{int}(f^{-1}(f(A))) \subseteq f^{-1}(\sigma_{a-\beta} - \text{int}(f(A)))$. This implies that $f(\tau_{a-\gamma} - \text{int}(A)) \subseteq f(\tau_{a-\gamma} - \text{int}(f^{-1}(f(A)))) \subseteq \sigma_{a-\beta} - \text{int}(f(A))$. Consequently, $f(\tau_{a-\gamma} - \text{int}(A)) \subseteq \sigma_{a-\beta} - \text{int}(f(A))$, for all $A \subseteq X$. By Theorem 4.7, $f$ is $\alpha$-($\gamma$, $\beta$)-open.

**Theorem 4.9** A mapping $f : (X, \tau) \to (Y, \sigma)$ is $\alpha$-($\gamma$, $\beta$)-open if and only if $f^{-1}(\sigma_{a-\beta} - \text{cl}(B)) \subseteq \tau_{a-\gamma} - \text{cl}(f^{-1}(B))$, for all $B \subseteq Y$.

**Proof:** Let $B$ be any subset of $Y$. By theorem 4.8 $\tau_{a-\gamma} - \text{int}(f^{-1}(Y - B)) \subseteq f^{-1}(\sigma_{a-\beta} - \text{int}(Y - B))$. Then $\tau_{a-\gamma} - \text{int}(X - f^{-1}(B)) \subseteq f^{-1}(\sigma_{a-\beta} - \text{int}(Y - B))$. As $\sigma_{a-\beta} - \text{int}(B) = Y - \sigma_{a-\beta} - \text{cl}(Y - B)$, therefore $X - \tau_{a-\gamma} - \text{cl}(f^{-1}(B)) \subseteq X - f^{-1}(Y - \sigma_{a-\beta} - \text{cl}(B))$ or $X - \tau_{a-\gamma} - \text{cl}(f^{-1}(B)) \subseteq X - f^{-1}(\sigma_{a-\beta} - \text{cl}(B))$. Hence $f^{-1}(Y - \sigma_{a-\beta} - \text{cl}(B)) \subseteq f^{-1}(Y - \sigma_{a-\beta} - \text{cl}(B))$.

Conversely, Let $B \subseteq Y$. By hypothesis, $f^{-1}(\sigma_{a-\beta} - \text{cl}(Y - B)) \subseteq \tau_{a-\gamma} - \text{cl}(f^{-1}(Y - B))$. Then
\[ X - \tau_{\alpha\gamma} - \text{cl}(f^{-1}(Y - B)) \subseteq X - f^{-1}(\sigma_{\alpha\beta} - \text{cl}(Y - B)). \] Hence \( X - \tau_{\alpha\gamma} - \text{cl}(f^{-1}(B)) \subseteq f^{-1}(Y - \sigma_{\alpha\beta} - \text{cl}(Y - B)). \) This gives that \( \tau_{\alpha\gamma} - \text{int}(f^{-1}(B)) \subseteq f^{-1}(\sigma_{\alpha\beta} - \text{int}(B)). \) Using Theorem 4.8, it follows that \( f \) is \( \alpha-(\gamma, \beta) \)-open.

**Theorem 4.10** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \rightarrow (Z, \zeta) \) be two mappings such that \( g \circ f : (X, \tau) \rightarrow (Z, \delta) \) be \( \alpha-(\gamma, \beta) \)-continuous mapping. Then

(i) If \( f \) is \( \alpha-(\beta, \delta) \)-open injection then \( f \) is \( \alpha-(\beta, \delta) \)-continuous;

(ii) If \( f \) is \( \alpha-(\gamma, \beta) \)-open surjection then \( g \) is \( \alpha-(\beta, \delta) \)-continuous;

**Proof:**

(i) Let \( U \in \sigma_{\alpha\beta} \). Since \( g \) is \( \alpha-(\beta, \delta) \)-open, then \( g(U) \in \zeta_{\alpha\delta} \). Since \( g \) is injective and \( g \circ f \) is \( \alpha-(\gamma, \delta) \)-continuous, we have \((g \circ f)^{-1}(g(U)) = (f^{-1} \circ g^{-1})(g(U)) = f^{-1}(g^{-1}g(U)) = f^{-1}(U) \) is \( \alpha-\gamma \)-open in \( X \). This proves that \( f \) is \( \alpha-(\gamma, \beta) \)-continuous.

(ii) Let \( V \in \zeta_{\alpha\delta} \). Since \( g \circ f \) is \( \alpha-(\gamma, \delta) \)-continuous, then \((g \circ f)^{-1}(V) \in \tau_{\alpha\gamma}(X) \). Also \( f \) is \( \alpha-(\gamma, \beta) \)-open, so \( f((g \circ f)^{-1}(V)) \) is \( \alpha-\beta \)-open in \( Y \). Since \( f \) is surjective, we obtain \((f \circ (g \circ f)^{-1})(V) = (f \circ f^{-1} \circ g^{-1})(V) = ((f \circ f^{-1}) \circ g^{-1})(V) = g^{-1}(V) \). It follows that \( g^{-1}(V) \in \sigma_{\alpha\beta} \). This proves that \( g \) is \( \alpha-(\beta, \delta) \)-continuous mapping.

## 5 \( \alpha-(\gamma, \beta) \)-closed mappings

In this section we introduce the concept of \( \alpha-(\gamma, \beta) \)-closed mappings and study some of its basic properties.

Let \((X, \tau)\) and \((Y, \sigma)\) be two topological spaces. \( \gamma : \tau \rightarrow P(X) \) and \( \beta : \sigma \rightarrow P(Y) \) be operations on \( \tau \) and \( \sigma \) respectively.

**Definition 5.1** A mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be \( \alpha-(\gamma, \beta) \)-closed if and only if the image set \( f(A) \) is \( \alpha-\beta \)-closed for each \( \alpha-\gamma \)-closed subset \( A \) of \( X \).

**Example 5.2** Let \( X = \{a, b, c\} \), \( Y = \{1, 2, 4\} \), \( \tau = \{\varnothing, X, \{a\}, \{c\}, \{a,b\}, \{a,c\}\} \) and \( \sigma = \{\varnothing, Y, \{1\}, \{4\}, \{1,2\}, \{1,4\}\} \). Define Operations \( \gamma : \tau \rightarrow P(X) \) and \( \beta : Y \rightarrow P(Y) \) by

\[
A^\gamma = \begin{cases} 
A & \text{if} A = \{a\} \\
A \cup \{c\} & \text{if} A \neq \{a\}
\end{cases}
\]

\[
B^\beta = \begin{cases} 
\text{cl}(B) & \text{if} b \in B \\
B & \text{if} b \notin B
\end{cases}
\]

Define \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = 1, f(b) = 2 \) and \( f(c) = 2 \). The image of every \( \alpha-\gamma \)-closed set in \( X \) is \( \alpha-\beta \)-closed in \( Y \) under \( f \). Hence \( f \) is \( \alpha-(\gamma, \beta) \)-closed.

**Remark 5.3** Every \( \alpha-(\gamma, \beta) \)-closed mapping is \( (\gamma, \alpha-\beta) \)-closed. But the converse need not be true.
Define Operations \( \gamma : \tau \rightarrow P(X) \) and \( \beta : Y \rightarrow P(Y) \) by

\[
A^\gamma = \begin{cases} 
A & \text{if } b \notin A \\
\text{cl}(A) & \text{if } b \in A
\end{cases}
\]

\[
B^\beta = \begin{cases} 
\text{cl}(B) & \text{if } b \notin B \\
B \cup \{c\} & \text{if } b \in B
\end{cases}
\]

Define \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = 1 \), \( f(b) = 3 \) and \( f(c) = 2 \). \( f \) is (\( \gamma \), \( \alpha \)-\( \beta \))-closed but not \( \alpha-(\gamma, \beta) \)-closed.

**Remark 5.4** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \alpha-(\gamma, \beta) \)-closed and \( g : (Y, \sigma) \rightarrow (Z, \zeta) \) is \( \alpha-(\beta, \delta) \)-closed, then \( g \circ f : (X, \tau) \rightarrow (Z, \delta) \) be \( \alpha-(\gamma, \delta) \)-closed.

**Definition 5.5** A mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be \( \alpha-(\gamma, \beta) \)-homeomorphism if \( f \) is bijective, \( \alpha-(\gamma, \beta) \)-continuous and \( f^{-1} \) is \( \alpha-(\gamma, \beta) \)-homeomorphism.

**Remark 5.6** From the definitions 6.1[3] and 5.1 every bijective, \( \alpha-(\gamma, \beta) \)-continuous and \( \alpha-(\gamma, \beta) \)-closed map is \( \alpha-(\gamma, \beta) \)-homeomorphism.

**Theorem 5.7** A mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \alpha-(\gamma, \beta) \)-closed if and only if \( \sigma_{\alpha-\beta} - \text{cl}(f(A)) \subseteq f(\tau_{\alpha-\gamma} - \text{cl}(A)) \), for every subset \( A \) of \( X \).

**Proof**: Suppose \( f \) is \( \alpha-(\gamma, \beta) \)-closed and let \( A \subseteq X \). Then \( f(\tau_{\alpha-\gamma} - \text{cl}(A)) \) is \( \alpha-\beta \)-closed in \( Y \). Since \( f(A) \subseteq f(\tau_{\alpha-\gamma} - \text{cl}(A)) \), we obtain \( \sigma_{\alpha-\beta} - \text{cl}(f(A)) \subseteq f(\tau_{\alpha-\gamma} - \text{cl}(A)) \).

Conversely, suppose \( A \) is a \( \alpha-\gamma \)-closed set in \( X \). By hypothesis, we obtain \( f(A) \subseteq \sigma_{\alpha-\beta} - \text{cl}(f(A)) \subseteq f(\tau_{\alpha-\gamma} - \text{cl}(A)) = f(A) \). Hence \( f(A) = \sigma_{\alpha-\beta} - \text{cl}(f(A)) \). Thus \( f(A) \) is \( \alpha-\beta \)-closed set in \( Y \). This proves that \( f \) is \( \alpha-(\gamma, \beta) \)-closed.

**Theorem 5.8** A mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \alpha-(\gamma, \beta) \)-closed if and only if \( \sigma_{\beta} - \text{cl}(\sigma_{\beta} - \text{int}(\sigma_{\beta} - \text{cl}(f(A)))) \subseteq f(\tau_{\alpha-\gamma} - \text{cl}(A)) \), for every subset \( A \) of \( X \).

**Proof**: Suppose \( f \) is \( \alpha-(\gamma, \beta) \)-closed and let \( A \subseteq X \). Then \( f(\tau_{\alpha-\gamma} - \text{cl}(A)) \) is \( \alpha-\beta \)-closed in \( Y \). This implies that \( \sigma_{\beta} - \text{cl}(\sigma_{\beta} - \text{int}(\sigma_{\beta} - \text{cl}(f(\tau_{\alpha-\gamma} - \text{cl}(A)))))) \subseteq f(\tau_{\alpha-\gamma} - \text{cl}(A)) \). Then \( \sigma_{\beta} - \text{cl}(\sigma_{\beta} - \text{int}(\sigma_{\beta} - \text{cl}(f(A)))) \subseteq \sigma_{\beta} - \text{cl}(\sigma_{\beta} - \text{cl}(f(\tau_{\alpha-\gamma} - \text{cl}(A)))) \) gives \( \sigma_{\beta} - \text{cl}(\sigma_{\beta} - \text{int}(\sigma_{\beta} - \text{cl}(f(A)))) \subseteq f(\tau_{\alpha-\gamma} - \text{cl}(A)) \).

Conversely, Suppose \( A \) is a \( \alpha-\gamma \)-closed set in \( X \). Then by hypothesis, \( \sigma_{\beta} - \text{cl}(\sigma_{\beta} - \text{int}(\sigma_{\beta} - \text{cl}(f(A)))) \subseteq f(\tau_{\alpha-\gamma} - \text{cl}(A)) \). Since \( A \) is \( \alpha-\gamma \)-closed, we obtain \( f(\tau_{\alpha-\gamma} - \text{cl}(A)) \subseteq f(A) \). Hence \( f(A) \) is \( \alpha-\beta \)-closed in \( Y \). This implies that \( f \) is \( \alpha-(\gamma, \beta) \)-closed.

**Theorem 5.9** A mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \alpha-(\gamma, \beta) \)-closed if and only if for each
subset B of Y and each \(\alpha\)-\(\gamma\)-open set A in X containing \(f^{-1}(B)\), there exists a \(\alpha\)-\(\beta\)-open set C in Y containing B such that \(f^{-1}(C) \subseteq A\).

**proof:** Let C = Y - f(X - A). Then \(f(X - A) \subseteq Y - B\). Since f is \(\alpha\)-\(\gamma\),\(\beta\)-closed , then C is \(\alpha\)-\(\beta\)-open and \(f^{-1}(C) = X - f^{-1}(f(X - A)) \subseteq X - (X - A) = A\). Conversely, suppose F is a \(\alpha\)-\(\gamma\)-closed set in X. Let B = Y - f(F). Then \(f^{-1}(B) \in X - f^{-1}(f(F)) \subseteq X - F\) and X - F is \(\alpha\)-\(\gamma\)-open in X. Hence by hypothesis, there exists a \(\alpha\)-\(\beta\)-open set C containing y such that \(f^{-1}(C) \subseteq X - F\). Then we have \(f^{-1}(C) \cap F = \phi\) and \(C \cap f(F) = \phi\). Therefore \(Y - f(F) \supseteq C \supseteq B = Y - f(F)\) and f(F) is \(\alpha\)-\(\beta\)-closed in Y. This proves that f is \(\alpha\)-\(\gamma\),\(\beta\)-closed.

**Theorem 5.10** Let \(f : (X, \tau) \to (Y, \sigma)\) be a bijective mapping. Then the following are equivalent:

(i) \(f\) is \(\alpha\)-\(\gamma\),\(\beta\)-closed.

(ii) \(f\) is \(\alpha\)-\(\gamma\),\(\beta\)-open.

(iii) \(f^{-1}\) is \(\alpha\)-\(\gamma\),\(\beta\)-continuous.

**Proof** (i) \(\Rightarrow\) (ii) Follows from the Definitions 7.1 and 8.1.

(ii) \(\Rightarrow\) (iii) Let A be a \(\alpha\)-\(\gamma\)-closed set in X. Then \(\tau_{\alpha} \gamma \subseteq A\). By condition (ii) and by Theorem 4.9, \(f^{-1}(\sigma_{\alpha} \gamma - cl(f(A))) \subseteq \tau_{\alpha} \gamma - cl(f^{-1}(f(A)))\) implies that \(\sigma_{\alpha} \gamma - cl(f(A)) \subseteq f(\tau_{\alpha} \gamma - cl(A))\). Thus \(\sigma_{\alpha} \gamma - cl((f^{-1})^{-1}(A)) \subseteq (f^{-1})^{-1}(A)\), for every subset A of X, it follows that \(f^{-1}\) is \(\alpha\)-\(\gamma\),\(\beta\)-continuous.

(iii) \(\Rightarrow\) (i). Let A be a \(\alpha\)-\(\gamma\)-closed set in X. Then X - A is \(\alpha\)-\(\gamma\)-open in X. Since \(f^{-1}\) is \(\alpha\)-\(\gamma\),\(\beta\)-continuous, \((f^{-1})^{-1}(X - A)\) is \(\alpha\)-\(\beta\)-open set in Y. But \((f^{-1})^{-1}(X - A)) = f(X - A) = Y - f(A)\). Thus f(A) is \(\alpha\)-\(\beta\)-closed in Y. This proves that f is \(\alpha\)-\(\gamma\),\(\beta\)-closed.

**Definition 5.11** Let id: \(\tau \to P(X)\) be the identity operation. A mapping \(f : (X, \tau) \to (Y, \sigma)\) is said to be \(\alpha\)-\(\beta\)-closed if for any \(\alpha\)-closed set F of X, f(F) is \(\alpha\)-\(\beta\)-closed in Y.

**Definition 5.12** If \(f\) is bijective mapping and \(f^{-1} : (Y, \sigma) \to (X, \tau)\) is \(\alpha\)-\(\beta\)-continuous , then f is \(\alpha\)-\(\beta\)-closed.

**Proof:** Follows from the Definitions 6.1[2], 5.1 and 5.5.

**Theorem 5.13** Suppose that f is \(\alpha\)-\(\gamma\),\(\beta\)-continuous mapping and A is \(\alpha\)-\(\gamma\),\(\beta\)-closed. Then

(i) For every \(\alpha\)-\(\gamma\) g-closed set A of \((X, \tau)\) the image f(A) is \(\alpha\)-\(\beta\) g-closed.

(ii) For every \(\alpha\)-\(\beta\) g-closed set B of \((Y, \sigma)\), the set \(f^{-1}(B)\) is \(\alpha\)-\(\gamma\) g-closed.
Proof: (i) Let V be any $\alpha$-$\beta$-open set in Y such that $f(A) \subseteq V$. By using Theorem 3.3 $f^{-1}(V)$ is a $\alpha$-$\gamma$-open set containing A. Therefore by assumption we have $\tau_{\alpha-\gamma} - cl(A) \subseteq f^{-1}(V)$, so $f((\tau_{\alpha-\gamma} - cl(A))) \subseteq V$. Since f is $\alpha$-$(\gamma, \beta)$-closed, $f((\tau_{\alpha-\gamma} - cl(A)))$ is a $\alpha$-$\beta$-closed set containing f(A), implies that $\sigma_{\alpha-\beta} - cl(f(A)) \subseteq \sigma_{\alpha-\beta} - cl(f(f_{\alpha-\gamma} - cl(A)))) = f((\tau_{\alpha-\gamma} - cl(A))) \subseteq V$. Hence f(A) is $\alpha$-$\beta$ g-closed.

(ii) Let U be a $\alpha$-$\gamma$-open set of (X, $\tau$) such that $f^{-1}(B) \subseteq U$ for any subset B in Y. Put $F = \tau_{\alpha-\gamma} - cl(f^{-1}(B)) \cap (X - U)$. It follows from remark 3.23 (ii) [8] and Theorem 3.4 [9] that F is $\alpha$-$\gamma$-closed set A in (X, $\tau$). Since f is $\alpha$-$(\gamma, \beta)$-closed, f(F) is $\alpha$-$\beta$-closed in (Y, $\sigma$). By using Theorem 4.8 [9], Theorem 3.4 (ii) and the following inclusion $f(F) \subseteq \sigma_{\alpha-\beta} - cl(B) - B$, it is obtained that $f(F) = \phi$, and hence $F = \phi$. This implies that $\tau_{\alpha-\gamma} - cl(f^{-1}(B)) \subseteq U$. Therefore $f^{-1}(B)$ is $\alpha$-$\gamma$ g-closed.

**Theorem 5.14** Let f : (X, $\tau$) → (Y, $\sigma$) is $\alpha$-($\gamma, \beta$)-continuous and $\alpha$-($\gamma, \beta$) closed. Then

(i) If f is injective and (Y, $\sigma$) is $\alpha$-$\beta$-$T_{\frac{1}{2}}$ then (X, $\tau$) is $\alpha$-$\gamma$-$T_{\frac{1}{2}}$ space.

(ii) If f is surjective and (X, $\tau$) is $\alpha$-$\gamma$-$T_{\frac{1}{2}}$ then (Y, $\sigma$) is $\alpha$-$\beta$-$T_{\frac{1}{2}}$ space.

**Proof:** (i) Let A is $\alpha$-$\gamma$ g-closed set in (X, $\tau$). Then by Theorem 5.13 (i) f(A) is $\alpha$-$\beta$ g-closed. Therefore by assumption A is $\alpha$-$\gamma$-closed in (X, $\tau$). Therefore (X, $\tau$) is $\alpha$-$\gamma$-$T_{\frac{1}{2}}$ space.

(ii) Let B be $\alpha$-$\beta$ g-closed set in (Y, $\sigma$). Then it follows from Theorem 5.13 (ii) and the assumption that $f^{-1}(B)$ is $\alpha$-$\gamma$-closed. Hence f is $\alpha$-($\gamma, \beta$)-closed map, implies that $f(f^{-1}(B)) = B$ is $\alpha$-$\gamma$-closed in (Y, $\sigma$). Therefore (Y, $\sigma$) is $\alpha$-$\beta$-$T_{\frac{1}{2}}$.

**Theorem 5.15** Let f : (X, $\tau$) → (Y, $\sigma$) is $\alpha$-($\gamma, \beta$)-homeomorphism. If (X, $\tau$) is $\alpha$-$\gamma$-$T_{\frac{1}{2}}$ then (Y, $\sigma$) is $\alpha$-$\beta$-$T_{\frac{1}{2}}$.

**Proof:** Let {y} be a singleton set of (Y, $\sigma$). Then there exists a point x of X such that y = f(x). By Theorem 4.10[9], it follows that the singleton set {y} is $\alpha$-$\beta$-open or $\alpha$-$\beta$-closed. Therefore (Y, $\sigma$) is $\alpha$-$\beta$-$T_{\frac{1}{2}}$ space.

**Theorem 5.16** Let f : (X, $\tau$) → (Y, $\sigma$) is $\alpha$-($\gamma, \beta$)-continuous, injective mapping. If (Y, $\sigma$) is $\alpha$-$\beta$-$T_1$ space (respectively $\alpha$-$\beta$-$T_2$) then (X, $\tau$) is $\alpha$-$\gamma$-$T_1$ space (respectively $\alpha$-$\gamma$-$T_2$).

**Proof:** Suppose (Y, $\sigma$) is $\alpha$-$\beta$-$T_2$ space and x, y be two distinct points in X. Then there exists two $\alpha$-$\beta$-open sets V and W of Y such that f(x) ∈ V and f(y) ∈ W and $V \cap W = \phi$. Since, f is $\alpha$-($\gamma, \beta$)-continuous for V and W there exists two $\alpha$-$\gamma$-open sets U and S such that x ∈ U and y ∈ S and f(U) ⊆ V and f(S) ⊆ W. Therefore $U \cap S = \phi$. Hence (X, $\tau$) is $\alpha$-$\gamma$-$T_2$ space. The proof of the case $\alpha$-$\gamma$-$T_1$ is proved similarly.

**Definition 5.17** If $\gamma : \tau \rightarrow P(X)$ is a regular operation then X is a $\alpha$-$\gamma$-$T_{\frac{1}{2}}$ space.
Proof: By proposition 2.9 [7], we have $(X, \tau_\gamma)$ is a topological space. To prove $X$ is $\alpha$-$\gamma$-$T_2$ space, it is enough to show that $\{x\}$ is $\alpha$-$\gamma$-open or $\alpha$-$\gamma$-closed.

Case (i): Suppose $\{x\} \in \tau_\gamma$, then by Theorem 3.17[9] $\{x\}$ is $\alpha$-$\gamma$-open.

Case (ii): Suppose $\{x\} \notin \tau_\gamma$, then $\tau_\gamma - \text{int}(\tau_\gamma - \text{cl}(\tau_\gamma - \text{int}({\{x\}}))) = \tau_\gamma - \text{cl}(\phi) = \phi \subseteq \{x\}$. Hence $\{x\}$ is $\alpha$-$\gamma$-closed.

Definition 5.18 Let $(X, \tau)$ be a topological space and $\gamma$ be an operation on $\tau$. Then a subset $A$ of $X$ is said to be $\alpha$-$\gamma$ generalized open set ($\alpha$-$\gamma$-g-open set) if $F \subseteq \tau_\gamma - \text{int}(A)$ whenever $F \subseteq A$ and $F$ is $\alpha$-$\gamma$-closed in $(X, \tau)$. A subset $A$ of $X$ is said to be $\alpha$-$\gamma$ g-closed if $X - A$ is $\alpha$-$\gamma$ g-open.

The family of all $\alpha$-$\gamma$ generalized open set in $(X, \tau)$ is denoted by $\tau_{\alpha-\gamma}$-g-open set and the family of all $\alpha$-$\gamma$ generalized closed set in $(X, \tau)$ is denoted by $\tau_{\alpha-\gamma}$-g-closed set.

Remark 5.19 The union of two disjoint $\alpha$-$\gamma$ g-closed set need not be a $\alpha$-$\gamma$ g-closed set.

Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$, define an operation $\gamma$ on $\tau$ such that

$$A^\gamma = \begin{cases} \text{cl}(A) & \text{if } b \in A \\ A & \text{if } b \notin A \end{cases}$$

then $\tau_{\alpha-\gamma} = \{\phi, X, \{a\}, \{b\}, \{a, c\}, \{b, c\}\}$, $\alpha$-$\gamma$ g-closed set = $\{\phi, X, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$. $A = \{a\}$ and $B = \{b\}$ are $\alpha$-$\gamma$ g-closed sets but $A \cup B = \{a, b\}$ is not a $\alpha$-$\gamma$ g-closed set.

6 γ-g α-open sets

Definition 6.1 Let $(X, \tau)$ be a topological space and $\gamma$ be an operation on $\tau$. Then a subset $A$ of $X$ is said to be $\gamma$ generalized $\alpha$-open set ($\gamma$-g $\alpha$-open set) if $F \subseteq \tau_{\alpha-\gamma} - \text{int}(A)$ whenever $F \subseteq A$ and $F$ is $\gamma$-closed in $(X, \tau)$. A subset $A$ of $X$ is said to be $\gamma$-g $\alpha$-closed if $X - A$ is $\gamma$-g $\alpha$-open.

Theorem 6.2 Let $(X, \tau)$ be a topological space and $\gamma$ be an operation on $\tau$. Then a subset $A$ of $X$ is said to be $\gamma$-g $\alpha$-closed set if and only if $\tau_{\alpha-\gamma} - \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\gamma$-open in $(X, \tau)$.

Proof: Proof follows from the Definition 6.1 and the results $\tau_{\alpha-\gamma} - \text{int}(A) = X - \tau_{\alpha-\gamma} - \text{cl}(A)$, $\tau_{\alpha-\gamma} - \text{cl}(A) = X - \tau_{\alpha-\gamma} - \text{int}(A)$

Remark 6.3 From the Definitions 4.6 [9], 5.18 and 6.1 have the following digrammatic implications:
Remark 6.4 The union of two disjoint $\gamma$-g $\alpha$-closed sets need not be a $\gamma$-g $\alpha$-closed set.

Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$, define an operation $\gamma$ on $\tau$ such that

$$A^\gamma = \begin{cases} cl(A) & \text{if } b \notin A \\ A & \text{if } b \in A \end{cases}$$

Then $\tau_{\alpha-\gamma} = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. $\alpha$-g closed set = $\{\phi, X, \{b\}, \{c\}, \{a, c\}\}$. $A = \{b\}$ and $B = \{c\}$ are $\alpha$-g closed sets. But $A \cup B = \{b, c\}$ is not a $\alpha$-g closed set.

Theorem 6.5 Let $(X, \tau)$ be a topological space and $\gamma$ be an operation on $\tau$. If A is $\gamma$-open and $\gamma$-g $\alpha$-closed set in $(X, \tau)$, then A is $\alpha$-$\gamma$-closed.

Proof: Since A is $\gamma$-open and $\gamma$-g $\alpha$-closed , $\tau_{\alpha-\gamma} - cl(A) \subseteq A$ and hence $\tau_{\alpha-\gamma} - cl(A) = A$. This implies that A is $\alpha$-$\gamma$-closed.

Theorem 6.6 Let $(X, \tau)$ be a topological space and $\gamma$ be an operation on $\tau$. If A is $\gamma$-g $\alpha$-closed set in $(X, \tau)$, then $\tau_{\alpha-\gamma} - cl(A) - A$ does not contain any nonempty $\gamma$-closed set.

Proof: Let F be a $\gamma$-closed sub set of $\tau_{\alpha-\gamma} - cl(A) - A$. This implies that $A \subseteq (X - F)$. Since A is $\gamma$-g $\alpha$-closed and $X - F$ is $\gamma$-open, implies $\tau_{\alpha-\gamma} - cl(A) \subseteq (X - F)$. Therefore we have $F \subseteq (X - \tau_{\alpha-\gamma} - cl(A)) \cap (\tau_{\alpha-\gamma} - cl(A)) = \phi$. Hence $F = \phi$.

Theorem 6.7 Let $(X, \tau)$ be a topological space and $\gamma$ be an operation on $\tau$. Then for each $x \in X$, $\{x\}$ is $\gamma$-closed or $X - \{x\}$ is $\gamma$-g $\alpha$-closed in $(X, \tau)$.
Proof: Suppose \( \{x\} \) is not \( \gamma \)-closed. Then \( X - \{x\} \) is not a \( \gamma \)-open set. Therefore \( X \) is the only \( \gamma \)-open set containing \( X - \{x\} \). Hence we have \( \tau_{\alpha - \gamma} - cl(X - \{x\}) \subseteq X \). This implies \( X - \{x\} \) is \( \gamma \)-\( \alpha \)-closed.

**Theorem 6.8** Let \((X, \tau)\) be a topological space and \( \gamma \) be a regular operation on \( \tau \). Then the following are equivalent:

(i) Every \( \gamma \)-\( \alpha \)-closed set of \((X, \tau)\) is \( \alpha \)-\( \gamma \)-closed.

(ii) For each \( x \in X \), \( \{x\} \) is \( \gamma \)-closed or \( \alpha \)-\( \gamma \)-open in \((X, \tau)\).

(iii) \((X, \tau)\) is \( \alpha \)-\( \gamma \)-\( T_{1/2} \)-space.

**Proof:**

(i) \( \rightarrow \) (ii) Suppose that for \( x \in X \), \( \{x\} \) is not \( \gamma \)-closed. By Theorem 6.7, \( X - \{x\} \) is a \( \gamma \)-\( \alpha \)-closed set. Therefore by assumption \( X - \{x\} \) is \( \alpha \)-\( \gamma \)-closed. Hence \( \{x\} \) is \( \alpha \)-\( \gamma \)-open.

(ii) \( \rightarrow \) (iii) By Theorem 6.7 \( X - \{x\} \) is \( \gamma \)-\( \alpha \)-closed, using Theorem 4.6 [8] and 4.10[8], \((X, \tau)\) is \( \alpha \)-\( \gamma \)-\( T_{1/2} \)-space.

(iii) \( \rightarrow \) (i) By Theorem 6.5.

**Definition 6.9** A topological space \((X, \tau)\) is said to be \( \gamma \)-\( \alpha \) \( T_b \) space (respectively \( \gamma \)-\( \alpha \) \( T_d \) space) if every \( \gamma \)-\( \alpha \)-closed set is \( \gamma \)-closed (respectively \( \gamma \)-\( g \)-closed).

**Theorem 6.10**

(i) If \((X, \tau)\) is \( \gamma \)-\( \alpha \) \( T_b \), then for each \( x \in X \), \( \{x\} \) is \( \alpha \)-\( \gamma \)-closed or \( \gamma \)-open.

(ii) If \((X, \tau)\) is \( \gamma \)-\( \alpha \) \( T_d \), then for each \( x \in X \), \( \{x\} \) is \( \gamma \)-closed or \( \gamma \)-\( g \)-open.

**Proof:**

(i) Suppose that for \( x \in X \), \( \{x\} \) is not \( \alpha \)-\( \gamma \)-closed, then by Theorem 6.7, \( X - \{x\} \) is \( \gamma \)-\( \alpha \)-closed. Therefore, by assumption \( X - \{x\} \) is \( \gamma \)-closed. Hence \( \{x\} \) is \( \gamma \)-open.

(ii) Suppose that, for \( x \in X \), \( \{x\} \) is not \( \gamma \)-closed. Then by Theorem 6.7 and by the assumption it follows that \( X - \{x\} \) is \( \gamma \)-\( \alpha \)-closed and \( X - \{x\} \) is \( \gamma \)-g-closed. Hence \( \{x\} \) is \( \gamma \)-g-open.

**Remark 6.11** Let \((X, \tau)\) be a topological space and \( \gamma \) be a regular operation on \( \tau \). Then every \( \gamma \)-\( \alpha \) \( T_b \) space is \( \gamma \)-\( \alpha \) \( T_d \) and \( \alpha \)-\( \gamma \)-\( T_{1/2} \)-space. However the converse need not be true.

**Proof:** Proof follows from the Definition 6.9 and Theorem 6.10.

Let \( X = \{a, b, c\} \), \( \tau = \{\varnothing, X, \{b\}, \{c\}, \{a, c\}, \{b, c\}\} \), define an operation \( \gamma \) on \( \tau \) such that

\[
A^\gamma = \begin{cases} 
cl(A) & \text{if } b \notin A \\
A & \text{if } b \in A 
\end{cases}
\]

Then \( \tau_\gamma = \{\varnothing, X, \{b\}, \{a, c\}, \{b, c\}\} \), \( \tau_{\alpha - \gamma} = \{\varnothing, X, \{b\}, \{a, c\}, \{b, c\}\} \). \((X, \tau)\) is a \( \alpha \)-\( \gamma \)-\( T_{1/2} \)-
space but not a $\gamma$-$\alpha$ $T_b$ and $\gamma$-$\alpha$ $T_d$ space.

**Remark 6.12** From the Definition 4.6[8], Definition 6.9 and the Remark 6.11, we have the following diagram implications:

![Diagram]

A $\longrightarrow$ B represents $A$ implies $B$ and
A $\longrightarrow$ B represents $A$ does not imply $B$. $\gamma$ is a regular operation on $\tau$.

**Definition 6.13** A topological space $(X, \tau)$ is called $\gamma$-$T_{g\alpha}$ space if for every $\gamma$-$g$ $\alpha$-closed set is $\alpha$-$\gamma$ $g$-closed.

**Remark 6.14** Let $(X, \tau)$ be a topological space and be $\gamma$ a regular operation on $\tau$. Then by Definitions 5.18, 6.1 and 6.13, every $\alpha$-$\gamma$-$T_{\frac{1}{2}}$-space is $\gamma$-$T_{g\alpha}$ space.

**Theorem 6.15** If $f : (X, \tau) \to (Y, \sigma)$ is $(\gamma, \beta)$ continuous and $\alpha$-$(\gamma, \beta)$-closed, then for every $\gamma$-$g$ $\alpha$-closed set $B$ of $(X, \tau)$, $f(B)$ is $\beta$-$g$ $\alpha$-closed in $(Y, \sigma)$.

*Proof:* Let $A$ be a $\beta$-open set such that $f(B) \subseteq U$. Then $B \subseteq f^{-1}(U)$. Since $f$ is $(\gamma, \beta)$-continuous and $B$ is $\gamma$-$g$ $\alpha$-closed set, implies $\tau_{\alpha, \gamma} - cl(B) \subseteq f^{-1}(U)$ and hence $f(\tau_{\alpha, \gamma} - cl(B)) \subseteq U$. Therefore it follows from the assumption that $\tau_{\alpha, \gamma} - cl(f(B)) \subseteq f(\tau_{\alpha, \gamma} - cl(B)) \subseteq U$. Hence $f(B)$ is $\beta$-$g$ $\alpha$-closed in $(Y, \sigma)$.

**Theorem 6.16** Let $f : (X, \tau) \to (Y, \sigma)$ is $\alpha$-$(\gamma, \beta)$-continuous and $\alpha$-$(\gamma, \beta)$-closed, then for every $\beta$-$g$ $\alpha$-closed set $A$ of $(Y, \sigma)$, $f^{-1}(A)$ is $\gamma$-$g$ $\alpha$-closed in $(X, \tau)$.

*Proof:* Let $A$ be a $\beta$-$g$ $\alpha$-closed set in $(Y, \sigma)$. Let $U$ be a $\gamma$-open set such that $f^{-1}(A) \subseteq U$. Since $f$ is $\alpha$-$(\gamma, \beta)$-continuous, $f(\tau_{\alpha, \gamma} - cl(f^{-1}(A))) \cap (X - U) \subseteq f(\tau_{\alpha, \gamma} - cl(f^{-1}(A))) \cap f(X - U) \subseteq \tau_{\alpha, \gamma} - cl(f^{-1}(A)) \cap (X - A) \subseteq \sigma_{\alpha, \gamma} - cl(A) - A$. Since $f$ is $\alpha$-$(\gamma$-$\beta)$-closed and $\sigma_{\alpha, \gamma} -$
\[ cl(f^{-1}(A)) \cap (X-U) \] is a \( \alpha - \gamma \)-closed set, implies \( \sigma_{\alpha-\beta} - cl(A) - A \) contains a \( \alpha-\beta \)-closed set \[ f(\tau_{\alpha-\gamma}-cl(f^{-1}(A))) \cap (X-U) \]. Hence by Theorem 4.7 [8] \[ f(\tau_{\alpha-\gamma}-cl(f^{-1}(A))) \cap (X-U) = \phi \). This implies that \( f(\tau_{\alpha-\gamma}-cl(f^{-1}(A))) \cap (X-U) = \phi \), hence \( \tau_{\alpha-\gamma} - cl(f^{-1}(A)) \subseteq U \). Therefore, \( f^{-1}(A) \) is \( \gamma-g \) \( \alpha \)-closed.

**Theorem 6.17** Let \( f : (X, \tau) \to (Y, \sigma) \) is a \( \alpha-(\gamma, \beta) \)-homeomorphism. If \( (X, \tau) \) is a \( \gamma-T_{\gamma \alpha} \) space then \( (Y, \sigma) \) is \( \beta-T_{\gamma \alpha} \) space.

**Proof:** Let \( F \) be a \( \beta-g \) \( \alpha \)-closed set in \( (Y, \sigma) \). Then by assumption and Theorem 6.16 we have \( f^{-1}(F) \) is \( \gamma-g \) \( \alpha \)-closed. Then by Theorem 6.15 \( f(f^{-1}(F)) = F \) is \( \beta-g \) \( \alpha \)-closed. Therefore \( (Y, \sigma) \) is \( \beta-T_{\gamma \alpha} \) space.

**Theorem 6.18** If \( (X, \tau) \) is \( \gamma-\alpha T_{b} \), \( f : (X, \tau) \to (Y, \sigma) \) is a \( \alpha-(\gamma, \beta) \)-homeomorphism and \( (\gamma, \beta) \)-closed map, then \( (Y, \sigma) \) is \( \beta-\alpha T_{b} \).

**Proof:** Let \( F \) be a \( \gamma-g \) \( \alpha \)-closed set in \( (Y, \sigma) \) then by Theorem 6.16 \( f^{-1}(F) \) is \( \gamma-g \) \( \alpha \)-closed in \( (X, \tau) \). Since \( (X, \tau) \) is \( \gamma-\alpha T_{b} \) and \( f \) is \( (\gamma, \beta) \)-closed, implies that \( f^{-1}(F) \) is \( \gamma \)-closed and hence \( f^{-1}(F) = F \) is \( \beta \)-closed in \( (Y, \sigma) \). Therefore \( (Y, \sigma) \) is \( \beta-\alpha T_{b} \).

**References**


[2] N.Kalaivani and G.Sai sundara Krishnan, Operation Approaches On (\( \alpha-\gamma, \beta \))-Continuous Mappings and \( \alpha-(\gamma, \beta) \)-Continuous Mappings(Submitted).


