Open Distance Pattern Coloring of Certain Classes of Graphs

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Abstract—Let $G$ be a connected graph with diameter $d(G)$, $X = \{1,2,3,...,d(G)\}$ be a non-empty set of colors of cardinality $d(G)$, and let $\phi \neq M \subseteq V(G)$. Let $f_M^{V}$ be an assignment of subsets of $X$ to the vertices of $G$ such that $f_M^{V}(u) = \{d(u,v), v \in M, u \neq v\}$, where $d(u,v)$ is the distance between $u$ and $v$. We call $f_M^{V}$ an $M$-open distance pattern coloring of $G$ if no two adjacent vertices have the same $f_M^{V}$ and if such an $M$ exists for a graph $G$, then $G$ is called an open distance pattern colorable (odpc) graph; the minimum cardinality of such an $M$ if it exists, is the open distance pattern coloring number of $G$ denoted by $\eta_M(G)$. In this paper, we study open distance pattern coloring of certain classes of graphs.

Index Terms — distance pattern coloring, open distance pattern of vertices, coloring, bipartite graphs, chain graphs, triangular snake, quadrilateral snake,

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1 INTRODUCTION

For all terms and definitions, not defined specifically in this paper, we refer to [10] and for more about graph labeling, we refer to [12]. Unless mentioned otherwise, all graphs considered here are simple, finite and connected. Let $G$ be a $(p,q)$-graph and let $X$, $Y$ and $Z$ be non-empty sets and $2^X, 2^Y$ and $2^Z$ be their power sets. Then, the functions $f : V(G) \rightarrow 2^X$ and $f : E(G) \rightarrow 2^Y$ are called the set assignments of vertices, edges and elements of $G$ respectively. By a set-assignment of a graph, we mean any one of them. A set-assignment $f : V(G) \rightarrow 2^X$ is called a set-labeling or a set-valuation if it is injective. A proper coloring of a graph $G$ is a function from the vertices of $G$ to a set of colors such that no two adjacent vertices have the same color. The chromatic number of a graph $G$ is the minimum number of colors required in its proper coloring. Graph coloring has been used as a model in many practical problems and has played a vital role in the development of graph theory. Using the concepts of graph coloring, distances in graphs and set-labeling of graphs, we defined the following in [7].

Definition 1.1 [6] Given a connected graph $G(V,E)$ of diameter $d(G)$, $\phi \neq M \subseteq V(G)$. Let $X = \{1,2,3,...,d(G)\}$ be nonempty set of colors of $G$ with cardinality $d(G)$. Let be $f_M^{V}$ an assignment of subsets of $X$ to the vertices of $G$ such that $f_M^{V}(u) = \{d(u,v), v \in M, u \neq v\}$, where $d(u,v)$ is the usual distance between $u$ and $v$. We call $f_M^{V}$ an $M$-open distance pattern coloring of $G$, if no two adjacent vertices have same $f_M^{V}$ and if such an $M$ exists for a graph $G$, then $G$ is called an open distance pattern colorable graph. An open distance pattern colorable graph is usually written in short as an odpc-graph. The minimum cardinality of such a set $M$, if it exists, is said to the open distance pattern coloring number (odpc-number, in short) of $G$, denoted by $\eta_M(G)$.

It has been proved, in [6], that for any graph $G$ $\eta_M(G) \geq 2$. Further, the following theorem has been proved in [6].

Theorem 1.2. [6] Every connected bipartite graphs are open distance pattern colorable.

In this paper, we study open distance pattern coloring of certain classes of graphs.

2 Main Results:

The graph obtained by identifying the end points of $b$ internally disjoint paths, each of length $a$, is denoted in [4], by $P_{a,b}$. The following proposition establishes the open distance pattern colorability of this graph class.

Proposition 2.1 $P_{a,b}$ is open distance pattern colorable.

Proof. Let the end points of $b$ internally disjoint paths of length $a$ are identified at $u$ and $v$. Hence, any cycle in the graph $P_{a,b}$ is of length $2a$. That is, the length of any cycle in $P_{a,b}$ is even and hence it is a bipartite graph. Therefore, by Theorem 1.2, is $o P_{a,b}$ open distance pattern colorable.
Theorem 2.2. The graph $G$ isomorphic to $n$ cycles $C_m$ all of which have one edge in common is odpc if and only if $m \geq 4$.

Proof. Let $V(G) = \{u_{ij}, 1 \leq i \leq n, 1 \leq j \leq m\}$ be the vertex set of $G$, where $u_{ij}$ is the vertex set of the $i$th copy of $C_m$.

Assume that $m \geq 4$. Then, we have the following cases.

Case 1: $m$ even. $G$ is bipartite. By theorem 1.2, $G$ is odpc.

Case 2: $m$ odd. Choose $M = \{u_{ij}, u_{ij+1/2}, u_{ij+1/2}\}$. Then

\[ f_m^o(u) = \left\lfloor \frac{m}{2} \right\rfloor, \quad f_m^o(v) = \left\lfloor \frac{m}{2} \right\rfloor. \]

Also

\[ f_m^o(u_{ij+1/2}) = f_m^o(u_{ij+1/2}) = \left\lfloor \frac{m}{2} \right\rfloor. \]

For $i = r, s$, $f_m^o(u_{ij}) = \left\lfloor \frac{m}{2} \right\rfloor + 1, i \neq j$,

\[ f_m^o(u_{ij}) = \left\lfloor \frac{m}{2} \right\rfloor + 1, i \neq j. \]

From all the above cases, it is evident that no two adjacent vertices of $G$ have the same $f_m^o$ value. Hence, $G$ is open distance pattern colorable.

Conversely, assume that $G$ is open distance pattern colorable. If possible, let $m = 3$.

Then $G \cong K_2 + K_1$. Let $V(K_2) = \{v_1, v_2\}$. $V(K_n) = \{v_1, v_2, ..., v_{m+1}, v_{m+2}\}$.

If we choose any number of vertices of $G$ to $M$, $f_m^o(v_1) = f_m^o(v_2) = 1$.

Therefore, $G$ is not open distance pattern colorable. This completes the proof.

Figure 1 depicts odpc labeling of 4 copies of $C_6$ which have one edge in common. The vertices in $M$ are represented by white circles in the figure.

Another interesting graph structure is the path union of a given graph $G$, which is defined as follows.

Definition 2.3. [12] Let $G_1, G_2, G_3, ..., G_n$ be $n$ copies of a given graph $G$. The graph obtained by adding an edge from $G_i$ to $G_{i+1}$ for all $i = 1, 2, ..., n-1$ is called the path union of $G$.

We now proceed to verify the open distance pattern colorability of the path union of $G$ in the following theorem.

Theorem 2.4. Let $G$ be the path union of $m$ copies ($m \geq 2$) of cycle $C_n$. Then, $G$ is open pattern distance colorable except when $m$ is even and $n = 3$.

Proof. Let $G$ be the path-union of $m$ copies of the cycle $C_n$. Consider the following cases.

Case 1: $n$ even. Then, $G$ can be considered as the union of even cycles and hence is a bipartite graph. Therefore, by Theorem 1.2, $G$ is open distance pattern colorable.

Case 2: $n$ odd. Here we have the following subcases.

Subcase 2.1: $m$ odd. Choose the set $M = \{v_1, v_2, ..., v_{m+2}\}$. For $i = 1, 2, 3, ..., m+2$,

\[ f_m^o(v_i) = f_m^o(v_i) = \left\lfloor \frac{m}{2} \right\rfloor. \]

For the vertex $v_{m+2}$, there are two adjacent vertices at distance diameter of the cycle. These two vertices have identical element $\left\lfloor \frac{m}{2} \right\rfloor$ in their $f_m^o$ value. By considering distance from these two vertices to other elements in $M$, they differ by 1.

Hence adjacent vertices have distinct $f_m^o$.

Subcase 2.2: $m$ is even and $n = 3$. Assume that $n = 3$. No vertices of the form $v_i$ can be an element of $M$, since if it is so, the vertices $v_{i+2}, v_{i+3}$ have $f_m^o(v_{i+2}) = f_m^o(v_{i+3})$ for any $i$. If we take any number of vertices of the form $v_i, j \neq 1$, then the vertices $v_{m/3}$ and $v_{m/3+1}$ have the same distance pattern. Therefore, $G$ is not open distance pattern colorable if $n = 3$.

Subcase 2.3: $m$ is even and $n \geq 5$. In this case, choose $M = \{v_1, v_{m/2}, v_{m/3}, v_{m/3+1}, ..., v_{m/3+k}, v_{m/3+k+1}\}$. Then, for $i = 1, 2, 3, ..., m$.

(a) If $i$ is odd, then 2 is an element of $f_m^o(v_i)$, but 1 is not in $f_m^o(v_i)$. Moreover, two vertices $v_i$ are equidistant from $v_{i+1}$ and distance of these vertices from other elements in $M$ differ by 1. Hence adjacent vertices have distinct $f_m^o$.

(b) If $i$ is even, then 1 is an element of $f_m^o(v_i)$, but 2 is not in $f_m^o(v_i)$. Moreover, $v_i$ for $j = \lfloor m/2 \rfloor + 1$ are equidistant from $v_{i+1}$ and distance of these vertices from other elements in $M$ differ by 1. Hence adjacent vertices have distinct $f_m^o$. This completes the proof.

Definition 2.5. [12] A triangular snake, denoted by $S_{3n}$, is the graph obtained from a path $P_n$ by replacing every edge of it by a cycle $C_3$.

Theorem 2.6. A triangular snake $S_{3n}$ is open distance pattern colorable if $n \neq 3$.

Proof. Let $P_n = u_0 u_1 u_2 ... u_n$. For $0 \leq i \leq n - 1$ the triangular snake $S_{3n}$ is the graph obtained by replacing every edge of $P_n$ by the triangle $u_{i-1} u_i u_{i+1}$. We prove the theorem in
three cases.

Case 1: When \( n = 2 \)
Choose \( M = \{ u_0, u_2 \} \). Then \( f_M^0(u_0) = \{ 1 \} \) and \( f_M^0(u_2) = \{ 1, 2 \} \). When \( n \geq 4 \). For the choice of \( M = \{ u_0, u_2, u_n \} \), we have 
\[
\begin{align*}
& f_M^0(u_0) = \{ 2, d(G) \}, \\
& f_M^0(u_1) = \{ 1, d(G) - 1 \}, \\
& f_M^0(u_2) = \{ 1, 2, d(G) - 1 \}, \text{ and for } \quad 3 \leq i \leq n,
\end{align*}
\]
With this choice of \( M, S_{2n} \neq 3 \) is odpc.

Case 3: If possible let \( n = 3, S_{2n} \neq 3 \) is odpc. Label the vertices as shown in Figure 2.

If neither \( u_0 \) nor \( u_1 \) is in \( M \), then \( f_M^0(u_0) = f_M^0(u_1) \). Hence, either \( u_0 \) or \( u_1 \) must be an element of \( M \). By the same argument we see that either \( u_2 \) or \( u_n \) belongs to \( M \). Now, let \( u_0 \) and \( u_2 \) be in \( M \). Then \( f_M^0(u_0) = f_M^0(u_2) = \{ 1, 2 \} \). Irrespective of the case whether \( u_0, u_2 \) are in \( M \) or not. Since \( u_0 \) is at distance 1 from \( u_2 \) and \( u_0 \) is at distance 2 from \( u_n \), and \( u_2 \) is at a distance 1 from \( u_0 \) and \( u_n \), and at distance 2 from \( u_2 \), \( u_n \) and \( u_3 \), and \( f_M^0(u_2) = f_M^0(u_3) = \{ 1, 2 \} \) in all possible cases. Hence \( S_{2n} \neq 3 \) is not odpc.

Analogous to triangular snake, a quadrilateral snake is defined as follows:

Definition 2.7. A quadrilateral snake, denoted by \( ana_{2n} \), is the graph obtained from a path \( P_n \) by replacing every edge of it by a cycle \( C_4 \).

Theorem 2.8. A quadrilateral snake is open distance pattern colorable.

Proof. A quadrilateral snake is a graph that has only cycles of length 4 and hence is bipartite. Therefore, by Theorem 1.2, \( G \) is open distance pattern colorable.

Another interesting graph we consider is a chain graph which is defined as follows:

Definition 2.9. [1] A chain graph is a graph with blocks \( B_1, B_2, B_3, ..., B_k \) such that for every \( i, B_i \) and \( B_{i+1} \) have a common vertex in such a way that the block cut point is a path.

Definition 2.10. [15] A chain graph with \( n \) blocks and the sequence of \( n \) blocks of complete graphs \( (K_{a_1}), (K_{a_2}), ..., (K_{a_n}) \) is called a Husimi Chain and is denoted by \( CH(n; (a_1, a_2, a_3, ..., a_n)) \).

If \( a_1 = a_2 = a_3 = ... = a_n = 2 \), then \( CH(n; (2, 2, 2, ..., 2)) = P_n \), a path of length \( n \geq 3 \) and if \( a_1 = a_2 = a_3 = ... = a_n = 3 \), then \( CH(n; (3, 3, 3, ..., 3)) = S_n \) a triangular snake with \( n \neq 3 \). In both cases, \( G \) is odpc, by Theorem 1.2 and Theorem 2.6 respectively. It is meaningless to say that \( P_n \) has an open distance pattern colorable for \( n \leq 2 \) and we have already proved in Theorem 2.6 that \( S_n \neq 3 \) is not odpc if \( n = 3 \). It remains to verify the other cases.

Theorem 2.11. \( G = CH(n; (a_1, a_2, a_3, ..., a_n)) \) is not an odpc-graph if \( a_i \geq 4 \) for some \( i, 1 \leq i \leq n \).

Proof. For some \( i, 1 \leq i \leq n \) assume that \( a_i \geq 4 \). Let \( u_1, u_2, u_3, ..., u_{a_i} \) be the vertices of the component \( K_{a_i} \). We consider the following cases.

Case 1: If \( K_{a_i} \) is an end component of \( G \), then exactly one vertex of \( K_{a_i} \) is common to another component \( K_{a_j} \) of \( G \). Without loss of generality, let \( u_1 \) be the vertex of \( K_{a_i} \) that is common to the component \( K_{a_j} \). Then, there are the following subcases.

Subcase 1.1: When \( u_2, u_3, ..., u_{a_i} \) are not the elements of \( M \).
In this case, for some positive integer \( k \), let \( i_1, i_2, i_3, ..., i_k \) be the set assignment \( f_M^0 \) of \( M \) with respect to \( M \). If \( u_i \in M \), for all \( 2 \leq r \leq a_i \), 
\[
\begin{align*}
& f_M^0(u_i) = \{ i_1, i_2, i_3, ..., i_k \} \\
& \text{That is, the adjacent vertices } u_i \text{ have the same set assignment for all } 2 \leq r \leq a_i. \text{ Hence if there exists an odpc set for the graph } G, \text{ then necessarily } u_1 \in M. \text{ Then, for positive integer } k, \text{ let } \\
& f_M^0(u_k) = \{ i_1, i_2, i_3, ..., i_k \} \\
& \text{Therefore } f_M^0(u_r) = \{ i_1 + 1, i_2 + 1, i_3 + 1, ..., i_k + 1 \} \quad 2 \leq r \leq a_i.
\end{align*}
\]

Subcase 1.2: When one of \( u_2, u_3, ..., u_{a_i} \) is in \( M \). In this case \( u_i \in M \). Without loss of generality, let \( u_2 \in M \). \( f_M^0(u_2) = \{ u_1, u_2, u_3, ..., u_{a_i} \} \). Then \( f_M^0(u_r) = \{ 1, i_1, i_2, i_3, ..., i_k \} \). If \( u_2 \in M \), let \( M \) be the left Husimi chain and \( K_{a_j} \) be the right Husimi chain of \( CH(n; (a_1, a_2, a_3, ..., a_n)) \). We can adopt the process, same as in case 1, for both the chains.

For \( 1 \leq l \leq a_i \), let \( u_l \) be a vertex of \( K_{a_i} \), which is not common to any other component of \( G \). If \( f_M^0(u_l) = \{ i_1, i_2, i_3, ..., i_k \} \), \( u_l \in K_{a_i} \) with respect to odpc set \( M_i \), in left Husimi chain and if \( f_M^0(u_l) = \{ i_1, i_2, i_3, ..., i_k \} \), \( u_l \in K_{a_i} \) with respect to the odpc set \( M_i \), in the right Husimi chain. Then, \( f_M^0(u_r) \) with respect to the set \( M = M_i \cup M_j = \{ i_1, i_2, i_3, ..., i_k \} \cup \{ j_1, j_2, j_3, ..., j_l \} \). In all these cases \( M \) cannot be an odpc-set.

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References


