Abstract:

A graph \( G = (V, E) \) with \( n \) vertices is said to admit prime labeling if its vertices can be labeled with distinct positive integers not exceeding \( n \) such that the label of each pair of adjacent vertices are relatively prime. A graph \( G \) which admits prime labeling is called a prime graph. And a graph \( G \) is said to be a strongly prime graph if for any vertex \( v \) of \( G \) there exists a prime labeling \( f \) satisfying \( f(v) = 1 \). In this paper we investigate prime labeling for some graphs related to \( H \)-graph, ladder graph, comb graph and also we prove that comb graph is a strongly prime graph.

Keywords: Prime Labeling, Prime Graph, Strongly Prime Graph, \( H \)-Graph, Ladder Graph, Comb Graph.

1. INTRODUCTION:

In this paper, we consider only simple, finite, undirected and non trivial graph \( G = (V(G), E(G)) \) with the vertex set \( V(G) \) and the edge set \( E(G) \). The set of vertices adjacent to a vertex \( u \) of \( G \) is denoted by \( N(u) \). For notations and terminology we refer to Bondy and Murthy [1].

Two integers \( a \) and \( b \) are said to be relatively prime if their greatest common divisor is 1. Relatively prime numbers play an important role in both analytic and algebraic number theory. The notion of a prime labeling was introduced by Roger Entringer and was discussed in a paper by Tout.A (1982 P 365-368) [7]. Many researchers have studied prime graph. For example Fu.H (1994 P 181-186) [3] have proved that path \( P_n \) on \( n \) vertices is a prime graph. Deresky.T (1991 P 359-369) [2] have proved that the \( C_n \) on \( n \) vertices is a prime graph. Lee.S (1998 P 59-67) [5] have proved that wheel \( W_n \) is a prime graph if \( n \) is even.

Around 1980 Roger Entringer conjectured that all trees having prime labeling which is not settled till today.


In [9] S.K.Vaidya and Udayan M.Prajapati have introduced Strongly prime graph and has proved the \( C_n, P_n \) and \( K_{1,n} \) are Strongly prime graphs and \( W_n \) is a Strongly prime graph for every even integer \( n \geq 4 \), in Some New Results On Prime Graph (2012 P 99-104). In [10] R.Vasuki and A.Nagarajan have proved Some Results On Super Mean Graphs Vol.3 (2009), 82-96.For latest Dynamic Survey On Graph Labeling we refer to [4] (Gallian .J.A., 2009). Vast amount of literature is available on different types of graph labeling more than 1000 research papers have been published so far in last four decades.

Definition 1.1:

Let \( G = (V(G), E(G)) \) be a graph with \( p \) vertices. A bijection \( f : V(G) \rightarrow \{1, 2, \ldots, p\} \) is called a prime labeling if for each edge \( e = uv \), \( \gcd(f(u), f(v)) = 1 \). A graph which admits prime labeling is called a prime graph.

Definition 1.2:

A graph \( G \) is said to be a strongly prime graph if for any vertex \( v \) of \( G \) there exists a prime labeling \( f \) satisfying \( f(v) = 1 \).

Definition 1.3:

The \( H \)-graph of a path \( P_n \) is the graph obtained from two copies of \( P_n \) in \( P \) with vertices \( u_1, u_2, \ldots, u_n \) and \( v_1, v_2, \ldots, v_n \) by joining the vertices \( u_{\frac{n+1}{2}} \) and \( v_{\frac{n+1}{2}} \) if \( n \) is odd and the vertices \( u_{\frac{n}{2}} \) and \( v_{\frac{n}{2}} \) if \( n \) is even.

Definition 1.4:

The corona of a graph \( G \) on \( p \) vertices \( v_1, v_2, \ldots, v_p \) is the graph obtained from \( G \) by adding \( p \) new vertices \( u_1, u_2, \ldots, u_p \) and the new edges \( u_i v_i \) for \( 1 \leq i \leq p \) then it is denoted by \( G \odot K_1 \). And \( G + K_1 \) is a graph obtained from \( G \) by adding only one new vertex \( v \) and join every vertex of \( G \) with \( v \), then the new edges are \( v_i u_i, v_i v \) for \( 1 \leq i \leq n \).

Definition 1.5:

The product of two graphs \( P_n \times P_n \) is called a ladder and it is denoted by \( L_n \).
**Definition 1.6:**
The graph \( P_n \sqcup K_1 \) is called a comb \( C_n \). In this paper we investigate prime labeling for some graphs related to \( H \)-graph, ladder graph, comb graph and also we prove that comb graph is a strongly prime graph.

### 2. Prime Labeling of Some Graphs

**Theorem 2.1:**
The \( H \)-graph of path \( P_n \) is a prime graph.

**Proof:**
Let \( H_n \) be the \( H \)-graph with Vertex set is \( \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\} \).

The edge set is \( E(H_n) = \{uv_{i+1}, v_{i+1}/1 \leq i \leq n-1\} \cup \{u_{n+1}v_{n+1}/2\} \)

if \( n \) is odd (or) \( \{u_{n+1}v_{n+1}/2\} \) if \( n \) is even.

Here \( |V(H_n)| = 2n \)

Define a labeling \( f : V(H_n) \rightarrow \{1, 2, \ldots, 2n\} \) by considering the following cases:

**Case (i):** When \( n \) is odd

- \( f(u_i) = i + 1 \) for \( 1 \leq i \leq n \),
- \( f(v_i) = n + i + 1 \) for \( 1 \leq i < n + 1, \frac{n + 1}{2} \),
- \( f(v_i) = n + i \) for \( \frac{n + 1}{2} \leq i \leq n, \)

\[
f\left(\frac{v_{i+1}}{2}\right) = 1,
\]

here \( \gcd(f(u_i), f(u_{i+1})) = 1 \) for \( 1 \leq i \leq n - 1, \)

\( \gcd(f(v_i), f(v_{i+1})) = 1 \) for \( 1 \leq i < \frac{n - 1}{2}, \)

\( \gcd(f(v_i), f(v_{i+1})) = 1 \) for \( \frac{n + 1}{2} < i \leq n, \)

Since they are consecutive integers.

- \( \gcd\left(f\left(\frac{v_{i+1}}{2}\right), f\left(\frac{v_{i+1}}{2}\right)\right) = \gcd\left(f\left(\frac{v_{i+1}}{2}\right), 1\right) = 1, \)
- \( \gcd\left(f\left(\frac{v_{i+1}}{2}\right), f\left(\frac{v_{i+1}}{2}\right)\right) = \gcd\left(1, f\left(\frac{v_{i+1}}{2}\right)\right) = 1, \)
- \( \gcd\left(f\left(\frac{u_{i+1}}{2}\right), f\left(\frac{v_{i+1}}{2}\right)\right) = \gcd\left(f\left(\frac{u_{i+1}}{2}\right), 1\right) = 1, \)

Clearly vertex label are distinct.
Thus labeling defined above gives a prime labeling for a graph \( H_n \) (for \( n \) is odd).

**Case (ii):** When \( n \) is even

- \( f(u_i) = i + 1 \) for \( 1 \leq i \leq n, \)
- \( f(v_i) = n + i + 1 \) for \( 1 \leq i < \frac{n}{2}, \)
- \( f(v_i) = n + i \) for \( \frac{n}{2} < i \leq n, \)

Now \( \gcd(f(u_i), f(u_{i+1})) = 1 \) for \( 1 \leq i \leq n - 1, \)

\( \gcd(f(v_i), f(v_{i+1})) = 1 \) for \( 1 \leq i \leq \frac{n}{2} - 1, \)

Since they are consecutive integers.

- \( \gcd\left(f\left(\frac{v_{i+1}}{2}\right), f\left(\frac{v_{i+1}}{2}\right)\right) = \gcd\left(f\left(\frac{v_{i+1}}{2}\right), 1\right) = 1, \)
- \( \gcd\left(f\left(\frac{v_{i+1}}{2}\right), f\left(\frac{v_{i+1}}{2}\right)\right) = \gcd\left(1, f\left(\frac{v_{i+1}}{2}\right)\right) = 1, \)
- \( \gcd\left(f\left(\frac{v_{i+1}}{2}\right), f\left(\frac{u_{i+1}}{2}\right)\right) = \gcd\left(1, f\left(\frac{u_{i+1}}{2}\right)\right) = 1, \)

\( \gcd(f(v_i), f(v_{i+1})) = 1 \) for \( \frac{n}{2} < i \leq n, \)

Since it is a consecutive integer.
Clearly vertex label are distinct. Thus labeling defined above gives a prime labeling for a graph \( H_n \) (for \( n \) is even).

Thus in both the cases \( f \) admits prime labeling. Hence \( H_n \) becomes a prime graph.
Theorem 2.2:
The graph $G \cong K_1$ is a prime graph where $G$ is a $H$-graph with $n$ vertices.

Proof:
Let $G$ be the graph with vertices $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$. Let $u'_1, u'_2, \ldots, u'_n$ and $v'_1, v'_2, \ldots, v'_n$ be the corresponding new vertices, join $u'_i u'_j$ and $v'_i v'_j$ in $G$. we get the graph $G_1$, i.e., $G \cong K_1$.

Now the vertex set of $G_1$ is

\begin{align*}
\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n, u'_1, u'_2, \ldots, u'_n, v'_1, v'_2, \ldots, v'_n\}
\end{align*}

and the edge set $E(G_1) = \{u_i u_{i+1}, v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{u_i, v_i / 1 \leq i \leq n\}$

\begin{align*}
&\cup \left\{u_i v_{i+1} / \text{if } n \text{ is odd} \right\} \quad \text{(or)} \\
&\cup \left\{u_i v_{i+1} / \text{if } n \text{ is even} \right\}
\end{align*}

Here $|V(G_1)| = 4n$

Define a labeling $f : V(G_1) \rightarrow \{1, 2, \ldots, 4n\}$ by considering the following cases:

Case (i): When $n$ is odd.

\begin{align*}
&f(u_i) = 2i + 1 \\
&f(u_i) = 2i \\
&f(v_i) = 2n + 2i + 1 \\
&f(v_i) = 2n + 2i - 1 \\
&f\left(\frac{v_{i+1}}{2}\right) = 1, \\
&f(v_i) = 2n + 2i \\
\end{align*}

Here $\gcd(f(u_i), f(u_{i+1})) = \gcd(2i + 1, 2i + 3) = 1$ for $1 \leq i \leq n - 1$,

\begin{align*}
\gcd(f(v_i), f(v_{i+1})) &= \gcd(2n + 2i - 1, 2n + 2i + 1) = 1 \\
&\text{for } \frac{n+1}{2} < i < n-1,
\end{align*}

Since they are all consecutive odd numbers.

\begin{align*}
\gcd(f(u_i), f(u_{i+1})) &= \gcd(2i + 1, 2i + 3) = 1 \\
&\text{for } 1 \leq i < \frac{n-1}{2},
\end{align*}

\begin{align*}
\gcd(f(v_i), f(v_{i+1})) &= \gcd(2n + 2i - 1, 2n + 2i + 1) = 1 \\
&\text{for } \frac{n+1}{2} < i < n-1,
\end{align*}

Since they are all consecutive integers.

\begin{align*}
\gcd\left(\left\frac{v_{i+1}}{2}\right, \left\frac{v_{i+1}}{2}\right\right) &= \gcd\left(\left\frac{u_{i+1}}{2}\right, \left\frac{u_{i+1}}{2}\right\right) = 1, \\
\gcd\left(\left\frac{v_{i+1}}{2}\right, \left\frac{u_{i+1}}{2}\right\right) &= \gcd\left(\left\frac{u_{i+1}}{2}\right, \left\frac{u_{i+1}}{2}\right\right) = 1.
\end{align*}

Clearly vertex labels are distinct.

Thus labeling defined above gives a prime labeling for a graph $G_1$, (for $n$ is odd).

Case (ii): When $n$ is even.

If $n$ is even then we join the vertices $u_{n+1}^4$ and $v_{n+1}$. Therefore in the above labeling $f$ defined in case (i) we have to change only these labels $f\left(\frac{v_{i+1}}{2}\right) = 1, \ f(v_i) = 2n + 2i + 1$ for $1 \leq i < \frac{n}{2}$, $f(v_i) = 2n + 2i - 1$ for $\frac{n}{2} < i \leq n$.

Hence $\gcd\left(\left\frac{v_{i+1}}{2}\right, \left\frac{u_{i+1}}{2}\right\right) = \gcd\left(\left\frac{u_{i+1}}{2}\right, \left\frac{u_{i+1}}{2}\right\right) = 1,$

\begin{align*}
\gcd\left(\left\frac{v_{i+1}}{2}\right, \left\frac{v_{i+1}}{2}\right\right) &= \gcd\left(\left\frac{u_{i+1}}{2}\right, \left\frac{v_{i+1}}{2}\right\right) = 1.
\end{align*}
gcd\left( f\left( v_{\frac{n}{2}} \right) , f\left( v_{\frac{n+1}{2}} \right) \right) = gcd\left( 1, f\left( v_{\frac{n+1}{2}} \right) \right) = 1,

Now clearly vertex labels are distinct.

Thus labeling defined above gives a prime labeling for a graph $G_1$ where $n$ is odd. Thus $G_1 \cong K_n$ is a prime graph.

Figure 4: Prime labeling of $G \cong K_4$ where $G = H_6$

Theorem 2.3:
The graph obtained by identifying the central vertex of $K_{1,2}$ at each pendent vertex of a comb $C_m$ is a prime graph.

Proof:
Let $P_n$ be the path $u_1, u_2, \ldots, u_n$. Let $v_i$ be a vertex adjacent to $u_i$, $1 \leq i \leq n$. The resultant graph is comb $C_m$. Let $x_i, w_i, y_i$ be the vertices of $i$'th copy of $K_{1,2}$ with $w_i$ the central vertex. Identify the vertex $w_i$ with $v_i$, $1 \leq i \leq n$. We get the graph $G$ whose vertex set is

$$\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}.$$

The edge set is

$$E(G) = \{u_i v_i, v_i x_i, x_i y_i, y_i / 1 \leq i \leq n\}.$$

Here $\left| V(G) \right| = 4n$

Define a function $f : V(G) \rightarrow \{1, 2, \ldots, 4n\}$ by

$$f(u_i) = 4i - 3 \quad \text{for} \quad 1 \leq i \leq n,$$

$$f(v_i) = 4i - 1 \quad \text{for} \quad 1 \leq i \leq n,$$

$$f(x_i) = 4i - 2 \quad \text{for} \quad 1 \leq i \leq n,$$

$$f(y_i) = 4i \quad \text{for} \quad 1 \leq i \leq n.$$

Here

$$gcd(f(u_i), f(u_{i+1})) = gcd(4i - 3, 4i + 1) = 1 \text{ for } 1 \leq i \leq n - 1,$$

as these two numbers are odd and their difference is 4.

$$gcd(f(v_i), f(v_{i+1})) = gcd(4i - 3, 4i - 1) = 1,$$

as these two numbers are consecutive odd numbers.

$$gcd(f(x_i), f(v_i)) = gcd(4i - 2, 4i - 1) = 1$$

and

$$gcd(f(y_i), f(v_i)) = gcd(4i, 4i - 1) = 1$$

Since both are consecutive positive integers.

Clearly vertex labels are distinct. Thus labeling defined above gives a function $f$ is a prime labeling of $G$. Thus $G$ is a prime graph.

Figure 5: Prime labeling of Comb identifying the central vertex of $K_{1,2}$ at each pendent vertex

Theorem 2.4:
The graph $G \cong K_1$ is a prime graph where $G$ is a Ladder graph with $n$ vertices.

Proof:
Let $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ be the two paths of equal length, join $u_i$ and $v_i$, $1 \leq i \leq n$. The resultant graph is Ladder graph $G$. Add two new vertices $x_i, y_i$ and join these vertices with $u_i$ and $v_i$ respectively, $1 \leq i \leq n$. we get the graph $G_1 \cong K_1$.

Now the vertex set is

$$\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}.$$

The edge set is

$$E(G_1) = \{u_i u_{i+1}, v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{u_i v_i, v_i x_i, x_i y_i, y_i / 1 \leq i \leq n\}.$$

Here $\left| V(G_1) \right| = 4n$

Define a function $f : V(G) \rightarrow \{1, 2, \ldots, 4n\}$ by

$$f(u_i) = 4i - 1 \quad \text{for} \quad 1 \leq i \leq n,$$

$$f(v_i) = 4i - 3 \quad \text{for} \quad 1 \leq i \leq n,$$

$$f(x_i) = 4i \quad \text{for} \quad 1 \leq i \leq n,$$

$$f(y_i) = 4i - 2 \quad \text{for} \quad 1 \leq i \leq n.$$

Here

$$gcd(f(u_i), f(u_{i+1})) = gcd(4i - 1, 4i + 3) = 1 \text{ for } 1 \leq i \leq n - 1,$$

as these two numbers are odd and their difference is 4.

$$gcd(f(v_i), f(v_{i+1})) = gcd(4i - 3, 4i - 1) = 1,$$

Since both are consecutive odd numbers.

$$gcd(f(x_i), f(v_i)) = gcd(4i - 2, 4i - 1) = 1$$

and

$$gcd(f(y_i), f(v_i)) = gcd(4i, 4i - 1) = 1$$

Since both are consecutive numbers.

Clearly vertex labels are distinct. Thus a labeling defined above gives a prime labeling for a graph $G_1$. Thus $G \cong K_1$ is a prime graph.
Theorem 2.5:
The graph $G + K_i$ is a prime graph where $G$ is a H-graph with $n$ vertices.

Proof:
Let $G$ be the graph with vertices $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$. Let $v_0$ be the new vertex, join $v_0 u_i$ and $v_0 v_i$ where $1 \leq i \leq n$ in $G$. We get the graph $G_1$, i.e., $G + K_i$.

Now the vertex set of $G_1$ is $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n, v_0\}$.

The edge set $E(G_1) = \{u_i u_{i+1}, v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{v_0 u_i, v_0 v_i / 1 \leq i \leq n-1\}$

$\cup \left\{ \frac{u_{i+1} v_{i+1}}{2} \text{ if } n \text{ is odd} \right\}$

$\cup \left\{ u_{i+1} v_i \text{ if } n \text{ is even} \right\}$

Here $V | G_1 | = 2n + 1$.

Define a labeling $f : V(G_1) \to \{1, 2, 3, \ldots, 2n+1\}$ by considering the following cases:

Case (i): When $n$ is odd.

$f(v_0) = 1$,

$f(v_i) = \frac{n+1}{2} i + 1$ for $1 \leq i \leq n$,

$f(u_i) = \frac{3n+1}{2} i + 1$ for $1 \leq i \leq \frac{n+1}{2}$,

$f(u_i) = i - \frac{n-3}{2}$ for $\frac{n+1}{2} \leq i \leq n$,

Here $\gcd(f(v_0), f(v_i)) = \gcd(1, f(v_i)) = 1$,

$\gcd(f(v_0), f(u_i)) = \gcd(1, f(u_i)) = 1$,

$\gcd(f(v_i), f(v_{i+1})) = \gcd\left(\frac{n+1}{2} i + 1, \frac{n+1}{2} i + 2\right) = 1$

for $1 \leq i \leq n-1$, $\gcd(f(u_i), f(u_{i+1})) = \gcd\left(\frac{3n+1}{2} i + 1, \frac{3n+1}{2} i + 2\right) = 1$

Case (ii): When $n$ is even.

$f(v_0) = 1$,

$f(u_i) = \frac{3n}{2} i + 1$ for $1 \leq i < \frac{n+1}{2}$,

$f(u_i) = i - \frac{n-2}{2}$ for $\frac{n+1}{2} \leq i \leq n$,

$f(v_i) = \frac{n}{2} i + 1$ for $1 \leq i \leq n$,

Similar to case (i) here also $\gcd(f(u_i), f(v_{i+1})) = 1$,

$\gcd(f(u_i), f(u_{i+1})) = 1$, $\gcd(f(v_0), f(u_i)) = 1$ and

$\gcd\left(\frac{u_{i+1}}{2}, \frac{v_{i+1}}{2}\right) = 1$.

Thus $G + K_i$ is a prime graph.
3. STRONGLY PRIME GRAPHS

Theorem 3.1:
The Comb graph $C_{bn}$ is a strongly prime graph.

Proof:
Let $C_{bn}$ be the Comb graph with vertex set
$$\{v_1, v_2, ..., v_n, v'_1, v'_2, ..., v'_n\}.$$ 
Let $E(C_{bn})$ be the edge set of the comb graph is
$$E(C_{bn}) = \{v_i v'_j / 1 \leq i \leq n \} \cup \{v_{i+1} v'_i / 1 \leq i \leq n-1\}.$$ 
Here $V[C_{bn}] = 2n$, where $n$ is a positive integer.
If $v$ is any arbitrary vertex of $C_{bn}$ then we have the following possibilities.

Case (i): When $v$ is of degree 2,3.
If $v = v_j$ for some $j \in \{1,2,3,...n\}$ then the function
$$f : V(C_{bn}) \rightarrow \{1,2,3,...2n\}$$ 
defined by
$$f(v_i) = \begin{cases} 2n + 2i - 2j + 1 & \text{if } i = 1,2,...,j-1; \\ 2i - j + 1 & \text{if } i = j+1, j+2,...,n, \end{cases}$$
$$f(v_j) = 1,$$
and
$$f(v'_j) = \begin{cases} 2n + 2i - 2j + 2 & \text{if } i = 1,2,...,j-1; \\ 2i - j + 2 & \text{if } i = j+1, j+2,...,n, \end{cases}$$
$$f(v'_j) = 2.$$ 

$$\gcd(f(v_i), f(v_{i+1})) = \gcd(2i - 2j + 1, 2i - 2j + 3) = 1$$
for $1 \leq i \leq n-1$.

Since they are all consecutive odd numbers.
$$\gcd(f(v_i), f(v'_j)) = \gcd(2n + 2i - 2j + 1, 2n + 2i - 2j + 2) = 1$$
for $1 \leq i \leq j-1$.
$$\gcd(f(v_i), f(v'_j)) = \gcd(2i - 2j + 1, 2i - 2j + 2) = 1$$
for $1 \leq i \leq n$.

Since they are consecutive integers.
Clearly vertex label are distinct.
Thus $C_{bn}$ is a prime labeling with $f(v) = f(v_j)=1$. Thus $f$ admits prime labeling as well as it is possible to assign label 1 to any arbitrary vertex of degree 2,3 in $C_{bn}$.

Case (ii): When $v$ is of degree 1.
Let $v = v_j$ for some $j \in \{1,2,3,...n\}$, let $f_2$ be the labeling obtained from $f$ in case (i) by interchanging the labels $f(v_j)$ and $f(v'_j)$ and for all other remaining vertices $f_2(v) = f(v)$. Then the resulting labeling $f_2$ is a prime labeling as well as it is possible to assign label 1 to any arbitrary vertex of $C_{bn}$. Thus from all the cases described above $C_{bn}$ is a strongly prime graph.

4. CONCLUDING REMARKS

The prime numbers and their behavior are of great importance as prime numbers are scattered and there are arbitrarily large gaps in the sequence of prime numbers. If these characteristics are studied in the framework of graph theory then it is more challenging and exciting as well. Here we investigate several results on prime graphs and we prove that comb graph is a strongly prime graph.
REFERENCES


