On Painlevé analysis for some non–linear evolution equations

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Abstract
In this paper, we present explicit Painlevé test for the potential Boussinesq equation, The murray equation, The (2 + 1) Calogero equation, The Rosenau – Hyman equation (RH), Cole – Hopf (CH) equation, The Fornberg – Whitham equation (FW). Some of these equations have shown to possess Painlevé property, therefore, are Painleve integrable while the rest did not pass the test and reasons for that are conjectured.

Keywords: Painlevé analysis method, The potential Boussinesq equation, The murray equation, The (2 + 1) Calogero equation, The Rosenau – Hyman equation (RH), Cole – Hopf (CH) equation, The Fornberg – Whitham equation (FW), integrability, Bäcklund transformation.

Introduction:
Nonlinear partial differential equations (NLPDEs) [1] are widely used to describe complex phenomena in various fields of sciences, especially in physics. Therefore solving nonlinear problems plays an important role in nonlinear sciences. In this direction, many effective methods for determining exact solutions of NLPDEs have been established and developed during the past few decades. Among the various different methods, the Lie symmetry method, also called Lie group method, is one of the most powerful methods to determine solutions of NLPDEs. The fundamental basis of this method is that when a differential equation is invariant under a Lie group of transformations [2–4], a reduction transformation exists. For PDEs with two independent variables, a single group reduction transforms the PDEs into ordinary differential equations (ODEs), which are generally easier to solve. In the recent past there have been considerable developments in symmetry methods for differential equations as is evident by the number of research papers, books and new symbolic software devoted to the subject.

The (2 + 1)–dimensional PKP equation [5]:

\[
\sigma_{tt} + \frac{3}{2} \sigma_t \sigma_{xx} + \frac{1}{4} \sigma_{xxx} + \frac{3}{4} \sigma_{xy} = 0, \tag{1.1}
\]

describes the dynamics of 2–dimensional, small, but finite amplitude waves and solitons in a variety of media, for example, in plasma physics, hydrodynamics and solid–state physics. Eq. (1.1) is also derived in various physical contexts assuming that the wave is moving along x and all changes in y are slower than in the direction of motion [6]. By using various techniques and methods exact traveling wave solutions, linear solitary wave solutions, soliton–like solutions and some numerical solutions were obtained in [7–10]. However, in multifarious real physical backgrounds, nonlinear partial differential equations with variable coefficients often provide more powerful and realistic models than their constant coefficient counterparts when the inhomogeneities of media is considered. So it is of great importance to find exact solutions of NLPDEs with variable coefficients and recently, many authors have researched in this direction [11–16].

The integrability of non–linear partial differential equations (NLPDEs) is an interesting topic in non–linear sciences. Many methods have been established by mathematicians and physicists to study the integrability of NLPDEs. Some of the most important methods and notations, for integrability are the bilinear method [17], the symmetry reductions [18], Bäcklund and Darboux transformations [19], the Painleve analysis method.
and the Lax pairs can be found from the Painlevé analysis [30], the Lax pairs of many Painlevé integrable models have not yet been found [25,26]. Therefore, when saying a model is integrable, we must say under what specific meaning(s). for example, we say a model is Painlevé integrable if the model has the Painlevé property, and a model is Lax or IST [27,28](inverse scattering transformation) integrable if the model has a Lax pair and can be solved by the IST approach.

### Painlevé analysis method:

Painlevé property is a method of investigation for the integrability properties of many NLEEs . If a PDE which has no points such as movable branch, algebraic and logarithmic then is called P–type. An ordinary differential equation (ODE) might still admit movable essential singularities without movable branch points. This method does not identify essential singularities and therefore it provides only necessary conditions for an ODE to be of P–type. Singularity structure analysis admitting the P–property advocated by Ablowitz et al. For ODEs and extended to PDE by Weiss, Tabor and Carnevale (WTC), plays a key role of investigating the integrability properties of many NLEEs. The well–known procedure of WTC requires:

1. The determination of leading orders Laurent series,
2. The identification of powers at which the arbitrary functions can enter into the Laurent series called resonances,
3. Verifying that , at the resonance values , sufficient number of arbitrary functions exist without introducing the movable critical manifold.

According to the WTC method, the general solution of PDE is in the below from

\[
u( x, t ) = \varphi^\alpha(x, t) \sum_{j=0}^{\infty} u_j ( x, t ) \varphi^j ( x, t), \quad (2.1)
\]

where \( \alpha \) is negative integer, \( \varphi( x, t ) = 0 \) is the equation of singular manifold. The functions \( u_j ( j = 0, 1, 2, ... ) \) have to be determined by substitution of expansion (2.1) into the PDE, So PDE becomes

\[
\sum_{j=0}^{\infty} E_j( u_0, ..., u_j, \varphi ) \varphi^{j+q} ( x, t ) = 0,
\]

where \( q \) is some negative constant . \( E_j \) depends on \( \varphi \) only by the derivatives of \( \varphi \). The successive practical steps of Painlevé analysis are the following:

1– Determine the possible leading orders \( \alpha \) by balancing two or more terms of the PDE and expressing that they dominate the other terms.
2– Solve equation \( E_0 = 0 \) for non–zero values of \( u_0 \); this may lead to several solutions, called branches.
3– Find the resonances, i.e. the values of \( j \) for which \( u_j \) cannot be determined from equation \( E_j = 0 \). This last equation has generally the form

\[
E_j = ( j + 1 ) p(j) \varphi_x^j \varphi^{n-i} u_j + Q ( u_0, ..., u_{j-1}, \varphi ) = 0, \quad \forall \ j > 0,
\]

where \( n \) is order of the PDE, \( 0 \leq i \leq n \) and \( p \) is a polynomial of degree \( n-1 \).

The values of the resonances are the zeros of \( p \).
4– Determine whether the resonances are compatible, or not. At resonance , after substitution in (2.3) of the previously computed \( u_i, i \leq j – 1 \), the function \( Q \) is either zero or non–zero then in the case \( u_i \) can be arbitrarily chosen and the expansion (2.2) does not exist for arbitrary \( \varphi \), so the resonance is called compatible.
5– All resonances occur at positive integer values of \( j \) and are compatible.

3. The potential Boussinesq equation :

\[
u_{tt} + u_xu_{xx} + u_{xxxx} = 0,
\]

(3.1)

We first present the Painlevé test of the potential Boussinesq equation. According to the WTC method, the general solution of PDE is in the form

\[
u( x, t ) = \varphi^\alpha(x, t) \sum_{j=0}^{\infty} u_j ( x, t ) \varphi^j ( x, t), \quad (3.2)
\]

where \( \alpha \) is negative , \( \varphi(x, t) = 0 \) is the equation of singular manifold.
The function \( u_j \) \((j = 0, 1, 2,...)\) have to be determined by substitution of expansion into the PDE, so it becomes
\[
\sum_{j=0}^{\infty} E_j (u_0, ..., u_j, \varphi) \varphi^{j+q}(x,t) = 0, \tag{2.2}
\]
where \( q \) is some negative constant. \( E_j \) depends on \( \varphi \) only by the derivatives of \( \varphi \).

The leading order of solution of equation (3.2) is assumed as
\[
u \approx u_0 \varphi^\infty, \tag{3.3}\]

Substituting Eq. (3.3) into (3.1) and equating the most dominant terms, the following results are obtained
\[
\alpha = -1, \quad u_0 = 12\varphi_x. \tag{3.4}
\]

For finding the resonances, the full Laurent series:
\[
u = u_0 \varphi^{-1} + \sum_{j=1}^{\infty} u_j \varphi^{-j}, \tag{3.5}\]
is substituted into Equation (3.1) and by equating the coefficients of \( \varphi^{-j} \), the polynomial equation in \( j \) is derivated as
\[
j^3 - 4j^2 - j + 4 = 0, \quad (j - 1)(j + 1)(j - 4) = 0. \tag{3.6}\]

Using the previous Eq. (3.4), the resonances are found to be \( j = -1, 1, 4 \).

As usual, the resonance at \( j = -1 \) corresponds to the arbitrariness of singular manifold \( \varphi(x, y, z, t) = 0 \). In order to check the existence of sufficient number of arbitrary functions at the other resonance values, the full Laurent expansion (3.5) is substituted in Eq. (3.1). From the coefficient of \( \varphi^{-5} \), the explicit value of \( u_0 \) is obtained as given in Eq. (3.4). Collecting the coefficient of \( \varphi^{-4} \), the result is obtained as zero. Absence of \( u_1 \) proves that \( u_1 \) is arbitrary. This corresponds to the resonance value at \( j = 1 \). As solving these algebraic equations by Maple program, we obtain the results: \( u_1, u_2, u_3, u_4, u_5, u_6 \). Collecting the coefficient of \( \varphi^{-3} \), the following equation is obtained to give \( u_2 \) as
\[
u_2 = \frac{1}{2} u_0 \frac{\varphi_x}{\varphi^3} \left( -8 u_0 \varphi_x \varphi_{x,x,x} + u_0 \varphi_x u_{0,x,x} + u_{0,x} u_0 \varphi_{x,x} + 2u_0^2 \varphi_x \varphi_x - 2u_{1,x} u_0 \varphi_x^2 - 6u_0 \varphi_x \varphi_x \varphi_{x,x} - 12 u_{0,x} \varphi_x^2 \right). \tag{3.7}\]

Proceeding further to the coefficient of \( \varphi^{-2} \), the value of \( u_3 \) is obtained as
\[
u_3 = \frac{1}{2} u_0 \frac{\varphi_x}{\varphi^3} \left( -6 u_{0,x} \varphi_x + u_0 \varphi_x u_{0,x,x} + 2u_0 \varphi_x u_2 \varphi_x + u_0 \varphi_x u_{1,x,x} - u_{0,x} u_0 \varphi_{x,x} + u_{1,x} u_0 \varphi_x \varphi_{x,x} + 2u_{1,x} \varphi_x u_{0,x} + 2u_2 u_{0,x} \varphi_x^2 + 4u_{0,x} \varphi_x \varphi_{x,x} \right). \tag{3.8}\]

Collecting the coefficient of \( \varphi^{-1} \), the result is obtained as zero. Absence of \( u_4 \) proves that \( u_4 \) is arbitrary. This corresponds to the resonance value at \( j = 4 \).

And so on. we conclude that the equation be amenable to integration possible.

To construct the Bäcklund transformation of Eq. (3.1), let us truncate the Laurent series
\[
u = \frac{u_0}{\varphi} + u_1 + u_2 \varphi + u_3 \varphi^2 + u_4 \varphi^3, \tag{3.7}
\]
Hence
\[
u = \frac{12\varphi_x}{\varphi} + u_1 + u_2 \varphi + u_3 \varphi^2 + u_4 \varphi^3. \tag{3.7}
\]
where the pair of function \((u,u_x)\) satisfy Eq. (3.1) and hence Eq. (3.7) may be the associated Bäcklund transformation of Eq. (3.1).

**4. The murrary equation:**
\[
u_{xx} + \lambda_1 \nu \nu_x + \lambda_2 \nu - \lambda_3 \nu^2 - \nu_t = 0, \tag{4.1}
\]

\( \lambda_1, \lambda_2, \lambda_3 \) are constant.

We first present the Painlevé test of the murrary equation. According to the WTC method, the general solution of PDE is in the form
\[ u( x, t ) = \phi^\alpha ( x, t ) \sum_{j=0}^{\infty} u_j ( x, t ) \phi_j( x, t ), \]

where \( \alpha \) is negative, and \( \phi( x, t ) = 0 \) is the equation of singular manifold.

The function \( u_j \) (\( j = 0, 1, 2, \ldots \)) have to be determined by substitution of expansion into the PDE, So it becomes
\[ \sum_{j=0}^{\infty} E_j( u_0, \ldots, u_j, \phi ) \phi^{j+q} ( x, t ) = 0, \]

where \( q \) is some negative constant. \( E_j \) depends on \( \phi \) only by the derivatives of \( \phi \).

The leading order of solution of equation (4.2) is assumed as
\[ u \approx u_0 \phi^\alpha. \]  

Substituting Eq. (4.3) into (4.1) and equating the most dominant terms, the following results are obtained
\[ \alpha = -1, \quad u_0 = \frac{2\phi_x}{\lambda_1}. \]  

For finding the resonances, the full Laurent series :
\[ u = u_0 \phi^{-1} + \sum_{j=1}^{\infty} u_j \phi^{-j} \]

is substituted into Equation (4.1) and by equating the coefficients of \( \phi^{-j} \) , the polynomial equation in \( j \) is

\[ ( j + 1 ) ( j - 2 ) \lambda_j = 0, \]

Using the previous Eq. (4.4), the resonances are found to be \( j = -1, 2 \).

As usual, the resonance at \( j = -1 \) corresponds to the arbitrariness of singular manifold \( \phi( x, y, z, t ) = 0 \). In order to check the existence of sufficient number of arbitrary functions at the other resonance values, the full Laurent expansion (4.5) is substituted in Eq. (4.1). From the coefficient of \( \phi^{-3} \), the explicit value of \( u_0 \) is obtained as given in Eq. (4.4). Collecting the coefficient of \( \phi^{-2} \), the following equation is obtained by Maple program , we obtain the results: \( u_1, u_2, u_3, u_4, u_5, u_6 \)

Collecting the coefficient of \( \phi^{-1} \), the result is obtained as zero. Absence of \( u_2 \) proves that \( u_2 \) is arbitrary. This corresponds to the resonance value at \( j = 2 \).

From the coefficient of \( \phi^0 \), the value of \( u_3 \) is obtained as
\[ u_3 = -\frac{1}{2\phi_x + u_0 \lambda_1} \left( u_2 \phi_x + 2 u_2 \phi_x + u_0 \lambda_1 u_2 + u_1 \lambda_1 u_1 + \frac{1}{3} u_1 \phi_x \right). \]

Proceeding further to the coefficient of \( \phi^{-2} \), the value of \( u_4 \) is obtained as
\[ u_4 = -\frac{1}{2 \phi_x + u_0 \lambda_1} \left( \lambda_1 u_2 \phi_x + 2 \phi_x + 2 \phi_x + u_3 \lambda_1 u_3 + 4 \phi_x \lambda_1 u_1 + 2 \lambda_1 u_1 \phi_x + u_2 \phi_x \right). \]

and so on. We conclude that the equation be possess to integration possible.

To construct the Bäcklund transformation of Eq. (4.1), let us truncate the Laurent series
\[ u = \frac{u_0}{\phi} + u_1 + u_2 \phi + u_3 \phi^2 + u_4 \phi^3, \]

Hence
\[ u = \frac{2\phi_x}{\lambda_1} + u_1 + u_2 \phi + u_3 \phi^2 + u_4 \phi^3. \]  

where the pair of function \((u,u_4)\) satisfy Eq. (4.1) and hence Eq. (4.7) may be the associated Bäcklund transformation of Eq. (4.1).

### 5. The ( 2 + 1 ) Calogero equation

\[ u_{xxxy} - 2 u_y u_{xx} - 4 u_x u_{xy} + u_{xy} = 0, \]

We first present the Painlevé test of the Calogero equation. According to the WTC method, the general solution of PDE is in the form
\[ u(x, t) = \varphi^\alpha(x, t) \sum_{j=0}^{\infty} u_j(x, t) \varphi^j(x, t), \] (5.2)

where \( \alpha \) is negative, \( \varphi(x, t) = 0 \) is the equation of singular manifold.

The function \( u_j \) \((j = 0, 1, 2, \ldots)\) have to be determined by substitution of expansion into the PDE, So it becomes
\[ \sum_{j=0}^{\infty} E_j(u_0, \ldots, u_j, \varphi) \varphi^{j+q}(x, t) = 0, \]

where \( q \) is some negative constant. \( E_j \) depends on \( \varphi \) only by the derivatives of \( \varphi \).

The leading order of solution of equation (5.2) is assumed as
\[ u \approx u_0 \varphi^\alpha. \] (5.3)

Substituting Eq. (5.3) into (5.1) and equating the most dominant terms, and so on, the following results are obtained
\[ \alpha = -1, \quad u_0 = -12\varphi_x. \] (5.4)

For finding the resonances, the full Laurent series:
\[ u = u_0 \varphi^{-1} + \sum_{j=1}^{\infty} u_j \varphi^{-j}, \] (5.5)
is substituted into Equation (5.1) and by equating the coefficients of \( \varphi^{-5} \), the polynomial equation in \( j \) is derivated as
\[ j^2 - 5j - 6 = 0, \quad (j + 1)(j - 6) = 0. \] (5.6)

Using the previous Eq. (5.4), the resonances are found to be \( j = -1, 6 \).

As usual, the resonance at \( j = -1 \) corresponds to the arbitrariness of singular manifold \( \varphi(x, y, z, t) = 0 \). In order to check the existence of sufficient number of arbitrary functions at the other resonance values, the full Laurent expansion (5.5) is substituted in Eq. (5.1). From the coefficient of \( \varphi^{-5} \), the explicit value of \( u_0 \) is obtained as given in Eq. (5.4). Collecting the coefficient of \( \varphi^{-1} \), the result is obtained as zero. Absence of \( u_1 \) proves that \( u_1 \) is arbitrary. This corresponds to the resonance value at \( j = 6 \). As solving these algebraic equations by Maple program, we obtain the results: \( u_1, u_2, u_3, u_4, u_5, u_6 \). Collecting the coefficient of \( \varphi^{-3} \), the following equation is obtained to give \( u_1 \) as solving these algebraic equations by Maple program, we obtain the results: \( u_1, u_2, u_3, u_4, u_5, u_6 \).

we conclude that the equation be satisfy to integration possible.

To construct the Bäcklund transformation of Eq. (5.1), let us truncate the Laurent series
\[ u = \frac{u_0}{\varphi} + u_1 + u_2 \varphi + u_3 \varphi^2 + u_4 \varphi^3, \]
Hence
\[ u = \frac{-12\varphi_x}{\varphi} + u_1 + u_2 \varphi + u_3 \varphi^2 + u_4 \varphi^3. \] (5.7)

where the pair of function \((u, u_4)\) satisfy Eq. (5.1) and hence Eq. (5.7) may be the associated Bäcklund transformation of Eq. (5.1), relating a solution \( u \) with a known solution \( u_1 \) of the Eq. (5.1) which can be taken to be a known solution.

6. The Rosenau – Hyman equation (RH):
\[ uu_t + 3 uu_x + uu_{xxx} - uu_{tx} = 0. \] (6.1)

We first present the Painlevé test of the Rosenau – Hyman equation. According to the WTC method, the general solution of PDE is in the form
\[ u(x, t) = \varphi^\alpha(x, t) \sum_{j=0}^{\infty} u_j(x, t) \varphi^j(x, t), \] (6.2)

where \( \alpha \) is negative, \( \varphi(x, t) = 0 \) is the equation of singular manifold. The function \( u_j \) \((j = 0, 1, 2, \ldots)\) have to be determined by substitution of expansion into the PDE, So it becomes
\[ \sum_{j=0}^{\infty} E_j(u_0, \ldots, u_j, \varphi) \varphi^{j+q}(x, t) = 0. \] (6.3)
where q is some negative constant. \( E_j \) depends on \( \phi \) only by the derivatives of \( \phi \).

The leading order of solution of equation (6.2) is assumed as

\[
u \approx u_0 \phi^\alpha.
\]

(6.4)

Substituting Eq. (6.4) into (6.1) and equating the most dominant terms, the wanted \( \alpha \) and \( u_0 \) results could not be computed because balancing (comparison) two or more terms of the PDE leads to a contradiction as

\[
2\alpha+1= 2\alpha-3
\]

\[
0 = -4
\]

contradiction

So It is not Painlevé integrable.

We note that the Rosenau–Hyman equations (RH) has no (does not contain) linear higher order derivative term.

7. Cole–Hopf (CH) equation:

\[
u u_{ut} - u_x u_t - u_x^2 = 0,
\]

(7.1)

We first present the Painlevé test of the A Cole–Hopf (CH) equation. According to the WTC method, the general solution of PDE is in the form

\[
u (x, t) = \phi^\alpha (x, t) \sum_{j=0}^{\infty} u_j (x, t) \phi^j (x, t).
\]

(7.2)

where \( \alpha \) is negative, \( \phi(x, t) = 0 \) is the equation of singular manifold. The function \( u_j (j = 0, 1, 2,...) \) have to be determined by substitution of expansion into the PDE, So it becomes

\[
\sum_{j=0}^{\infty} E_j (u_0, \ldots, u_j, \phi) \phi^{j+q} (x, t) = 0.
\]

(7.3)

where q is some negative constant. \( E_j \) depends on \( \phi \) only by the derivatives of \( \phi \).

The leading order of solution of equation (7.2) is assumed as

\[
u \approx u_0 \phi^\alpha.
\]

(7.4)

Substituting Eq. (7.4) into (7.1) and equating the most dominant terms, the following results are not obtained because balancing two or more terms of the PDE more terms of the PDE leads to a contradiction as

\[
2\alpha-2= 2\alpha-2
\]

\[
0 = 0
\]

contradiction

So It is not Painlevé integrable.

We note that the Cole–Hopf (CH) equation has no (does not contain) linear higher order derivative term.

8. The Fornberg–Whitham equation (FW):

\[
u u_t - u_{xxx} + u_x - u u_{xxx} + u u_x - 3 u_x u_{xx} = 0,
\]

(8.1)

We first present the Painlevé test of the Fornberg–Whitham (FW) equation. According to the WTC method, the general solution of PDE is in the form

\[
u (x, t) = \phi^\alpha (x, t) \sum_{j=0}^{\infty} u_j (x, t) \phi^j (x, t).
\]

(8.2)

where \( \alpha \) is negative, \( \phi(x, t) = 0 \) is the equation of singular manifold. The function \( u_j (j = 0, 1, 2,...) \) have to be determined by substitution of expansion into the PDE, So it becomes

\[
\sum_{j=0}^{\infty} E_j (u_0, \ldots, u_j, \phi) \phi^{j+q} (x, t) = 0.
\]

(8.3)

where q is some negative constant. \( E_j \) depends on \( \phi \) only by the derivatives of \( \phi \).

The leading order of solution of equation (8.2) is assumed as

\[
u \approx u_0 \phi^\alpha.
\]

(8.4)
Substituting Eq. (8.3) into (8.1) and equating the most dominant terms, the wanted $\alpha$ and $u_0$ results are could not be computed because balancing (comparison) two or more terms of the PDE leads to a contradiction as

$$2\alpha - 1 = 2\alpha - 3$$
$$0 = -2$$

contradiction

So It is not Painlevé integrable.

We note that for the Fornberg–Whitham (FW) equation the linear higher order derivative term and the non-linear term are of the same derivative orders.

References:
[36] Steeb W.-H., Kloeck M., Speker B.M., Liouville equation, Painlevé property and Bäcklund transforma