OSCILLATION PROPERTIES OF CERTAIN TYPES OF FIRST ORDER NEUTRAL DELAY DIFFERENCE EQUATIONS

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ABSTRACT

In this paper some sufficient condition for the oscillation of first order neutral delay difference equation were obtained.

KEYWORDS:

Neutral Delay Difference Equation, Oscillation, Nonoscillation, Eventually positive.

Introduction 1.1

In this paper some sufficient condition for the oscillation of first order neutral delay difference equation of the form

\[ \Delta (a_n x_n - p_n x_{n-k}) + q_n f(x_{n-\ell}) = 0, \quad n \in \mathbb{N}(n_0) \]  

(1.1.1)

and

\[ \Delta (x_n + p_n x_{n-k}) + q_n f(x_{n-\ell}) = 0, \quad n \in \mathbb{N}(n_0) \]  

(1.1.2)

were obtained with the assumption of e the following conditions.

H1: \( \{p_n\} \) is a positive sequence.

H2: \( f \) is a continuous function such that \( uf(u) \geq 0 \).

H3: If there exists a function \( w \) such that \( w(u) > 0 \), for \( u > 0 \) and \( f(ufv) \leq w(u)f(v) \).

H4: If there exists a function \( \phi \) such that \( \phi(u) \) is increasing and \( u\phi(u) > 0 \), for \( u \neq 0 \) & \( |\phi(u + v)| \leq |f(u)f(v)| \).

1.2 Existence of Oscillatory Solutions

In this section, I obtain some sufficient condition for the oscillatory solutions of the equation (1.1.1) and (1.1.2)
**Theorem 1.2.1**

Assume that \( \frac{p_n}{a_{n-k}} \leq 1 \)
and \( x_n \) be an eventually positive solution of
the equation (1.1.1) and

\[ y_n = (a_n x_n - p_n x_{n-k}) \].

Then eventually \( y_n > 0 \).

**Proof**

Let us consider \( x_n > 0, \ x_{n-1} > 0, \ x_{n-k} > 0 \) for
some \( n > n_1 \).

From the equation (1.1.1),

\[ \Delta y_n = -q_n f(x_{n-1}) < 0. \]

Hence \( y_n \) is a
decreasing function.

Suppose \( y_n \) is not eventually positive, then
eventually \( y_n < 0 \).

Hence there exists \( n_2 > n_1 \) and \( M > 0 \), such
that \( y_n < -M \).

Let, \( z_n = a_n x_n > 0 \).

Then, \( z_n = y_n + p_n x_{n-k} \).

\[ z_n < -M + \frac{p_n}{a_{n-k}} z_{n-k} \].

Hence \( z_n \to -\infty \), \( n \to \infty \)

as \( n \to \infty \). Which contradicts the fact that
\( z_n \) is eventually positive. Hence the proof.

**Theorem 1.2.2**

Assume that \( p_n, q_n > 0 \) and \( \frac{p_n}{a_{n-k}} \leq 1 \).

\[ \lim_{n \to \infty} \inf \frac{q_n^* (1 + \frac{p_{n-1} a_{n-k}}{q_{n-k}})}{q_{n-k}} > 0 \],

If,

where \( q_n^* = \frac{q_n}{a_{n-1}} \), then every solution of
equation (1.1.1) is an oscillatory solution.

**Proof**

Let us assume the contradiction that
equation (1.1.1) has an non oscillatory
solution. Let us consider \( x_n \) is eventually
positive.

Let us consider \( x_n > 0, \ x_{n-1} > 0, \ x_{n-k} > 0 \) for
some \( n > n_1 \).

By theorem 1.2.1, \( y_n \) is eventually positive.

Also we have
\[ \Delta y_n = -q_n f(x_{n-1}) \]

\[ y_n = a_n x_n - p_n x_{n-k} \]

\[ \Delta y_n \leq -q_n x_{n-l} \quad (1.2.1) \]

\[ \Delta y_n \leq -q_n \frac{y_{n-l} + p_{n-l} x_{n-k-l}}{a_{n-l}} \]

\[ \Delta y_n = -q_n \frac{y_{n-l}}{a_{n-l}} - q_n \frac{p_{n-l} x_{n-k-l}}{a_{n-l}} \]

From the equation (1.2.1),

\[ \Delta y_n \leq -q_n \frac{y_{n-l} + p_{n-l} \Delta y_{n-k}}{a_{n-l}} - q_n \frac{p_{n-l} \Delta y_{n-k}}{a_{n-l} q_{n-k}} \]

Hence \( y_n \) satisfies the inequality,

\[ \Delta y_n + q_n \frac{y_{n-l}}{a_{n-l}} - q_n \frac{p_{n-l} \Delta y_{n-k}}{a_{n-l} q_{n-k}} \leq 0. \]

Let \( \lambda_n = \frac{-\Delta y_n}{y_n} \), then

\[ \lambda_n y_n \geq q_n \frac{y_{n-l}}{a_{n-l}} - q_n \frac{p_{n-l} \Delta y_{n-k}}{a_{n-l} q_{n-k}} \]

\[ q_n^* = \frac{q_n}{a_{n-l}} \], Hence we have,

\[ \lambda_n \geq q_n^* + \frac{p_{n-l} \lambda_{n-k} q_n^*}{q_{n-k}} \]

\[ \text{Hence } \lim_{n \to \infty} \inf q_n^*(1 + \frac{p_{n-l} \lambda_{n-k}}{q_{n-k}}) \leq \lambda_n, \]

\[ \lim_{n \to \infty} \inf q_n^*(1 + \frac{p_{n-l} \lambda_{n-k}}{q_{n-k}}) \leq 0. \]

Therefore,

Which contradicts the given condition of the theorem. Hence every solution of the equation (1.1.1) is an oscillatory solution.

**Theorem 1.2.3**

Suppose that 
\[ \text{Then } a_n = 1, \text{ for } n = 1, 2, 3, \ldots \]

\[ \Delta \left( x_n + p_n x_{n-k} \right) + q_n f(x_{n-l}) = 0, n \in N(n_0) \]

is oscillatory if there exists a function \( \lambda \) such that \( 0 \leq \lambda_n \leq 1 \) for \( n \geq n_0 \) and the difference inequality

\[ \Delta z_n + Q_n \phi(z_{n-l+k}) \leq 0, \]

(1.2.2)

has oscillatory solution where,

\[ Q_n = \min \left( \lambda_n q_n, \frac{(1 - \lambda_{n-k}) q_{n-l}}{wp_{n-l}} \right) \]

**Proof**
Suppose to the contrary that there is a non-oscillatory solution $x_n$. Assume that, $x_n > 0$, For all $n > n_0$. Let 

$$y_n = x_n + p_n x_{n-k}$$

$$\Delta(y_n) = -q_n f(x_{n-i}) < 0.$$ 

Also $y_{n+1} < y_n$, $y_n$ is decreasing function.,

Hence $y_{n+1} + q_n f(x_{n-i}) = y_n$

$$y_n > q_n f(x_{n-i}), n \geq n_0.$$ 

Taking summation from $n_0$ to $m$, $m > n_0$.

$$\sum_{n=n_0}^{m} y_n > \sum_{n=n_0}^{m} q_n f(x_{n-i})$$

$$\sum_{n=n_0}^{m} y_n > \sum_{n=n_0+k}^{m} Q_n \{f(x_{n-i}) + f(p_{n-i} x_{n-k-i})\}$$

Let $z_n = \sum_{n=n_0+k}^{m} Q_n \phi(y_{n-i}) > 0$.

Then $\Delta z_n = z_{n+1} - z_n$

$$\Delta z_n = \sum_{n=n_0}^{m} (Q_{n+1} \phi(y_{n+1-i}) - Q_n \phi(y_{n-i}))$$

$$\Delta z_n = Q_{m+1} \phi(y_{m+1-i}) - Q_{n_0+k} \phi(y_{n_0+k-i})$$

$$\Delta z_n > -Q_n \phi(y_{n-i})$$

$$\Delta z_n > -Q_n \phi(z_{n-i+k})$$

$$\Delta z_n + Q_n \phi(z_{n-i+k}) > 0.$$ This condition holds when $z_n$ is eventually positive solution. This is a contradiction to the equation (1.2.2).
Hence the proof completes. Similarly we prove that, when $x_n$ is eventually negative.

### 2.1 Examples

**Example 2.1.1**

Consider the first order neutral delay difference equation

$$
\Delta (nx_n - x_{n-1}) + (2n+3)x_{n-2}^3 = 0, n > 0
$$

Here

$$
a_n = n, k = 1, l = 2, p_n = -1, q_n = (2n+3)
$$

All the conditions of the theorem 1.2.2 are satisfied. Hence all its solutions are oscillatory. One such solution is $(-1)^n$.

**Example 2.1.2**

Consider the first order neutral delay difference equation

$$
\Delta \left( \frac{1}{n-1} x_{n-1} \right) + \frac{2n+3}{(n-2)^3} x_{n-2}^3 = 0, n > 2
$$

Here

$$
a_n = 1, k = 1, l = 2, p_n = -\frac{1}{n-1}, q_n = \frac{2n+3}{(n-2)^3}
$$

Hence all the conditions of the theorem 1.2.2 are satisfied. Hence all its solutions are oscillatory. One such solution is $n(-1)^n$.

### References


