New families of Super Mean Graphs

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Abstract - Let G be a (p, q) graph and let \( f : V(G) \rightarrow \{1, 2, ..., p+q\} \) be an injection. For each edge \( e = uv \), let \( f^*(e) = (f(u)+f(v))/2 \) if \( f(u)+f(v) \) is even and \( f^*(e) = ((f(u)+f(v))+1)/2 \) if \( f(u)+f(v) \) is odd. Then \( f \) is called a super mean labeling if \( f(V) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, ..., p+q\} \). A graph that admits a super mean labeling is called a super mean graph.

In this paper, we prove that \( C_n+v_1v_3 \) (\( n \geq 4 \)), Cube \( Q_3 \), Octahedron, the balloon of the triangular snake \( T_n(C_m) \) \( n \geq 2, m \geq 3, m \neq 4 \), \((2G, v_1, v_2)\) are super mean graphs.

Key words - labeling, mean labeling, super mean labeling, mean graph, super mean graph.

1 INTRODUCTION

Throughout this paper, by a graph we mean a finite, undirected, simple graph. Let \( G(V,E) \) be a graph with \( p \) vertices and \( q \) edges. For notations and terminology we follow [2]. Path on \( n \) vertices is denoted by \( P_n \) and a cycle on \( n \) vertices is denoted by \( C_n \). A triangular snake \( T_n \) is obtained from a path \( v_1, v_2, ... , v_n \) by joining \( v_i \) and \( v_{i+1} \) to a new vertex \( u_i \) for \( 1 \leq i \leq n-1 \), that is, every edge of a path is replaced by a triangle \( C_3 \). For example, \( T_6 \) is shown in Figure 1.

The graph \( C_n+v_1v_3 \) is obtained from the cycle \( C_n: v_1v_2 ... v_nv_1 \) by joining the vertices \( v_1 \) and \( v_3 \) by means of an edge. For example, \( C_5+v_1v_3 \) is shown in Figure 2.

The graph \( C_5+v_1v_3 \) is shown in Figure 2.

The graph \( P_2 \times P_2 \times P_2 \) is called the cube and is denoted by \( Q_3 \). Cube \( Q_3 \) is shown in Figure 3.

An octahedron is a polyhedron with 8 faces. An octahedron is shown in Figure 4.
The balloon of the triangular snake $T_n(C_m)$ is the graph obtained from $C_m$ by identifying an end vertex of the basic path in $T_n$ at a vertex of $C_m$. For example, $T_3(C_6)$ is shown in Figure 5.

![Figure 5](image_url)

Let $G_1$ and $G_2$ be two graphs with fixed vertices $v_1$ and $v_2$ respectively. Then $(G_1, G_2, v_1, v_2)$ is the graph obtained from $G_1$ and $G_2$ by identifying the vertices $v_1$ and $v_2$. For example, the graph $(C_6, P_5, v_1, v_2)$ is shown in Figure 6.

![Figure 6](image_url)

If $G_1 = G_2$, then $(G, G, v_1, v_2)$ is denoted by $(2G, v_1, v_2)$. For example, $(2C_6, v_1, v_2)$ is shown in Figure 7.

![Figure 7](image_url)

A vertex labeling of $G$ is an assignment $f : V(G) \rightarrow \{1, 2, 3, \ldots, p + q\}$ be an injection. For a vertex labeling $f$, an induced edge labeling $f^*$ is defined by

$$f^*(e) = \begin{cases} 
\frac{f(u) + f(v)}{2} & \text{if } f(u) \text{ and } f(v) \text{ are of same parity} \\
2 & \text{otherwise}
\end{cases}$$

A vertex labeling $f$ is called a super mean labeling of $G$ if its induced edge labeling $f(V) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \ldots, p + q\}$. If a graph has a super mean labeling, then we say that $G$ is a super mean graph.

The concept of mean labeling was first introduced by S. Somasundaram and R. Ponraj [7]. They have studied in [5, 6, 7, 8] the mean labeling of some standard graphs.

The concept of super mean labeling was first introduced by R. Ponraj and D. Ramya [3]. They have studied [3, 4] the super meanness of some standard graphs like $P_n, C_{2n+1}, n \geq 1, K_n(n \leq 3), K_{1,n}(n \leq 3), T_n, C_m \cup P_n(m \geq 3, n \geq 1), B_{mn}(m=n, n+1)$ etc. They have proved that the union of two super mean graphs is also a super mean graph and $C_4$ is not a super mean graph. Also they determined all super mean graph of order $\leq 5$. R. Vasuki and A. Nagarajan [10] proved that the super meanness of the graph $C_{2n}$ for $n \geq 3$, the H-graph, carona of a H-graph, 2 - carona of a H-graph, carona of cycle $C_n$ for $n \geq 3$, $mC_n$ - snake for $m \geq 1$, $n \geq 3$ and $n \neq 4$ and $C_m \times P_n$ for $m = 3, 5$. In [1], the meanness of the following graphs have been proved: $C_m \times P_n$; the caterpillar $P(n,2,3)$; $Q_3 \times P_{2n}$; carona of a H-graph; $mC_n$; $C_n \cup K_{1,m}$ ($n \geq 3, 1 \leq m \leq 4$); $mC_3 \cup K_{1,m}$ ($1 \leq m \leq 4$); the dragon $P_n(C_m)$ and some standard graphs.
In this paper, we prove the super meanness of the graph $C_{n+v_1v_3}$ ($n \geq 4$), Cube $Q_3$, Octahedron, the balloon of the triangular snake $T_n(C_m)$ $n \geq 2$, $m \geq 3$, $m \neq 4$, $(2G, v_1, v_2)$.

2 SUPER MEAN GRAPHS

Theorem 2.1 $C_{n+v_1v_3}$ is a super mean graph for $n \geq 4$.

Proof Let $C_n$ be a cycle with vertices $v_1, v_2, v_3, \ldots, v_n$ and edges $e_1, e_2, e_3, \ldots, e_n, e_{n+1}$.

Define $f : V(C_{n+v_1v_3}) \rightarrow \{1, 2, \ldots, n+1\}$ as follows:

Case 1 when $n$ is odd, $n = 2m+1$, $m = 2, 3, 4, \ldots$

For $m = 2$, a super mean labeling of $C_{5+v_1v_3}$ is shown in Figure 8.

For $m \geq 3$,

$f(v_i) = 2i-1, 1 \leq i \leq 2$;

$f(v_{i+2}) = 2i+5, 1 \leq j \leq 2$;

$f(v_{m+i+2}) = 4m-4k+8, 1 \leq k \leq m-2$;

$f(v_{2m+1}) = 10$.

Then the induced edge labels are

$f^*(e_1) = 2$;

$f^*(e_i) = 3i-1, 2 \leq i \leq 3$;

$f^*(e_{i+4}) = 4j+7, 1 \leq j \leq m-2$;

$f^*(e_{m+i+2}) = 4m-4k+4, 1 \leq k \leq m-2$;

$f^*(e_{2m}) = 6$;

Then the induced edge labels are

Clearly $f$ is a super mean labeling of $C_{n+v_1v_3}$.

For example, the super mean labelings of $C_{7+v_1v_3}$ and $C_{10+v_1v_3}$ are shown in Figure 10.

Case 2 when $n$ is even, $n = 2m$, $m = 2, 3, 4, \ldots$

For $m = 2$, a super mean labeling of $C_{4+v_1v_3}$ is shown in Figure 9.

For $m \geq 3$,

$f(v_i) = 2i-1, 1 \leq i \leq 2$;

$f(v_{j+3}) = 4j+5, 1 \leq j \leq m-1$;

$f(v_{m+k+2}) = 4m-4k+2, 1 \leq k \leq m-2$.

Then the induced edge labels are

$f^*(e_1) = 2$;

$f^*(e_i) = 3i-1, 2 \leq i \leq 3$;

$f^*(e_{i+3}) = 4j+7, 1 \leq j \leq m-2$;

$f^*(e_{m+k+1}) = 4m-4k+4, 1 \leq k \leq m-2$;

$f^*(e_{2m}) = 6$;

$f^*(v_{1v_3}) = 4$.

(a) $C_{7+v_1v_3}$
Theorem 2.2  Cube $Q_3$ is a super mean graph.

Proof  Let $u_1, u_2, u_3, u_4$ and $v_1, v_2, v_3, v_4$ be the vertices of $Q_3$.

Define $f : V(Q_3) \rightarrow \{1, 2, 3, ..., 20\}$ as follows:


declare f(u_i) = 2i-1, 1 \leq i \leq 2; \quad f(u_4) = 18; \quad f(v_i) = 4i-3, 2 \leq i \leq 3; \quad f(v_4) = 16.

Then the induced edge labels are


declare f^*(u_1u_2) = 2; \quad f^*(u_iu_{i+1}) = 7i-2, 2 \leq i \leq 3; \quad f^*(v_1v_2) = 8;
\quad f^*(v_i,v_{i+1}) = 6i-5, 2 \leq i \leq 3; \quad f^*(v_1v_4) = 14;
\quad f^*(u_1v_1) = 6; \quad f^*(u_2v_2) = 4;
\quad f^*(u_3v_3) = 15; \quad f^*(u_4v_4) = 17.

Clearly $f$ is a super mean labeling of $Q_3$.

For example, a super mean labeling of $Q_3$ is shown in Figure 11.

Theorem 2.3  Octahedron is a super mean graph.

Proof  Let $u_1, u_2, u_3$ and $v_1, v_2, v_3$ be the vertices of the octahedron.

Define $f : V(G) \rightarrow \{1, 2, 3, ..., 18\}$ as follows:


declare f(u_1) = 6; \quad f(u_2) = 1; \quad f(u_3) = 13; \quad f(v_1) = 3;
\quad f(v_2) = 15; \quad f(v_3) = 18.

Then the induced edge labels are


declare f^*(u_1u_3) = 10; \quad f^*(v_i,v_{i+1}) = 8i+1, 1 \leq i \leq 2;
\quad f^*(v_1v_3) = 11; \quad f^*(u_iu_{i+1}) = 3i+1, 1 \leq i \leq 2;
\quad f^*(u_3v_3) = 16; \quad f^*(v_1u_2) = 2; \quad f^*(v_2u_3) = 14;
\quad f^*(v_3u_1) = 12.

Clearly $f$ is a super mean labeling of the Octahedron.

For example, a super mean labeling of the Octahedron is shown in Figure 12.
Theorem 2.4 \( T_n(C_m) \) is a super mean graph for \( n \geq 2 \), \( m \geq 3 \), \( m \neq 4 \).

**Proof** Let \( v_1, v_2, v_3, \ldots, v_m \) be the vertices of \( C_m \) and \( u_1, u_2, u_3, \ldots, u_n ; w_1, w_2, w_3, \ldots, w_{n-1} \) be the vertices of \( T_n \).

Then define \( g \) on \( T_n(C_m) \) as follows:

**Case 1** when \( m \) is even, \( m = 2k \), \( k = 3, 4, 5, \ldots \)

\[
\begin{align*}
g(v_i) &= f(v_i), 1 \leq i \leq m; \\
g(u_i) &= 2m+5i-5, 1 \leq i \leq n; \\
g(w_i) &= 2m+5i-3, 1 \leq i \leq n-1.
\end{align*}
\]

Then the induced edge labels are

\[
\begin{align*}
g^*(e_i) &= f(e_i), 1 \leq i \leq m; \\
g^*(u_iu_{i+1}) &= 2m+5i-2, 1 \leq i \leq n-1; \\
g^*(w_iu_{i+1}) &= 2m+5i-1, 1 \leq i \leq n-1.
\end{align*}
\]

**Case 2** when \( m \) is odd, \( m = 2k+1 \), \( k = 2, 3, 4, \ldots \)

\[
\begin{align*}
g(v_i) &= f(v_i), 1 \leq i \leq m; \\
g(u_i) &= 2m+5i-5, 1 \leq i \leq n; \\
g(w_i) &= 2m+5i-3, 1 \leq i \leq n-1.
\end{align*}
\]

Then the induced edge labels are

\[
\begin{align*}
g^*(e_i) &= f(e_i), 1 \leq i \leq m; \\
g^*(u_iu_{i+1}) &= 2m+5i-2, 1 \leq i \leq n-1; \\
g^*(u_iw_i) &= 2m+5i-4, 1 \leq i \leq n-1; \\
g^*(w_iu_{i+1}) &= 2m+5i-1, 1 \leq i \leq n-1.
\end{align*}
\]

Clearly \( g \) is a super mean labeling of \( T_n(C_m) \).

For example, the super mean labelings of \( T_5(C_6) \) and \( T_5(C_9) \) are shown in Figure 13.

\[\text{Figure 12}\]

\[\text{Figure 13}\]
Clearly $g$ is a super mean labeling of $(2G, v_1, v_2)$.

For example, a super mean labeling of $(2Q_3, v_1, v_2)$ is shown in Figure 14.

Figure 14

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