Nature of Composite Numbers that obey Fermat’s property & some properties of a Prime

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ABSTRACT: Any prime number ‘m’ obeys the Fermat property with respect to any number ‘a’ which is prime to m. But if m is composite it may be or may not be true. The composite numbers that obey Fermat property can be said as Fermat’s composite number or simply FC Numbers. This paper contains some natures of so called FC numbers. The existence of such numbers was first detected by the American mathematician Carmichael in 1899. So far divisibility is concerned a prime number possesses several properties out of which we can recall the famous property of Wilson’s theorem i.e. p divides (p – 1)! + 1. The proof of this theorem given by the great mathematician Lagrange also indicates some other divisibility properties of a prime which have been lying hidden to the proof itself and never day-lighted. My paper contains the proof of those hidden properties along with the fact that Wilson theorem is a particular case of a general property. My paper contains an important theorem regarding divisibility of twin primes.

INTRODUCTION: Before we investigate the nature of a FC-number and to extract some properties of a prime it is felt necessary to indicate the meanings of some usual notations and to highlight one useful common theorem.

\[\prod(p_i)\] denotes the product sequence of odd primes i.e. \(p_1p_2p_3 \ldots\)

\([x, y]\) denotes the LCM of \(x_1, x_2, x_3 \ldots\)

\((x, y)\) denotes the GCF of \(x_1, x_2, x_3 \ldots\)

If \(x + y = z\) where \((x, y) = 1\) then obviously \((x, z) = (y, z) = 1\)

For a number \(N = 2^e \cdot p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots\) Degree of intensity (DOI) with respect to any base prime is defined as DOI(N)_2 = n or symbolically \(\uparrow(N)_2 = n\)

\(\uparrow(N)_{|p \vert} = n\), \(\uparrow(N)|_{|p \vert} = n_1\), \(\uparrow(N)|_{|p \vert} = n_2\), and so on.

Obviously, for an odd integer \(\uparrow(N)_2 = 0\)

If \(N\) is an odd integer of \(1^{st}\) kind i.e. in the form of \(4x - 1\), \(\uparrow(N - 1)_2 = 1\) & for \(2^{nd}\) kind i.e. \(4x + 1\) form \(\uparrow(N - 1)_2 > 1\).

\(a \mid b\) is the symbol for a divides b.

KEY WORDS: FC-number, Degree of intensity (DOI)

1. Any FC-number is a product sequence of odd primes only i.e. \(\prod(p_i)\)

According to Fermat Theorem \(a^{m-1} \equiv 1 \pmod{m}\) where \(m\) is prime & \((a, m) = 1\). When \(m\) is composite it may be or may not be true. For a composite number \(m = p_1^{a_1}p_2^{a_2}p_3^{a_3} \ldots\) it will obey the Fermat’s property if and only if \(m - \lambda[p_1^{a_1}, p_2^{a_2}, p_3^{a_3}, \ldots] = 1\) where \(\lambda\) is any positive integer & \(\phi(x^y) = x^y - x^{y-1}\). The proof can be easily understood by an example given below.

1105 = 5.13.17 Let a be any integer relatively prime to 1105.

Then by Fermat’s theorem \(a^1 \equiv 1 \pmod{5}\), \(a^{12} \equiv 1 \pmod{13}\), \(a^{16} \equiv 1 \pmod{17}\)

This implies \(a^{12} \equiv 1 \pmod{5, 13, 17}\) i.e. \(a^{16} \equiv 1 \pmod{1105}\) & raising both sides to the power 23 we have \(a^{16^{23}} \equiv 1 \pmod{1105}\) i.e. \(a^{16^{23}} \equiv 1 \pmod{m}\) where \(m\) is composite.

\(\Rightarrow (p_1^{a_1}p_2^{a_2}p_3^{a_3} \ldots) - \lambda[p_1^{a_1}, p_2^{a_2}, p_3^{a_3}, \ldots] = 1\)

\(\Rightarrow (p_1^{a_1}p_2^{a_2}p_3^{a_3} \ldots) - \lambda[p_1^{a_1}, p_2^{a_2} - 1, p_3^{a_3} - 1, \ldots] \{(p_1 - 1), (p_2 - 1), (p_3 - 1), \ldots\} = 1\)

\(\Rightarrow (p_1^{a_1} - 1, p_2^{a_2} - 1, p_3^{a_3} - 1, \ldots)\{ (p_1p_2p_3 \ldots) - \lambda[p_1 - 1, (p_2 - 1), (p_3 - 1), \ldots] = 1\) which is quite impossible unless all \(a_i = 1\). Hence, proved that all FC-numbers are the product of primes only.
2. The composite numbers which are the product of twin primes cannot be FC-numbers
i.e. \( a^{p(p + 2)} \equiv 1 \pmod{p(p + 2)} \) where \( p \) & \( (p + 2) \) both are primes & \( \langle a, p \rangle = 1 = \langle a, p + 2 \rangle \)

Here, \( m = p(p + 2) \) & \( a \equiv [p - 1, p + 1] = 1 \pmod{m} \)
This implies \( p(p + 2) - \lambda(p - 1)/2(p + 1)/2 = 1 \) this implies \( (2 - \lambda)p^2 + 4p + (\lambda - 1) = 0 \)
Obviously, \( \lambda < 2 \) & \( m \) has no existence.

3. For a FC-number \( N = \Pi(P) \) where \( i \geq 3 \), \( \uparrow(p_i - 1) \), cannot be all different.

If all \( \uparrow(p_i - 1) \) are different with respect to any base \( x \), say \( \min\{\uparrow(p_i - 1)\} = n \) and \( \max\{\uparrow(p_i - 1)\} = m \)
\( \Rightarrow \lambda \) is in the form of \( x^{|d_1| + |d_2|}/x^{|d_3|d_4} \) where all \( d \) are some integers free from \( x \).
\( \Rightarrow \lambda \) cannot be integer as \( n \) \( < \) \( m \).

4. For a FC-number \( N = \Pi(P) \) where \( i \geq 3 \), if \( \min\{\uparrow(p_i - 1)\} = n \) & \( \max\{\uparrow(p_i - 1)\} = m \) then \( n \) must be repeated even number of times irrespective of the fact that other DOI of 2 is repeated or not. For odd numbers of repetitions of \( n \), FC-number cannot exist.

This is simply because of the following fact.
Say, \( v_1 + v_2 = v_3 \) where \( v \) denotes even integers.
\( \Rightarrow \uparrow(v_1) = \uparrow(v_2) \Rightarrow \uparrow(v_3) = x \) if \( \uparrow(v_1) \leq \uparrow(v_2) \) then \( \uparrow(v_3) = \uparrow(v_2) \)
If \( \uparrow(v_3) = x \) for \( i = 1, 2, 3, \ldots \) then \( \uparrow(\sum v_i) = x \) where \( i = 1, 2, 3, \ldots, (2r - 1) \) & \( > x \) where \( i = 1, 2, 3, \ldots, 2r \).
Now, if \( n \) is repeated odd nos. of times then numerator of \( \lambda \) contains single term as minimum DOI of 2 i.e. \( n \).
As a result, \( \lambda \) is in the form of \( 2^{|d_1|}/2^{|d_2|} \) for some integers of \( d_1 \) & \( d_2 \) where \( n \) \( < \) \( m \).
\( \Rightarrow \lambda \) cannot be an integer.
But if \( n \) is repeated even nos. of times \( n \) may go on increasing by chain rule with other DOI and as a result we may get \( \lambda \) in the form of \( 2^{|d_1|}/2^{|d_2|} \) where \( r \geq m \) \( \Rightarrow \lambda \) is an integer & FC-number is a product of at least 3 primes.

5. If \( p \) is an odd prime then

5.1 \( p | \sum x^m \) where \( x = 1, 2, 3, \ldots \), \( p - 1 \) \& \( m \neq k(p - 1) \)

5.2 \( p | \sum x^m \) where \( x = \) product of integers among \( 1, 2, \ldots \), \( p - 1 \) taken two at a time for \( 2m \neq k(p - 1) \)

5.3 \( p | \sum x^m \) where \( x = \) product of integers among \( 1, 2, 3, \ldots \), \( p - 1 \) taken \( r \) at a time \& \( r < p - 1 \)

5.4 \( p | \sum x^m \) where \( x = \) product of integers among \( 1, 2, \ldots \), \( p - 1 \) taken \( P - 1 \) at a time \& \( m \neq k(p - 1) \)

5.5 \( p | \sum x^m \) where \( x = \) product of integers among \( 1, 2, \ldots \), \( p - 1 \) taken \( P - 3 \) at a time \& \( 2m \neq k(p - 1) \)

Say, \( f(x) = (x - 1)(x - 2)(x - 3) \ldots \ldots (x - p + 1) = x^{p - 1} - a_1x^{p - 2} + a_2x^{p - 3} - \ldots \ldots - a_{p - 2}x + (p - 1)! \)
Where \( a_n \) denotes the sum of the product among \( 1, 2, 3, \ldots \), \( p - 1 \) taken \( r \) at a time for \( r < p - 1 \) & according to Lagrange theorem \( p | \) all \( a_n \)

Now, by logarithmic differentiation,
\( f'(x)/f(x) = (x - 1)^{-1} + (x - 2)^{-1} + \ldots \ldots + (x - (p - 1))^{-1} \)
\( = ((p - 1)x^{p - 2} - a(p - 2)x^{p - 3} + \ldots \ldots - a_{p - 2})/f(x) \)
\( \Rightarrow (\sum \alpha^0)x^{-1} + (\sum \alpha^1)x^{-2} + (\sum \alpha^2)x^{-3} + \ldots \ldots + (\sum \alpha^m)x^{-(m + 1)} + \ldots \ldots \)
where \( \alpha \) varies from 1 to \( (p - 1) = \)
\( (p - 1)x^{p - 1} + bx^{p - 2} + bx^{p - 3} + \ldots \ldots + bx^{-(p - 3)} + \ldots \ldots + b_0x^{-1} + \ldots \ldots \)
By algebraic division of \( f(x)/f(x) \)
Here, all the coefficients of \( b \) are the expression of \( \alpha \) not containing any free constant excepting the cases
\( \lambda \) where \( m \) is multiple of \( (p - 1) \) i.e. \( m \neq k(p - 1) \). By algebraic division it can be easily observed.

Hence, in all other cases \( p | \) \( b_0 \) & equating the coefficients on both sides we can say,
\( P | 1^m + 2^m + 3^m + \ldots \ldots + (p - 1)^m \) i.e. \( p | \sum x^m \) where \( x = 1, 2, 3, \ldots \ldots, p - 1 \) for \( m \neq k(p - 1) \)

Now, squaring both sides \( p | \sum x^m \) where \( y \) is the product among \( 1, 2, 3, \ldots \ldots, (p - 1) \) taken two at a time.
\( \Rightarrow p | \sum y^m \) as \( \sum y^m \) is divisible by \( p \) for \( 2m \neq k(p - 1) \) and hence proved 6.2

Now, \( p | \sum x \) i.e. \( p | \sum x \) say, \( p | (c_1 + c_2 + c_3 + \ldots) \) where \( c \) is the product of \( r \) different integers.
\( \Rightarrow p | (c_1 + c_2 + c_3 + \ldots)^p \) \( \Rightarrow p | (c_1^p + c_2^p + c_3^p + \ldots) \) + \( K \) multiple of \( p \).
⇒ p | (c₁p + c₂p + c₃p + …….) ⇒ p | ∑xᵢ where xᵢ is the product of r different integers & r < (p – 1)
Hence, proved 5.3
Replacing x by 1/x we have, f(x) = [(p – 1)!xᵖ⁻¹ – a₀p⁻¹xᵖ⁻² + a₁p⁻³xᵖ⁻³ – ……… + aₓ² – aₓ + 1 where roots of the
equation f(x) = 0 are 1, 1/2, 1/3, ………, 1/(p – 1)
Now, considering the same logics on f(x) / f(x) we can say,
P | Numerator of [1/1ᵐ + 1/2ᵐ + 1/3ᵐ + ……… + 1/(p – 1)ᵐ] for m ≠ k(p – 1)
⇒ p | ∑xᵐ where x is the product of (p – 2) integers at a time.
Again, P | Numerator of [1/1ᵐ + 1/2ᵐ + 1/3ᵐ + ……… + 1/(p – 1)ᵐ]
⇒ P | Nʳ of {∑xᵐ} where x = 1, ½, 1/3, ……. 1/(p – 1) & y is the product of two fractions at a time.
⇒ p | ∑yᵐ for 2m ≠ k(p – 1) ⇒ p | Nʳ of ∑yᵐ ⇒ p | ∑xᵐ where x is the product taken (p – 3) integers at a time
for 2m ≠ k(p – 1) and hence proved.

Note for shifting phenomenon: above mentioned all the theorems are also true if the original set 1, 2, 3, …… p
- 1 is replaced by 1.μ + pλ, 2.μ + pλ, 3.μ + pλ, ………, (p – 1).μ + pλ where λ, μ are positive integers. It happens
due to same logics applied in f(x) where roots of the equation f(x) = 0 have been increased
by μ times & then added by pλ in the equation f(x/μ – pλ) = 0.
e.g. for m ≠ 10k, 11 | 1ᵐ + 2ᵐ + 3ᵐ + ……… + 10ᵐ considering μ = 1 & λ = 0, 12ᵐ + 13ᵐ + 14ᵐ + …….. + 21ᵐ
considering μ = 1 & λ = 1, 25ᵐ + 28ᵐ + 31ᵐ + …….. + 52ᵐ for μ = 3, λ = 2 & so on.
Against any two fixed integers λ & μ and with respect to any odd prime p the set of integers pλ ± xμ where x
varies from 1 to (p – 1) will satisfy all the said theorems and also the Lagrange coefficients. All the sets under
this particular class are the complete reduced system of (mod p) while (1, 2, 3, ……, p – 1) is the least residue
of (mod p).
Hence, in general, p | [pλ + xμ] + μᵖ⁻¹ & for a particular case where μ = 1 & λ = 0, p | (p – 1)! + 1 which
is known as Wilson’s theorem. It is obtained simply by putting x = μ in the identity of f(x) after replacement of
all the roots by pλ + xμ. It is observed LH side is divisible by p & RH side is partially divisible by p. non-
divisible part on RH side is ∑((pλ + xμ) + μᵖ⁻¹) where x varies from 1 to p – 1. Hence, non-divisible part must be
divisible by p.
If (μ, p) = 1 then obviously p | [(pλ ± xμ) + 1 according to Fermat property and after multiplication all the
factors p | μᵖ⁻¹(p – 1)! + 1
⇒ p | (µᵖ⁻¹ + µᵖ⁻¹ + µᵖ⁻¹ + …….. p terms).(p – 1)! where (μ, p) = 1
⇒ p | (∑µᵖ⁻¹)(p – 1)! ⇒ p | ∑(µᵖ⁻¹)
In view of the above we can establish one important theorem given below.

6. If p & p + 2 are twin primes then p(p + 2) | ∑ xᵖ⁻¹
   x=1
It is quite obvious that if p + 2 is a prime then (p + 2) | ∑ xᵖ⁻¹ where x varies from 1 to p + 1 as per Th-5.1
Here, in ∑ xᵖ⁻¹ sum of all the terms excluding pᵖ⁻¹ is also divisible by p. Hence, ∑ xᵖ⁻¹ is also divisible by p

6.1 Converse of the theorem is also true.

As there exists infinitely many primes we can assume (p + 2) is a prime and accordingly p will be either prime
or a FC-number.
Say, p is a FC-number & p | X where X = 1ᵖ⁻¹ + 2ᵖ⁻¹ + 3ᵖ⁻¹ + …….. + (p – 1)p⁻¹ + pᵖ⁻¹ + (p + 1)p⁻¹, where obviously
(p + 1, p) = 1
As per Theorem-1 & 4, p must be product of at least 3-primes, say p = αβγ
Now, α | (1ᵖ⁻¹ + 2ᵖ⁻¹ + 3ᵖ⁻¹ + …….. + (α – 1)p⁻¹ + αᵖ⁻¹) + ((α + 1)p⁻¹ + (α + 2)p⁻¹ + (α + 3)p⁻¹ + …….. +
(2α − 1)p−1 + (2α)p−1 + …… βγ brackets.
⇒ α | [1p−1 + 2p−1 + 3p−1 + …… + (p−1)p−1]

Now, if p | X means α | X ⇒ α | (p + 1)p−1 which is impossible as (p + 1, p) = 1 i.e. (p + 1, α) = 1.

Hence, p cannot be a composite number obeying Fermat property.

\[ p^p \]

So, if it is found that \( \sum x^{p−1} \) is divisible by \( p + 2 \) both then \( p, p + 2 \) must be twin primes.

*Apart from the twin prime factors X also contains all the prime factors of \( p + 1 \) say \( α, i = 1, 2, 3, \ldots \).

Provided \( p − 1 \neq k(α−1) \) as per the next theorem.

It is to be noted that in between twin primes there cannot exist an even number in the form of \( 2^n \)

7. **For a prime number p if all the prime factors of \( p − 1 \) are \( α, i = 1, 2, 3, \ldots \), then \( X = \sum x^m \) must be divisible by \( p \) and all \( α \) provided \( m \neq k(p−1) \) or \( k(α−1) \).

Say \( p − 1 = αy \) where \( α \) is a prime.

\[ \Rightarrow α | [1^m + 2^m + 3^m + \ldots + (α − 1)^m + α^m] + [(α + 1)^m + (α + 2)^m + (α + 3)^m + \ldots + (2α)^m] + \ldots \]

\( y \) brackets

\[ \Rightarrow α | [1^m + 2^m + 3^m + \ldots + (p − 1)^m] \]

\( \Rightarrow α | X \) and similarly, for other prime factors of \( p − 1 \).

The following two theorems which are easy to establish are also felt necessary to mention.

8.1 \( (x | p) | x^p + (x ± α)^p + (x ± 2α)^p + \ldots \ldots \ldots \ldots p \) terms

8.2 \( (x | p) | x^p + (x ± α)^p + (x ± 2α)^p + \ldots \ldots \ldots \ldots \) up to any odd terms when \( p \) is a prime factor of \( x \).

**Few examples in favor of theorem 6 & 7**

Once again we can redefine clubbing the theorems 6 & 7 as: If \( p \) & \( p + 2 \) are twin primes where obviously \( p + 1 \) is in the form of \( 2(\text{odd}) + 1 \), \( X \) is always divisible by 5.

Let us consider the twin prime \( (101, 103) \) where \( p + 1 = 102 = 2 \times 3 \times 17 \) & \( p = 100 = 2^2 \times 5^2 \).

\[ \Rightarrow \{\uparrow(100): 2\} < \{\uparrow(17 − 1): 4\} \]. Hence \( (101.103.17) \) | \( X \) where \( X = \sum x^{100} \), \( x \) varies from 1 to 102.

For all the twin primes of the form \( (u \times u) \) where \( u \) is in the form of \( 2(\text{odd}) + 1 \), \( X \) is always divisible by 5.

\( \{u_1, u_2\} \) denotes a prime having unit digit \( x\).

Say the twin prime \( (59, 61) \) where \( p + 1 = 102 = 2 \times 3 \times 17 \) & \( p = 100 = 2^2 \times 5^2 \).

\[ \Rightarrow \{\uparrow(59): 1\} = \{\uparrow(61): 1\} \]. Hence, \( (59.61.17) \) | \( X \) where \( X = \sum x^{58} \), \( x \) varies from 1 to 60.

\( \{\uparrow(137): 1\} \) where \( 138 = 2 \times 3 \times 23 \) & \( \{\uparrow(139): 1\} = 0 \) \( \Rightarrow \{\uparrow(23): 1\} = 1 \).

Hence, \( (137.139.23) \) | \( X \) \( x \) varies from 1 to 138.

It is quite evident that if \( m \) is odd, then \( \{\uparrow(p_1) \} | \{\uparrow(m) \} = 0 \) but \( \{\uparrow(m) \} = 0 \).

9. **For a prime number p if \( (m, p) = 1 \) then**

9.1 \( p | X \) where \( X = 1 + m^d + (m^d)^2 + \ldots \ldots \ldots \ldots (p−1)/d \) terms, \( d \) is any factor of \( p−1 \) including one but excluding \( (p−1) \) itself, provided \( (p, m^d−1) = 1 \).

9.2 \( d \) is odd \( \Rightarrow p | Y \) where \( Y = 1 + m^d + (m^d)^2 + \ldots \ldots \ldots \ldots (p−1)/d \) terms, provided \( (p, m^d + 1) = 1 \).

9.3 \( p | XY \) without any condition i.e. only for \( (m, p) = 1 \) where \( m \) or \( p ≠ 2 \).

9.4 \( (p, m^d ± 1) = 1 \) for \( d \) is odd, \( p | 1 + (m^d)^2 + (m^d)^3 + \ldots \ldots \ldots \ldots (p−1)/2d \) terms

We have \( X = (m^d)^{d−1}/(d−1)/(d−1)/(m^d−1) \) \( \Rightarrow p | X \) for \( (p, m^d − 1) = 1 \).

If \( d \) is odd then replacing \( m \) by \(-m\) we have \( Y = - (m^d−1)/(m^d + 1) \) \( \Rightarrow p | Y \) for \( (p, m^d + 1) = 1 \).

Now, there cannot exist any common odd factor in between two consecutive even or odd numbers.

So, \( p \) cannot divide \( m^d ± 1 \) both and hence, \( p | XY \) without any condition except \( (m, p) = 1 \).

e.g. for any prime \( p, p | 1 + m^{2(n−k)} + m^{3(n−k)} + \ldots \ldots \ldots \ldots 2^kb \) terms where \( p − 1 = 2^k \) \& \( (p, m) = 1 \),
(p, m^{2^{(n-k)}} - 1) = 1, 0 \leq k \leq n \Rightarrow \text{for } m = 2 \& k = 0, (p, \prod(F_{n-i})) = 1, F \text{ denotes the Fermat number.}

Say the prime number 13 where p - 1 = 3.4

\Rightarrow 13 \mid (1 + 3^3 + 3^6 + 3^{12})(1 - 3^3 + 3^6 - 3^{12}) \text{ where } m^4 = 3^3 \text{ or } 13 \mid (1 + 7^3 + 7^6 + 7^{12})(1 - 7^3 + 7^6 - 7^{12}) \text{ where } m^4 = 7^3 & \text{so on}

Now, multiplying XY we get p \mid (r^2 - 1)(1 + r^2 + r^4 + \ldots \ldots (p - 1)/2d \text{ terms})

\Rightarrow p \mid 1 + r^2 + r^4 + \ldots \ldots (p - 1)/2d \text{ terms} \text{ where } r = m^4 \text{ for } (p, m^4 \pm 1) = 1

References: any text book in the field of Number Theory.

Conclusion: I believe that with the help of these newly invented theorems it will be possible to extract many more properties of prime numbers. Presently one important question excites our mind regarding existence of other prime factors of X in theorem 7 apart from p & all p\text{,}. If it exists, say p\text{-group what is the logic behind its existence? It seems p\text{,} is a FC-number as a whole i.e. } \prod(p\text{,}) \text{ or product of several FC-numbers i.e. } \prod(p\text{,}) \text{...... where some FC-numbers may contain few primes from p\text{-group e.g. say p\text{,} consists a single prime q}1\text{ & p\text{,} also consists a single prime q}2\text{ then pq}1q2\text{ must be a FC-number as FC-nos. is a product of minimum three primes. If p\text{,} consists two primes q}2\text{ & q}3\text{ then either pq}2q3\text{ or q}1q2q3\text{ or pq}1q2q3\text{ is a FC-number. Of course, it needs further investigation to prove.}

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