Matching Dominating Sets of Strong Product Graph of Euler Totient Cayley Graphs with Arithmetic $\nu_n$ Graphs

M.Manjuri and B. Maheswari

Abstract — Theory of domination in graphs introduced by Ore and Berge is an emerging area of research in graph theory today. It has been studied extensively and finds applications to various branches of Science & Technology.

Products are often viewed as a convenient language with which one can describe structures, but they are increasingly being applied in more substantial ways. Every branch of mathematics employs some notion of product that enables the combination or decomposition of its elemental structures.

In this paper, we consider strong product graph of Euler totient Cayley graphs with Arithmetic $\nu_n$ graphs and present some results on matching domination parameter of these graphs.

Index Terms — Arithmetic $\nu_n$ graph, Matching dominating set, Euler totient Cayley graph, Strong product graph.

Subject Classification: 68R10

1 INTRODUCTION

Graph Theory has been realized as one of the most useful branches of Mathematics of recent origin with wide applications to combinatorial problems and to classical algebraic problems. Graph theory has applications in diverse areas such as social sciences, linguistics, physical sciences, communication engineering etc.

The theory of domination in graphs is an emerging area of research in graph theory today. It has been studied extensively and finds applications to various branches of Science & Technology.

The strong product was studied by Nesetril [1]. Occasionally it is also called as strong direct product or symmetric composition. This product is commutative and associative as an operation on isomorphism classes of graphs [2].

Strong product graph of $G(Z_n, \varphi)$ with $G(V_n)$

In this paper we consider the strong product graph of Euler totient Cayley graph with Arithmetic $\nu_n$ graph. The properties of strong product graph are studied by Uma Maheswari [3].

Let $G_1$ and $G_2$ be two simple graphs with their vertex sets as $V_1 = \{u_1, u_2, \ldots, u_i\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$ respectively. Then the strong product of these two graphs denoted by $G_1 \boxtimes G_2$ is defined as a graph whose vertex set is $V_1 \times V_2$ and any two distinct vertices $(u_1, v_1)$ and $(u_2, v_2)$ of $G_1 \boxtimes G_2$ are adjacent if

(i) $u_1 = u_2$ and $v_1, v_2 \in E(G_2)$, or

(ii) $u_1 u_2 \in E(G_1)$ and $v_1 = v_2$, or

(iii) $u_1 u_2 \in E(G_1)$ and $v_1 v_2 \in E(G_2)$.

Let $G_1$ denote Euler totient Cayley graph graph and $G_2$ denote Arithmetic $V_n$ graph. Then $G_1$ and $G_2$ have no loops and multiple edges. Hence by the definition of strong product, $G_1 \boxtimes G_2$ is also a simple graph. The strong product $G_1 \boxtimes G_2$ is a complete graph, if $n$ is a prime.

2 EULER TOTIENT CAYLEY GRAPH $G(Z_n, \varphi)$

For any positive integer $n$, let $Z_n$ be the additive group of integers modulo $n$ and let $S$ be the set of all numbers less than $n$ and relatively prime to $n$. That is $S = \{r/1 \leq r < n\text{ and }GCD(r, n) = 1\}$. Then $|S| = \varphi(n)$, where $\varphi$ is the Euler totient function. We can see that $S$ is a symmetric subset of the group $(Z_n, \varphi)$.

The Euler totient Cayley graph $G(Z_n, \varphi)$ is defined as the graph whose vertex set $V$ is given by $Z_n = \{0, 1, 2, \ldots, n-1\}$ and the edge set is $E = \{(x, y)/x - y \in S \text{ or } y - x \in S\}$. This graph is denoted by $G(Z_n, \varphi)$.

The matching domination parameter of these graphs is studied by the authors [4] and we require the following results and we present them without proofs.

Theorem 2.1: If $n = p$, then matching domination number of $G(Z_n, \varphi)$ is 2.

Theorem 2.2: If $n = 2p$, where $p$ is an odd prime, then matching domination number of $G(Z_n, \varphi)$ is 4.

Theorem 2.3: Let $n$ be neither a prime nor 2p and $n = \prod_{i=1}^{k} p_i^{a_i} p_1^{b_1} p_2^{b_2} \ldots p_k^{b_k}$, where $p_1, p_2, \ldots, p_k$ are
primes and \(a_1, a_2, \ldots, a_k\) are integers \(\geq 1\). Then the matching domination number of \(G(Z_n, \varphi)\) is given by

\[
y_m(G(Z_n, \varphi)) = \begin{cases} 
\lambda + 1 & \text{if } \lambda \text{ is odd,} \\
\lambda + 2 & \text{if } \lambda \text{ is even.}
\end{cases}
\]

where \(\lambda\) is the length of the longest stretch of consecutive integers in \(V\) each of which shares a prime factor with \(n\).

### 3 Arithmetic \(V_n\) Graph

Let \(n\) be a positive integer such that \(n = p_1^{a_1}p_2^{a_2} \ldots p_k^{a_k}\). Then the Arithmetic \(V_n\) graph is defined as the graph whose vertex set consists of the divisors of \(n\) and two vertices \(u\) and \(v\) are adjacent in \(V_n\) graph if and only if \(\gcd(u, v) = p_i\) for some prime divisor \(p_i\) of \(n\).

In this graph, vertex 1 becomes an isolated vertex.

In this chapter, we made an attempt to study some domination parameters of these graphs. In doing so, we have deleted the vertex 1 from the graph as the contribution of this isolated vertex is nothing, when domination parameters are enumerated.

Clearly, \(V_n\) graph is a connected graph.

The matching domination parameter of these graphs is studied by the authors [5] and we require the following results and we present them without proofs.

**Theorem 3.1:** If \(n\) is a prime, then matching domination number does not exist for the graph \(G(V_n)\). 

**Theorem 3.2:** If \(n \neq p\) and \(n = p_1^{a_1}p_2^{a_2} \ldots p_k^{a_k}\), where \(a_i \geq 1\) then the matching domination number of \(G(V_n)\) is given by

\[
y_m(G(V_n)) = \begin{cases} 
k & \text{if } k \text{ is even,} \\
k + 1 & \text{if } k \text{ is odd.}
\end{cases}
\]

where \(k\) is the core of \(n\).

### 4 Matching Domination in Strong Product Graph

In this section, we find minimum matching dominating sets of strong product graph of \(G(Z_n, \varphi)\) with \(G(V_n)\) graph and obtain its matching domination number in various cases.

**Perfect Matching**

A matching \(M\) in \(G\) is called a perfect matching if every vertex of \(G\) is incident to some edge of \(M\).

**Matching Domination**

A dominating set \(D\) of a graph \(G\) is said to be a matching dominating set if the induced subgraph \((D)\) admits a perfect matching.

The minimum cardinality of a matching dominating set is called the matching domination number of \(G\) and is denoted by \(y_m(G)\).

**Theorem 4.1:** If \(n\) is a prime, then the matching domination number of \(G_1 \boxtimes G_2\) is 2.

**Proof:** Let \(n\) be a prime. Then the graph \(G_1 \boxtimes G_2\) is a complete graph and hence every vertex is of degree \(n - 1\). So, any single vertex dominates all other vertices in \(G_1 \boxtimes G_2\). Let \(D = \{(0, p)\}\), where 0 is a vertex in \(G_1\) and \(p\) is a vertex in \(G_2\). Then every vertex in \(V - D\) is adjacent to the vertex \(((0, p))\) in \(D\).

Therefore \(y(G_1 \boxtimes G_2) = 1\).

For any \(t \in \{1, 2, \ldots, n - 1\}\) in \(G_1\), the vertex \((t, p)\) is adjacent to the vertex \((t, p)\), since the graph \(G_1 \boxtimes G_2\) is a complete graph.

Hence \(D\) is a minimal matching dominating set of \(G_1 \boxtimes G_2\).

Therefore \(y_m(G_1 \boxtimes G_2) = 2\).

**Theorem 4.2:** Let \(n \neq p\) and \(n = p_1^{a_1}p_2^{a_2}p_3^{a_3} \ldots p_k^{a_k}\), where \(a_i \geq 1\). Then the matching domination number of \(G_1 \boxtimes G_2\) is given by

\[
y_m(G_1 \boxtimes G_2) = \begin{cases} 
\frac{(\lambda + 1) \cdot (k - 1)}{\lambda \cdot k} & \text{if } a_i = 1 \text{ for more than one } i, \\
\frac{(\lambda + 1) \cdot k}{\lambda \cdot k} & \text{otherwise.}
\end{cases}
\]

if \(\lambda\) is odd and

\[
y_m(G_1 \boxtimes G_2) = \begin{cases} 
\frac{(\lambda + 2) \cdot (k - 1)}{\lambda \cdot k} & \text{if } a_i = 1 \text{ for more than one } i, \\
\frac{(\lambda + 2) \cdot k}{\lambda \cdot k} & \text{otherwise.}
\end{cases}
\]

if \(\lambda\) is even, where \(\lambda\) is the core of \(n\) and \(\lambda\) is the length of the longest stretch of consecutive integers in \(V\) of \(G_1\) each of which shares a prime factor with \(n\).

**Proof:** Suppose \(n\) is not a prime and \(n = p_1^{a_1}p_2^{a_2} \ldots p_k^{a_k}\), where \(a_i \geq 1\).

Consider the graph \(G_1 \boxtimes G_2\). Let \(V_1, V_2, V\) denote the vertex sets of \(G_1, G_2\) and \(G_1 \times G_2\) respectively.

By Theorem 2.3, we know that \(D_1 = \{u_{d_1}, u_{d_2}, \ldots, u_{d_i}\}\) is a dominating set of \(G_1\) with cardinality \(\lambda + 1\), where \(\lambda \neq 2p\). But it can be easily seen that this set also becomes a dominating set even in the case \(n = 2p\).

The following cases arise.

**Case 1:** Suppose \(a_i > 1\) for all \(i\) or \(a_i = 1\) for only one \(i\).

We know that \(D_2 = \{p_1, p_2, \ldots, p_k\}\) is a dominating set of \(G_2\) as in the Remark of Theorem 3.6.4 of Chapter 3 [6], we know that \(D_2 = \{p_1, p_2, \ldots, p_k\}\) is a dominating set of \(G_2\) with cardin-
Let $D = D_1 \times D_2 = \{u_{d_1}, u_{d_2}, \ldots, u_{d_\lambda}\} \times \{p_1, p_2, \ldots, p_k\}$

We now show that $D$ is a dominating set of $G_1 \boxtimes G_2$.

Let $(u, v)$ be any vertex in $V - D$. Then $u$ is adjacent to $u_{d_i}$ for some $i \leq \lambda + 1$, because $D_1$ is a dominating set of $G_1$. Since $G_2$ is adjacent to any vertex $p_1, p_2, \ldots, p_k$, $D_2$ is a dominating set of $G_2$. Hence by the definition of strong product, every vertex $(u, v)$ of $V - D$ is adjacent to at least one vertex $(u_{d_i}, p_m)$ in $D$.

Thus $D$ becomes a dominating set of $G_1 \boxtimes G_2$.

Suppose we delete a vertex $(u_{d_i}, p_m)$ from $D$. Let vertex $u_{d_i}$ be adjacent to the vertices $u_{d_1}, u_{d_2}, \ldots, u_{d_\lambda}$ as degree of each vertex in $G_2$ is $\varphi(n)$. Here all the vertices $u_{d_1}, u_{d_2}, \ldots, u_{d_\lambda}$ are adjacent to the vertices of $D_1 - \{u_{d_i}\}$.

If this is happened, then $D_1 - \{u_{d_i}\}$ becomes a dominating set of $G_1$, a contradiction to the minimality of $D_1$.

Therefore there is at least one vertex in $\{u_{d_1}, u_{d_2}, \ldots, u_{d_\lambda}\}$ say $u_{d_i}$, which is not adjacent to any vertex of $D_1 - \{u_{d_i}\}$.

Since $p_1, p_2, \ldots, p_k$ are distinct primes, there is no adjacency between these vertices. Then by the definition of strong product, vertex $(u_{d_i}, p_m)$ is not adjacent to any vertex of $D - \{(u_{d_i}, p_m)\}$. Hence $D - \{(u_{d_i}, p_m)\}$ is not a dominating set of $G_1 \boxtimes G_2$.

Thus $D$ becomes a dominating set of $G_1 \boxtimes G_2$ with minimum cardinality $(\lambda + 1)k$.

Now we show that $D$ is a matching dominating set of $G_1 \boxtimes G_2$.

Two cases arise.

Case (i) : Suppose $\alpha$ is odd. The vertices $u_{d_1}, u_{d_2}, \ldots$ are adjacent since they are consecutive vertices in $G_1$. Hence by the definition of strong product, the vertices $(u_{d_i}, p_1),(u_{d_i}, p_2)$ are adjacent for $i = 1, 2, \ldots, \lambda$ in $G_1 \boxtimes G_2$. Further the edges in $F = \{(u_{d_i}, p_1),(u_{d_i}, p_2), \ldots, (u_{d_i}, p_k)\}$ for $i = 1, 2, \ldots, \lambda$ are pairwise non-adjacent, since there is no edge between the primes $p_1, p_2, \ldots, p_k$. Hence $F$ forms a matching in $G_1 \boxtimes G_2$.

Now every vertex in $G_1 \boxtimes G_2$ is incident with some edge in $F$ as $D$ is a dominating set. Since $\lambda$ is odd there are an even number of edges in the above set. Thus $< F >$ admits a perfect matching and hence $< D >$ becomes a matching dominating set of $G_1 \boxtimes G_2$ with cardinality $(\lambda + 1)k$.

Hence $\gamma_m(G_1 \boxtimes G_2) = (\lambda + 1)k$.

Case (ii) : Suppose $\alpha$ is even. Then $|F|$ is an odd number. So we cannot get a perfect matching in this case. Hence to get a matching dominating set in $G_1 \boxtimes G_2$, we include a vertex $u_{d_{\lambda+1}}$ to the dominating set $D_1$ of $G_1$ such that $u_{d_{\lambda+1}}$ and $u_{d_{\lambda+1}}$ are adjacent in $G_2$.

Let $D_1 = D_1 \cup\{u_{d_{\lambda+1}}\} = \{u_{d_1}, \ldots, u_{d_{\lambda+1}}\}$. Obviously $D_1$ is a dominating set of $G_1$ as $D_1$ is a dominating set of $G_2$.

Let $D' = D_1 \times D_2 = \{u_{d_1}, \ldots, u_{d_{\lambda+1}}\} \times \{p_1, p_2, \ldots, p_k\}$.

Obviously $D'$ is a dominating set of $G_1 \boxtimes G_2$ as $D$ is a dominating set of $G_1 \boxtimes G_2$.

By using Case (i), we see that the edges in $F = \{(u_{d_1}, p_1),(u_{d_1}, p_2), \ldots, (u_{d_\lambda}, p_k)\}$ for $i = 1, 2, \ldots, \lambda + 1$ are pair wise non-adjacent. Hence this set of edges forms a matching in $G_1 \boxtimes G_2$.

Now every vertex in $G_1 \boxtimes G_2$ is incident with some edge in $F$, as $D'$ is a dominating set.

Since $\lambda$ is even there are an even number of edges in the above set $F$. Thus $< F >$ admits a perfect matching and hence $< D' >$ becomes a matching dominating set of $G_1 \boxtimes G_2$ with cardinality $(\lambda + 2)k$.

Now we claim that $D'$ is minimum. Suppose we delete a vertex $(u_{d_i}, p_k)$ from $D'$. Then $|D'|$ is an odd number which contradicts that $D'$ is a matching dominating set.

Therefore $D'$ is a minimum matching dominating set of $G_1 \boxtimes G_2$.

Hence $\gamma_m(G_1 \boxtimes G_2) = (\lambda + 2)k$.

Case 2: Suppose $\alpha = 1$ for more than one $\lambda$.

We know that $D = \{u_{d_1}, \ldots, u_{d_\lambda}\}$ is a dominating set of $G_2$.

By Case 1 in Theorem 3.6.2 and Case 1 in Theorem 3.6.5 of Chapter 3 [6], we know that $D_2 = \{p_1, p_2, \ldots, p_{\lambda+1}\}$ is a dominating set of $G_2$ with cardinality $k$.

Let $D = D_1 \times D_2 = \{u_{d_1}, u_{d_2}, \ldots, u_{d_{\lambda+1}}\} \times \{p_1, p_2, \ldots, p_{\lambda+1}\}$.

Now we show that $D$ is a minimum dominating set of $G_1 \boxtimes G_2$.

Let $(u, v)$ be any vertex in $V - D$. Then $u$ is adjacent to $u_{d_i}$ for some $i$ where $1 \leq i \leq \lambda + 1$.
The vertex $v$ in $G_2$ is adjacent to some vertex in $D_2$, say $p_m$. Hence by the definition of strong product, every vertex $(u, v)$ of $V - D$ is adjacent to at least one vertex $(u_d, p_m)$ in $D$.

Thus $D$ becomes a dominating set of $G_1 \boxtimes G_2$.

Now we show that $D$ is minimal. Suppose we delete a vertex, say $(u_{d_j}, p_l)$, from $D$. Let vertex $u_{d_j}$ be adjacent to the vertices $u_1, u_2, u_3, \ldots, u_k$ of $G_1$. Since $G_2$ is a set of prime vertices, there is no adjacency between these vertices. Then by the definition of strong product, vertex $(u_{d_j}, p_l)$ is not adjacent to any vertex of $D - \{(u_{d_j}, p_l)\}$. Hence $D - \{(u_{d_j}, p_l)\}$ is not a dominating set of $G_1 \boxtimes G_2$. Similar is the case with the deletion of any other vertex in $D$.

Thus $D$ becomes a minimal dominating set of $G_1 \boxtimes G_2$ with cardinality $(\lambda + 1)(k - 1)$. Now we show that $D$ is a matching dominating set of $G_1 \boxtimes G_2$. Two cases arise.

Case (i): Suppose $\lambda$ is odd. The vertices $u_{d_1}, u_{d_2}, \ldots$ are adjacent since they are consecutive vertices in $G_1$. Hence by the definition of strong product, the vertices $(u_{d_1}, p_l), (u_{d_2}, p_l), \ldots$ are adjacent for $i = 1, 2, \ldots, k$. Thus $D$ is a matching dominating set of $G_1 \boxtimes G_2$.

Consider the set $F$ of edges

$$\{(u_{d_i}, p_l), (u_{d_{i+1}}, p_l), \ldots, (u_{d_{k-2}}, p_l), (u_{d_{k-1}}, p_l)\}$$

for $i = 1, 3, \ldots, \lambda$ and $j = 1, 2, \ldots, i - 2, i + 1, \ldots, k$. No two edges in $F$ are adjacent since there is no edge between the primes $p_1, p_2, \ldots, p_{i-2}, p_{i+1}, \ldots, p_k$ and $\gcd(p_i, p_{i-1}, p_l) = 1$. Hence $F$ forms a matching in $G_1 \boxtimes G_2$.

Now every vertex in $G_1 \boxtimes G_2$ is incident with some edge in $F$, as $D$ is a dominating set. Since $\lambda$ is odd, there is an even number of edges in $F$. Thus $< F >$ admits a perfect matching and hence $< D >$ becomes a matching dominating set of $G_1 \boxtimes G_2$ with cardinality $(\lambda + 1)(k - 1)$. Hence $\gamma_m(G_1 \boxtimes G_2) = (\lambda + 1)(k - 1)$.

Case (ii): Suppose $\lambda$ is even. In similar lines to Case (ii) of Case (i), we include vertex $u_{d_{k-1}}, \ldots, u_{d_1}$, which is adjacent to $u_{d_{k-1}}, \ldots, u_{d_1}$, to the dominating set $D_2$ of $G_2$.

Let $D'_2 = D_2 \cup \{u_{d_{k-1}}, \ldots, u_{d_1}\}$. Obviously $D'_2$ is a dominating set of $G_2$ such that $D'_2$ is a dominating set of $G_1 \boxtimes G_2$.

Let $F = D'_2 \times D = \{u_{d_i}, \ldots, u_{d_{k-1}}\} \times \{p_1, p_2, \ldots, p_l, p_{l-1}, \ldots, p_k\}$. Obviously $F$ is a dominating set of $G_1 \boxtimes G_2$ as $D$ is a dominating set of $G_1 \boxtimes G_2$.

Now we claim that $D'$ is minimal. Suppose we delete a vertex $(u_{d_i}, p_l)$ from $D$. Then $|D'|$ is an odd number which contradicts that $D$ is a matching dominating set.

Thus $D'$ is minimal matching dominating set of $G_1 \boxtimes G_2$ with cardinality $(\lambda + 2)(k - 1)$. Hence $\gamma_m(G_1 \boxtimes G_2) = (\lambda + 2)(k - 1)$.
### 6 REFERENCES