Magnetohydrodynamic Couette Flow Of A Non-Newtonian Fluid In A Rotating System With Heat And Mass Transfer

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Abstract— An analytical solution is obtained for the steady Couette flow of an electrically conducting fluid between two parallel plates when the fluid and the plates rotate in unison about an axis normal to the plates. The equations of motion and energy have been solved with the help of complex variable technique pertaining to the imposed boundary conditions. After necessary computation with the various numerical values of the fluid parameters the expressions for primary velocity, secondary velocity of flow and temperature have been obtained. Then, the expressions for shear stresses and the rates of heat transfer have been derived. The results obtained are discussed with the help of graphs and tables to observe the effects of various parameters. It has been found that the primary velocity decreases with the increase of rotation parameter whereas the secondary velocity rises in reverse direction. The increase in the value of Prandtl number increases the rate of heat transfer at the stationary plate but reduces at the moving plate. These results are in good agreement with earlier results.

Index Terms—MHD, Couette flow, Non-Newtonian fluid, Rotating, Heat transfer, Mass transfer

1 Introduction

The flow between two parallel plates with one plate moving (Couette flow) is of general interest. Pai[1], Lehnert[2], Bleviss[3] have considered the magnetohydrodynamic Couette flow and heat transfer. However, Couette flow in a rotating frame of reference does not seem to have received much attention. Such a study will have some bearing in MHD power generator, in cooling turbine blades, etc. Jana and Datta[4] have considered the hydrodynamic Couette flow in a rotating frame of reference. The problem of unsteady Couette flow of an incompressible viscous liquid between two plates occurring due to the sudden motion of one of the plates has already been studied by Pai[5]. The same flow through a porous channel, has been investigated by Nanda[6], Katagiri[7], Muhuri[8] and Rath et. al[9] have extended the above problem with heat transfer in case of unsteady Couette flow between two parallel walls having different temperature. Mishra[10] has analysed the plane Couette flow of Oldroyd fluid with suction or injection at the stationary wall. Further, Soundalgekar[11] has discussed the plane Couette flow of Walters B’ liquid with equal rate of injection at one wall and suction at the other (moving wall). The commencement of unsteady Couette flow in case of second order liquid has been analysed by Padhy[12].
Dash and Biswal[13] have investigated the problem of commencement of Couette flow in Oldroyd liquid through a porous channel in the presence of heat sources.

From technological view point, the study of both Newtonian and non-Newtonian Couette flow problem in the presence of a uniform transverse magnetic field is very important. Consequently, the literature is replete with copious instances of such investigations on MHD Couette flows. The object of the present study is to extend the above investigation in the hydromagnetic case of a non-Newtonian fluid in a rotating system with heat and mass transfer.

2 Formulation of the Problem

Consider the steady Couette flow of an electrically conducting fluid between two parallel plates when the fluid and the plates rotate in unison about an axis normal to the plates. A uniform magnetic flux $B_0$ acts normal to the plates parallel to the $y$-axis. The $xz$ – plane coincides with the stationary plate and the plate $y = d$ moves with a uniform velocity $U_0$ in the $x$-direction.

In a rotating frame of reference, the equation of continuity and the momentum equation are

$$\nabla \cdot \bar{q} = 0,$$

$$\bar{q} \cdot \nabla \bar{q} + 2\Omega \hat{k} \times \bar{q} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla^2 \bar{q} + \frac{k_0}{\rho} \nabla^2 \bar{q} + \frac{1}{\rho} \bar{J} \times \bar{B},$$

(2.2)

Maxwell’s equations along with Ohm’s law are

$$\nabla \bar{B} = 0$$

(2.3)

$$\nabla \times \bar{B} = \mu_e \bar{J},$$

(2.4)

$$\nabla \times \bar{E} = 0,$$

(2.5)

where $\bar{q}$ is the velocity vector, $\bar{B}$ is the magnetic induction vector, $\bar{J}$ is the current density, $\bar{E}$ is the electric field relative to the rotating frame, $\rho$ is the fluid density, $v$ is the kinematic coefficient of viscosity, $\mu_e$ is the magnetic electrical conductivity of the fluid $\Omega$ is the angular velocity of rotation.
Since the plates are infinite along x and z direction, all physical quantities will be functions of y only. It may be easily shown that the following assumptions are compatible with the equations (2.1) – (2.5).

\[
\bar{q} = (u^*, 0, w^*) ; \quad \bar{B} = (B_x^*, B_0, B_z^*) \quad \bar{E} = (E_x^*, E_y^*, E_z^*) ; \quad \bar{J} = (J_x^*, 0, J_z^*)
\]

(2.7)

It follows from equation (2.5) that \(E_x^*\) and \(E_z^*\) are constants. The velocity components given by (2.7) are consistent with the fact that rotations induce a cross-flow \(w^*(y)\) as shown by Batchelor[14]. It may be seen that the assumptions (2.7), equations (2.1), (2.3) and (2.5) are satisfied automatically while equations (2.2), (2.4) and (2.6) give

\[
\begin{align*}
J_x^* &= \sigma \left( E_x^* - B_0 w^* \right), \\
O &= E_y^* + (B_x^* w^* - B_y^* u^*), \\
J_z^* &= \sigma \left( E_z^* + B_0 u^* \right)
\end{align*}
\]

(2.13)  (2.14)  (2.15)

In equations (2.8) and (2.10), the pressure gradient terms are neglected since the motion is supported by the movement of the upper plate. However, due to rotation; a pressure gradient is induced along y-axis.

Introducing non-dimensional variables,

\[
\begin{align*}
\eta &= \frac{y}{d} , \\
P &= \frac{P}{\rho U_0^2} , \\
(u, w) &= \left( \frac{u^*, 0, w^*}{U_0} \right) , \\
(J_x, J_z) &= \left( \frac{J_x^*, 0, J_z^*}{\sigma U_0 B_0} \right) , \\
(b_x, b_z) &= \left( \frac{B_x^*, B_0, B_z^*}{B_0} \right) , \\
(E_x, E_y, E_z) &= \left( \frac{E_x^*, E_y^*, E_z^*}{U_0 B_0} \right)
\end{align*}
\]

(2.16)

Equations (2.8) – (2.15) become

\[
\begin{align*}
\frac{d^2 u}{d\eta^2} - R \frac{d^3 u}{d\eta^3} - 2a^2 w - M^2 J_z &= 0, \\
0 &= - R \frac{\partial p}{\partial \eta} + M^2 (b_x J_z - b_z J_x) , \\
\frac{d^2 w}{d\eta^2} - R \frac{d^3 w}{d\eta^3} + 2a^2 u + M^2 J_x &= 0 ,
\end{align*}
\]

(2.17)  (2.18)  (2.19)
\[ R_m J_x = \frac{db_x}{d\eta}, \]

(2.20)

\[ R_m J_z = -\frac{db_z}{d\eta}, \]

(2.21)

\[ J_z = E_x - w, \]

(2.22)

\[ 0 = E_y + (b_x w - ub_z), \]

(2.23)

\[ J_x = E_z + u, \]

(2.24)

Where \( M = B_0 d \left( \sigma / \rho \nu \right)^{1/2} \), the Hartmann number.

\[ R = \frac{U_0 d}{\nu} \] is the Reynolds number,

\[ R_m = \sigma \mu_e U_0 d \] is the magnetic Reynolds number,

\[ R_e = \frac{K_0 U_0^2}{\eta_0 \nu} \] is the elastic parameter,

\[ a^2 = \Omega d^2 / \nu \] is the rotation parameter.

The boundary conditions for the velocity field are

\[ w(0) = u(0) = 0 \text{ and } w(1) = 0, u(1) = 1, \]

(2.25)

Combining equations (2.17), (2.20), and (2.22) with equations (2.19), (2.21) and (2.24) respectively, we get

\[ \frac{d^2 F}{d\eta^2} - R_e \frac{d^3 F}{d\eta^3} + 2ia^2 F - M^2 J = 0, \]

(2.26)

\[ R_m J = -\frac{db}{d\eta}, \]

(2.27)

\[ J = E_0 + F, \]

(2.28)

where

\[ F = u + iw, \quad J = J_z - iJ_x, \]

\[ E_0 = E_z - iE_x, \quad b = b_x + ib_z, \]

(2.29)

Eliminating \( J \) from equations (2.26), (2.27) and (2.28), we have

\[ \frac{d^2 F}{d\eta^2} - R_e \frac{d^3 F}{d\eta^3} + (M^2 - 2ia^2) F = M^2 E_0 \]

(2.30)

\[ R_m (E_0 + F) = -\frac{db}{d\eta}, \]

(2.31)

Using (2.29), the boundary conditions (2.25) become

\[ F(0) = 0 \text{ and } F(1) = 1 \]

(2.32)

3 Solutions of the Equations

Now, we assume that the applied electric field components \( E_x \) and \( E_z \) are zero, so that \( E_0 = 0 \). This assumption is justified if we consider

\[ b(1) = 0 \text{ and } \frac{db(0)}{d\eta} = 0 \]

(3.1)

Putting \( E_0 = 0 \) in equation (2.30), the equation becomes
\[
\frac{d^2 F}{d\eta^2} - R_c \frac{d^3 F}{d\eta^3} - (M^2 - 2ia^2) F = 0,
\]
(3.2)

or
\[
R_c \frac{d^3 F}{d\eta^3} - \frac{d^2 F}{d\eta^2} + (M^2 - 2ia^2) F = 0,
\]
(3.3)

Solving equation (3.3), we get
\[
F = -\frac{1}{2} C_1 \eta^2 + C_2 \eta + C_3 \frac{R_c}{R_c - \eta + A_1 \eta^3}
\]
(3.4)

where
\[
A_1 = \frac{M^2 - 2ia^2}{6}
\]

Applying the boundary conditions (2.32) to (3.4), we have
\[
F = \frac{-\frac{1}{2} A_6 \eta^2 - \frac{1}{R_c} \eta}{R_c - \eta + A_1 \eta^3}
\]
(3.5)

Solving equation (3.5) and separating real and imaginary parts, we get
\[
F = u + iw
\]

where
\[
u = \frac{A_7 A_8 + 4a^4 R_c^2 \eta^5}{A_8^2 + 4a^4 R_c^2 \eta^6}
\]

and
\[
w = \frac{2a^2 R_c \eta^3 A_7 - 2a^2 R_c \eta^2 A_8}{A_8^2 + 4a^4 R_c^2 \eta^6}
\]

Skin friction:

The expressions for skin-frictions at the lower plate and upper plate for primary flow are given by
\[ +192 A_y^3 a^{10} R_c^5 + 960 A_y^3 a^{10} R_c^6 - 384 A_y a^{14} R_c^7 \\
+1536 A_y a^{14} R_c^8 + 512 A_y a^{14} R_c^7 + 12 A_y^6 a^2 R_c^2 \\
-384 A_y A_y^3 a^6 R_c^4 + 384 A_y^5 a^6 R_c^5 - 4 A_y a^2 R_c^2 \\
\left( A_y^3 + 16 a^8 R_c^4 + 8 A_y^2 a^4 R_c^2 \right)^2 \]

(3.39b)

**Heat transfer:**

For the fully developed steady flow discussed above, the energy equation is

\[ 0 = \alpha^* \frac{d^2 T}{dy^2} + \frac{\mu}{\rho C_p} \left( \left( \frac{du^*}{dy} \right)^2 + \left( \frac{dv^*}{dy} \right)^2 \right) \]

\[ + \frac{1}{\mu^2 \rho \sigma C_p} \left( \left( \frac{dB_x^*}{dy} \right)^2 + \left( \frac{dB_y^*}{dy} \right)^2 \right) \]

(3.40)

Where \( C_p \) the specific heat at constant pressure and \( \alpha^* \) is the thermal conductivity. The last two terms within parameters are the viscous dissipation and Joule heating term respectively.

The boundary conditions for \( T \) are taken as

\[ T = T_0 \quad \text{at} \quad y = 0 \quad \text{and} \quad T = T_1 \quad \text{at} \quad y = d \]

(3.41)

Where \( T_0 \) and \( T_1 \) \((T_1 > T_0)\) denote the uniform temperature of the stationary and the moving plate respectively.

Introducing

\[ \theta(\eta) = \frac{(T - T_0)}{(T_1 - T_0)} , \quad E = \frac{U_0^2}{C_p(T_1 - T_0)} , \quad P_r = \frac{\nu}{\alpha} \]

(3.42)

In equation (3.40), we get on using equation (2.16)

\[ \frac{d^2 \theta}{d\eta^2} + P_r E \left[ \left( \frac{du}{d\eta} \right)^2 + \left( \frac{dv}{d\eta} \right)^2 \right] + \frac{M^2}{R_m^2} \left[ \left( \frac{db_x}{d\eta} \right)^2 + \left( \frac{db_y}{d\eta} \right)^2 \right] = 0 \]

(3.43)

The boundary conditions (3.41) become

\[ \theta(0) = 0 \quad \text{and} \quad \theta(1) = 1 \]

(3.44)

Substituting the values of \( u(\eta) \), \( w(\eta) \), \( bx(\eta) \) and \( bz(\eta) \) from equation (3.5) and simplifying we obtain the value of \( \theta \).

\[ \therefore \quad \text{Nu}_0 = -\frac{d\theta}{d\eta}_{\eta=0} = C_1 \]

(3.45)

**Mass transfer:**

The equation of concentration is given by

\[ V \frac{dc}{dy} = D \frac{d^2 c}{dy^2} \]

(3.46)

Where \( D \) is the chemical molecular diffusivity.

With the boundary conditions

\[ Y' = 0, C' = C_0 \]
\[ Y' = d, C' = C_1 \]

(3.47)

Introducing the non-dimensional parameter

\[ C = \frac{C - C_0}{C_1 - C_0} \]

(3.48)
In equation (3.46), we get the non-dimensional form of the concentration equation as

\[
\frac{d^2 C}{d\eta^2} + RS_c \frac{dC}{d\eta} = 0
\]

(3.49)

Where \( S_c = \frac{v}{D} \), Schmidt parameter

With the modified boundary condition

When \( \eta = 0 \), \( C = 0 \)
\[
\eta = 1, \quad C = 1
\]

(3.50)

Solving the equation (3.49), we have

\[
C = \frac{1 - e^{RS_c (1-\eta)}}{1 - e^{RS_c}}
\]

(3.51)

Concentration gradient

\[
CG_i = -\frac{dC}{d\eta} \quad \text{(where } i=1,2)\]

\[
CG_1 = -\frac{dC}{d\eta} \bigg|_{\eta=0} = - \left( \frac{RS_c}{1 - e^{RS_c}} \right) e^{RS_c}
\]

\[
CG_2 = -\frac{dC}{d\eta} \bigg|_{\eta=1} = - \left( \frac{RS_c}{1 - e^{RS_c}} \right)
\]

4 Results and discussion

In the present investigation, magnetohydrodynamic Couette flow a non-Newtonian fluid in a rotating system with heat and mass transfer has been studied. The effects of various fluid parameters like the Hartmann number \( (M) \), Reynolds number \( (R) \), magnetic Reynolds number \( (R_m) \), elastic parameter \( (R_e) \) and the rotation parameter \( (a) \) have been displayed through graphs and tables. The new findings have been explained below.

Here, we have extended the problem of Jana, Datta and Majumdar, who have analysed the case of magnetohydrodynamic Couette flow of viscous fluid and heat transfer in a rotating system. They have not taken into consideration the case of non-Newtonian fluid. We have studied the effects of visco-elastic parameter \( (R_e) \) on the flow pattern.
Skin friction :

Table -1

Values of the skin-frictions & $\tau_s$ for $E = 0.001$ & $P = 2.0$

<table>
<thead>
<tr>
<th>$M^2$</th>
<th>R</th>
<th>$R_c$</th>
<th>$R_m$</th>
<th>$a^2$</th>
<th>$\tau^*_p$</th>
<th>$\tau_p$</th>
<th>$\tau^*_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>9</td>
<td>0.05</td>
<td>0.2</td>
<td>1</td>
<td>-2.2935</td>
<td>-0.6667</td>
<td>-0.4379</td>
</tr>
<tr>
<td>10</td>
<td>9</td>
<td>0.05</td>
<td>0.2</td>
<td>1</td>
<td>-3.7203</td>
<td>-0.6667</td>
<td>0.5581</td>
</tr>
<tr>
<td>10</td>
<td>12</td>
<td>0.05</td>
<td>0.2</td>
<td>1</td>
<td>-3.7203</td>
<td>-0.6667</td>
<td>0.5581</td>
</tr>
<tr>
<td>10</td>
<td>12</td>
<td>0.10</td>
<td>0.2</td>
<td>1</td>
<td>-1.6430</td>
<td>-0.6667</td>
<td>0.5685</td>
</tr>
<tr>
<td>10</td>
<td>12</td>
<td>0.10</td>
<td>0.4</td>
<td>1</td>
<td>-1.6430</td>
<td>-0.6667</td>
<td>0.5685</td>
</tr>
<tr>
<td>10</td>
<td>12</td>
<td>0.10</td>
<td>0.4</td>
<td>2</td>
<td>-1.8862</td>
<td>-1.3333</td>
<td>0.7238</td>
</tr>
</tbody>
</table>

The values of the skin-friction for the various values of the fluid parameters have been entered in the table-1. It is seen that there is no primary skin-friction at the fixed plate i.e. $\tau_p = 0$ at $\eta = 0$. But the primary skin-friction decreases with the rise of Hartmann number ($M$), at the movable plate ($\eta = 1.0$). Increase in Reynolds number does not vary the primary skin-friction at the movable plate. Increase in elastic parameter increases $\tau^*_p$ at $\eta = 1.0$. Magnetic Reynolds number does not change the value of $\tau^*_p$. However, the increase in rotational parameter reduces the value of primary skin-friction at the movable plate.

Secondary skin-friction $\tau_s$ at the fixed plate does not change with the change of $M^2$, $R$, $R_c$ and $R_m$, but varies with the rotational parameter ($a^2$). It is observed that increase in the value of rotational parameter reduces the secondary skin-friction at the fixed plate.

Secondary skin-friction at the movable plates rises with $M^2$, but does not vary with $R$, $R_c$ and $R_m$. However, $\tau^*_s$ increases with the increase of the rotation parameter ($a^2$)
Rate of Heat Transfer:

Table 2: Values of the rate of heat transfer $Nu_0$ & $Nu_1$

<table>
<thead>
<tr>
<th>$Pr$</th>
<th>$E$</th>
<th>$M^2$</th>
<th>$a^2$</th>
<th>$Nu_0$</th>
<th>$Nu_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.001</td>
<td>5</td>
<td>1</td>
<td>2.42495</td>
<td>0.87195</td>
</tr>
<tr>
<td>4</td>
<td>0.001</td>
<td>5</td>
<td>1</td>
<td>3.26639</td>
<td>0.43995</td>
</tr>
<tr>
<td>4</td>
<td>0.002</td>
<td>5</td>
<td>1</td>
<td>4.00003</td>
<td>0.23810</td>
</tr>
<tr>
<td>4</td>
<td>0.002</td>
<td>10</td>
<td>1</td>
<td>4.00003</td>
<td>0.23810</td>
</tr>
</tbody>
</table>

Various values of the rate of heat transfer $Nu_0$ and $Nu_1$ have been entered in the table-2. It is marked that the increase in the value of Prandtl number ($Pr$) increases $Nu_0$ and decreases $Nu_1$. The same result is obtained for Hartmann number ($M$) and the rotation parameter ($a^2$). But the rise in Eckert number raises $Nu_0$ and reduces $Nu_1$.

Concentration gradient:

Table 3: Effect of $Sc$ on concentration gradient for $R=5.0$, $R_c=0.05$, $G_r=5.0$, $G_c=2.0$, $P=2.0$, $S=0.5$, $M=5.0$, $K^*=1$

<table>
<thead>
<tr>
<th>$Sc$</th>
<th>2.13</th>
<th>2.5</th>
<th>3.0</th>
<th>5.0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$CG_1$</td>
<td>1.0650250E+01</td>
<td>1.2500050E+01</td>
<td>1.5000000E+01</td>
</tr>
<tr>
<td></td>
<td>$CG_2$</td>
<td>2.5241980E-04</td>
<td>4.6583340E-05</td>
<td>4.5885360E-06</td>
</tr>
</tbody>
</table>

Table 3 illustrates the behaviour of the concentration gradients $CG_1$ and $CG_2$ at the lower and upper walls of the channel respectively with the increase in the value of Schmidt number $Sc$. It is marked that $CG_1$ increases and $CG_2$ decreases with $Sc$ keeping other parameters of the fluid constant.
Fig. 1 explains the effects of Hartmann number ($M^2$) and non-Newtonian parameter ($R_c$) on the primary velocity ($u$). It is observed that the primary velocity increases with $M^2$ up to the distance $\eta = 0.45$ from the fixed plate in $xz$-plane as the moving plate advances in the $x$-direction. After $\eta = 0.45$, the primary velocity of flow decreases for $M^2 = 15$. Likewise, beyond $\eta = 0.57$, the primary velocity $u$ falls for $M^2 = 10$. At comparatively large distance from the fixed plates the velocity $u$ attains negative values. Increase in the value of $R_c$, the primary velocity $u$ reduces and attains negative values beyond $\eta = 0.75$ from the fixed plate of the channel (curve IV).
Fig. 2 shows the effects $M^2$ and $R_c$ on the secondary velocity $w$ for fixed values of other variables. It is seen that the secondary velocity attains negative values at all points from the fixed plate. As the magnetic parameter increases, the secondary velocity rises in the reverse direction. But the rise in the elastic parameter $R_c$ reduces the secondary velocity appreciably.
The effects of rotation parameter $a^2$ on the primary velocity $u$ have been displayed in Fig.3. It is observed that primary velocity $u$ decreases with the increase of the rotation parameter at distances beyond $\eta = 0.1$ from the fixed plate and at $\eta = 1.0$, there is no change in primary velocity $u$ for any values of the rotation parameter.
Fig. 4 presents the effects of the rotation parameter $a^2$ on the secondary velocity $w$, keeping all the fluid parameters fixed. It is observed that the increase in rotation parameter raises the negative values of the secondary velocity. But, the peak values of $w$ always leans towards the movable plate which is at a distance $\eta = 0.1$ from the fixed plate.
The temperature profiles have been presented in Fig. 5 to glean the effects of Prandtl number \((P_r)\) and Eckert number \((E)\) while fixing the values of the other fluid parameters like the rotation parameter \(a^2\) and Hartmann number \(M\). It is seen that the rise in the values of raises the temperature of the fluid. The increase in Prandtl number further raises the value of \(\theta\). However, at a distance \(\eta=1.0\), the temperature attains a fixed value for both Eckert number and Prandtl number i.e. steady thermal state is reached for the fluid flow.
Fig. 6 shows the effects of rotation parameter $a^2$ and magnetic parameter $M$ on the temperature of the fluid. It is observed that the rise in $a^2$ lowers the temperature. Same effect is also marked for the rise of $M$. Again a steady state of temperature is reached at a distance $\eta = 1.0$ from the fixed plate.
The effect of Schmidt number $S_c$ on the concentration $C$ have been illustrated in Fig. 7. It is observed that the rise in the values of $S_c$ decreases the concentration which attains negative values for $S_c=5.0$ (Curve IV) in between $\eta = 0.2$ and 0.4.
5 Conclusions

Above discussions on the flow and temperature profiles illuminate the following new findings of MHD Coquette flow of a non-Newtonian fluid in a rotating system.

(i) Increase in the value of non-Newtonian parameter reduces the primary velocity \( (u) \) attaining negative values beyond \( \eta = 0.75 \).

(ii) The rise in magnetic parameter raises the secondary velocity in the reverse direction.

(iii) The primary velocity decreases with the increase of rotation parameter.

(iv) The increase in rotation parameter raises the secondary velocity in reverse direction.

(v) The increase in Prandtl number increases the temperature \( \theta \), while the rise in the rotation parameter lowers the value of temperature.

(vi) Magnetic Reynolds number does not change the value of primary skin-friction at the movable plate.

(vii) Increase in the value of rotational parameter reduces the secondary skin-friction at the fixed plate. However, the secondary skin-friction at the movable plate increases with the increase of the rotation parameter \( (a^2) \).

(viii) The increase in the value of Prandtl number \( (P_r) \) increases the rate of heat transfer \( N u_0 \) but reduces \( N u_1 \).

References


