Multi-Stage Retrial Queueing System with Bernoulli Feedback

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Abstract — An M/G/1 retrial queueing system with k stages of heterogeneous services and feedback is considered. Primary customers get into the system according to Poisson process. If the server is free, an arriving customer receives first stage service immediately otherwise, he enters a retrial orbit. After the completion of the first stage, the customer may proceed to second stage with probability θ₁, or feedback to the retrial group with probability p₁, or depart the system with probability 1 – θ₁ – p₁ = q₁. In general, after the completion of the ith stage, (i = 1, 2, ..., k−1) stage, the customer may opt (i+1)th stage with probability θᵢ, or feedback to the retrial group with probability pᵢ or depart the system with probability 1 – θᵢ – pᵢ = qᵢ. The customer in final stage will feedback to the retrial group with probability p_k or depart the system with probability 1 – p_k = q_k. It is assumed that the service times and the retrial times are arbitrarily distributed. The condition under which the steady state exists is investigated. Performance measures are obtained. A stochastic decomposition is presented.

Key Words — Feedback, Heterogeneous Service, Multi Stage, Retrial Queue, Stochastic Decomposition.

1 INTRODUCTION

The retrial queueing system has been studied extensively due to its wide applicability in telephone switching system, telecommunication and computer networks. These systems are characterized by the feature that arrivals who find the server busy join the retrial queue (orbit) to try again for their requests or leave the service area immediately. For comprehensive survey on retrial queues refer [1, 2, 4] and references therein.

Recently considerable attention has been devoted to the queueing system with two or more stages of heterogeneous service. Choudhury and Paul [3] have inspected the M/G/1 system with two stages of heterogeneous service and Bernoulli feedback.

Shahkar and Badamchizadeh [7] have studied a single server general service queue with Poisson input and k-stages of service. Salehirad and Badamchizadeh [6] have analysed multi stage M/G/1 queueing system with feedback.

In many examples such as production system, bank services, computer and communication networks, the service of customers may be repeated. In this paper with this motivation a single server retrial queue with Poisson input, k stages of heterogeneous service and feedback is analyzed.

2 MODEL DESCRIPTION

Assume that the customers arrive at the system in accordance with a Poisson process with rate λ. If an arriving customer finds the server idle, the customer enters the service immediately for first stage service. If the server is found to be blocked, the arriving customer enters a retrial group. The retrial time of customer is generally distributed with distribution function A(s) and Laplace Stieltjes transform

\[ A^*(s). \]

The server provides k stages of heterogeneous service in succession. The service discipline is assumed to be first come first served. Service time of ith stage is denoted by the random variable Bᵢ, having Laplace Stieljes transform \( B_i^*(s) \) and first two moments \( \mu_{1i} \) and \( \mu_{2i}, i = 1, 2, \ldots, k \).

After completion the ith stage the customer may move to (i+1)th stage with probability θᵢ or feedback to the retrial queue with probability pᵢ or depart the system with probability qᵢ = 1 – θᵢ – pᵢ for i = 1, 2, ..., k−1. The customer in the final stage k may feedback to the retrial queue with probability p_k or depart the system with complementary probability. According to the model, the time required by a customer to complete the service cycle is a random variable B given by

\[ B = \sum_{i=1}^{k} \Theta_{i-1} B_i \]

having Laplace Stieltjes transform \( B^*(s) = \prod_{i=1}^{k} \Theta_{i-1} B_i^*(s) \) and expected value

\[ E(B) = \sum_{i=1}^{k} \Theta_{i-1} E(B_i) \]

where \( \Theta_t = \theta_t, \theta_{t-1}, \ldots, \theta_0 = 1 \).

The functions \( \eta(x) = \frac{dA(x)}{1-A(x)} \) and \( \mu_i(x) = \frac{dB_i(x)}{1-B_i(x)} \),

\[ i = 1, 2, \ldots, k \]

are the conditional completion rates (at time x) for repeated attempts and for services.

Define \( \Lambda_i = B_1^* B_2^* \ldots B_i^* \) and \( \Lambda_0^* = 0 \).

The first moment \( M_{1i} \) of \( \Lambda_i^* \) is given by

\[ M_{1i} = \lim_{z \to 1} dz \int \frac{1}{(1-z)^{1-j}} d \Lambda_i^* \] for i = 1, 2, ..., k

The second moment \( M_{2i} \) of \( \Lambda_i^* \) is

\[ M_{2i} = \lim_{z \to 1} d^2 \int \frac{1}{(1-z)^{1-j}} dz^2 \] for i = 1, 2, ..., k.
\[ \psi_j = \sum_{i=1}^{k} \Theta_{i-1} M_{1i} - \sum_{i=1}^{k-1} \Theta_{i} M_{1i} + \sum_{i=1}^{k} p_i \Theta_{i-1} - \Lambda'(\lambda) \] for all \( j \geq 1 \), where \( j \) denotes the number of customers in the orbit, we have the reasonable conclusion.

The term \( \sum_{i=1}^{k} \Theta_{i-1} M_{1i} - \sum_{i=1}^{k-1} \Theta_{i} M_{1i} \) represents the mean number of customers entering orbit due to the server being busy with \( i \) stage service, \( i = 1, 2, \ldots, k \). The second term, \( \sum_{i=1}^{k} p_i \Theta_{i-1} \) is arrival due to feedback. Further \( \Lambda'(\lambda) \) provides the expected number of orbiting customers who enter service successfully. For stability the new customers arrive during a service time more slowly than customers from the orbit who enters service successfully. That is \( \sum_{i=1}^{k} \Theta_{i-1} M_{1i} - \sum_{i=1}^{k-1} \Theta_{i} M_{1i} + \sum_{i=1}^{k} p_i \Theta_{i-1} < \Lambda'(\lambda) \) implying \( \psi_j < 0 \) for \( j \geq 1 \).

### 4 Steady State Distribution

For the process, \( \{N(t) ; t \geq 0\} \), define the probabilities,
\[
I_0(t) = P\{J(t) = 0, X(t) = 0\} \\
I_n(t, x) = P\{J(t) = i, X(t) = n, x < \xi_0(t) < x + dx\}, \quad n \geq 1 \\
W_{n}(t, x) = P\{J(t) = i, X(t) = n, x < \xi_0(t) < x + dx\}, \quad n \geq 0 ; i = 1, 2, \ldots, k
\]
Let \( I_0, I_n(x) \) and \( W_{n}(x) \) are the limiting densities of \( I_0(t), I_n(t, x) \) and \( W_{n}(t, x) \).

### 5 Steady State Probability Generating Function

The steady state equations for the model under consideration are,
\[
\lambda = \sum_{i=1}^{k} q_i \int_{0}^{\infty} \mu_i(x) W_{0}(x) \, dx + (1 - p_k) \int_{0}^{\infty} \mu_k(x) W_{0}(x) \, dx
\]
\[
\frac{dI_n(x)}{dx} = -(\lambda + \eta(x)) I_n(x), \quad n \geq 1
\]
\[
\frac{dW_{0,i}(x)}{dx} = -(\lambda + \mu_i(x)) W_{0}(x), \quad i = 1, 2, \ldots, k
\]
\[
\frac{dW_{n,i}(x)}{dx} = -(\lambda + \mu_i(x)) W_{n}(x) + \lambda W_{n-1,i}(x), \quad n \geq 1, i = 1, 2, \ldots, k
\]
with boundary conditions
\[
I_0(0) = \sum_{i=1}^{k} q_i \int_{0}^{\infty} \mu_i(x) W_{0}(x) \, dx + (1 - p_k) \int_{0}^{\infty} \mu_k(x) W_{0}(x) \, dx
\]
\[
+ \sum_{i=1}^{k} p_i \int_{0}^{\infty} \mu_i(x) W_{n-1,i}(x) \, dx, \quad n \geq 1
\]
\[ W_{0,1}(x) = \lambda W_0 + \int_0^\infty W_1(x) \eta(x) \, dx \] (6)

\[ W_{n,1}(x) = \lambda \int_0^\infty W_n(x) \, dx + \int_0^\infty W_{n+1}(x) \eta(x) \, dx, \quad n \geq 1 \] (7)

\[ W_{n,1}(x) = \theta_{n-1} \int_0^\infty \mu_{n-1}(x) \, W_{n-1}(x) \, dx, \quad n \geq 0, \quad i = 2, 3, \ldots, k \] (8)

The normalizing condition is given by the equation

\[ \int_0^\infty \int_0^\infty W_n(x) \, dx + \sum_{n=0}^\infty \int W_{n,1}(x) \, dx = 1 \] (9)

Define probability generating functions,

\[ I(z, x) = \sum_{n=0}^\infty W_{n,1}(x) z^n \] and

\[ W_1(x) = \sum_{n=0}^\infty W_{n,1}(x) z^n, \quad i = 1, 2, 3, \ldots, k \]

Multiplying the equations (1) - (5) with suitable powers of \( z \), taking the sum and solving the differential equations so obtained, we get

\[ I(z, x) = I(0, z) e^{-\lambda x} \left[ 1 - A(x) \right] \] (10)

\[ W_i(z, x) = \Theta_i \int_0^\infty \mu_i(\lambda) \, W_{i-1}(x) \, dx \sum_{i=1}^\infty \lambda^i \left[ 1 - A(x) \right] \] (11)

\[ I(z, x) = \sum_{i=1}^k \left[ \lambda + (1 - z) A_i \right] W_i(z, x) + \sum_{i=1}^k \left[ \lambda + (1 - z) A_i \right] W_i(z, x) = 1 \] (12)

Using equations (10) and (12), the equations (6) and (7) yield

\[ W_i(z, 0) = \sum_{i=1}^k \left[ \lambda + (1 - z) A_i \right] W_i(z, 0) + \sum_{i=1}^k \left[ \lambda + (1 - z) A_i \right] W_i(z, 0) \] (13)

and equation (8) yields

\[ W_i(z, 0) = \Theta_i \int_0^\infty \mu_i(\lambda) \, W_{i-1}(x) \, dx \sum_{i=1}^k \lambda^i \left[ 1 - A(x) \right] \] (14)

Substituting the expressions in equation (13) in to equation (14), and solving we get

\[ W_i(z, 0) = \Theta_i \int_0^\infty \mu_i(\lambda) \, W_{i-1}(x) \, dx \sum_{i=1}^k \lambda^i \left[ 1 - A(x) \right] \] (15)

where \( D(z) = \sum_{i=1}^k \left[ \lambda + (1 - z) A_i \right] \) \( \sum_{i=1}^k \lambda^i \left[ 1 - A(x) \right] \)

Substituting the expression of \( W_i(z, 0) \), \( i = 1, 2, \ldots, k \) in equation (12), we have

\[ I(z, 0) = \lambda I_0 + \left[ \sum_{i=1}^k \left[ \lambda + (1 - z) A_i \right] \right] \Theta_i \int_0^\infty A_i \left[ 1 - (1 - z) \right] ] / D(z) \] (16)

The partial generating function of the orbit size when the server is busy in providing \( n \)-th stage service is given by,

\[ W_i(z) = \int_0^\infty W_i(z, x) \, dx \]

\[ = \frac{I_0 \lambda^i \Theta_i \left[ 1 - A_i \right] (1 - z)}{D(z)} \] (17)

The steady state probability that the server is in \( i \)-th stage service, is

\[ W_i(1) = \frac{\Theta_i}{T_i} \] (18)

Substituting the expressions of (1) and \( W_i(1) \) for \( i = 1, 2, \ldots, k \) in the normalizing condition \( I_0 + I(1) + \sum_{i=1}^k W_i(1) = 1 \).

\[ I(0) = \frac{I_0 \lambda^i \Theta_i \left[ 1 - A_i \right] (1 - z)}{D(z)} \] (19)

The mean number of customers in the system is

\[ K(z) = I_0 \int_0^\infty \sum_{i=1}^k W_i(z) \] (20)

\[ = I_0 \lambda^i \Theta_i \left[ 1 - A_i \right] \left[ 1 - \sum \left[ \lambda + (1 - z) A_i \right] \right] / D(z) \]

where \( L_s = K'(1) = \frac{N_2}{T_2} + \frac{N_1 T_3}{T_1 T_2} \) (21)
\[ T_3 = [1 - A^*(\lambda)] [A^*(\lambda) - T_1] + \sum_{i=1}^{k} p_i \Theta_{i-1} M_i \]
\[ + \frac{1}{2} \left[ \sum_{i=1}^{k} \Theta_{i-1} M_{2i} - \sum_{i=1}^{k-1} \Theta_i M_{2i} \right] \]

The probability generating function for the number of customers in the queue is
\[ H(z) = I_0 + I(z) + \sum_{i=1}^{k} W_i(z) \]
\[ = I_0 A^*(\lambda) \left[ \sum_{i=1}^{k} [q_i + p_i z] \Theta_{i-1} A_i^*(\lambda (1 - z)) \right. \]
\[ \left. + \sum_{i=1}^{k} [1 - B_i^*(\lambda (1 - z))] \Theta_{i-1} A_i^{*(1)} (\lambda (1 - z)) - z \right] / D(z) \]
(21)

The mean number of customers in the queue is
\[ L_q = H'(1) = L_s - \sum_{i=1}^{k} \Theta_{i-1} \lambda \mu_i \] / T_2 \] (22)

6 STOCHASTIC DECOMPOSITION

Stochastic decomposition has been widely observed among M/G/1 type queues. The decomposition property states that the number of customers in the system in steady state at a random point of time is distributed as the sum of two independent random variables, one of which is the number of customers in the corresponding standard queueing system in steady state at a random point in time, the other random variable may have different probabilistic interpretation in specific cases depending on the vacation scheduled.

Let \( \pi(z) \) be the probability generating function of the number of customers in the classical M/G/1 queue with k-stages of service facility and feedback. Then
\[ \pi(z) = [1 - T_1 - A^*(\lambda)] \left[ \sum_{i=1}^{k} [q_i + p_i z] \Theta_{i-1} A_i^*(\lambda (1 - z)) \right. \]
\[ \left. + z \sum_{i=1}^{k} [1 - B_i^*(\lambda (1 - z))] \Theta_{i-1} A_i^{*(1)} (\lambda (1 - z)) - z \right] / D_1(z) T_2 \]
where \( D_1(z) = \sum_{i=1}^{k} [q_i + p_i z] \Theta_{i-1} A_i^*(\lambda (1 - z)) - z \)

If the server is idle either due to retrial of customers from the orbit or due to empty system, we say that the server is on vacation. Let \( \psi(z) \) be the probability generating function of the number of customers in the system at a random point of time given that the server is on vacation. Then
\[ \psi(z) = \frac{I_0 + I(z)}{I_0 + I(1)} \]
\[ = T_2 \left[ \sum_{i=1}^{k} [q_i + p_i z] \Theta_{i-1} A_i^*(\lambda (1 - z)) - z \right] \]
\[ / [D(z) [1 - T_1 - A^*(\lambda)]] \]

From equation (19) it is observed that the probability generating function of the number of customers in the system \( K(z) \) is decomposed as
\[ K(z) = \pi(z) \psi(z) \]
This shows that the decomposition law is valid for the model under consideration.

REFERENCES