Heat Conduction Problem for an Finite Elliptical Cylinder

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Abstract- This paper contains a heat conduction problem for an finite elliptical cylinder to determine the temperature distribution with the help of Mathieu transform and Marchi-Fasulo transform techniques

Key Words- Heat conduction problem, Finite elliptic cylinder, Mathieu transforms, Marchi-Fasulo transform

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1 INTRODUCTION

Integral transform technique plays important role in solving problem of applied Mathematics. Such problems have been out by Sneddon [3], Tranter [5] and Olcer [2]. Hankel Transform is used to solve circular boundary value problems. Analogous to Hankel Transform, Gupta [1] and Sharma [4] have investigated finite Mathiue Transforms.

In this paper, we have generalized the problem of Sneddon [3]. We consider the heat conduction in a finite elliptic cylinder

2. STATEMENT OF THE PROBLEM

Heat conduction equation in elliptical co-ordinates \((\xi, \eta)\) for elliptic cylinder as Mclachlan [8] is

\[
\frac{1}{k} \frac{\partial u}{\partial t} = \frac{2n^{-2}}{(\cosh 2\xi - \cos 2\eta)} \left( \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} \right) + \frac{\partial^2 u}{\partial z^2}
\]

\( \varphi(\xi, \eta, z, t) = a, u(a, \eta, z, t) = f(\eta, z, t) \)

\[
[u + k_1 \frac{\partial u}{\partial \xi}]_{\xi = -h} = 0
\]

\[
[u + k_2 \frac{\partial u}{\partial \xi}]_{\xi = h} = 0
\]

Require result : Finite Mathieu Transform,

If the function \( T(\xi, \eta) \) is continuous and single valued in the region

\[ 0 \leq \xi \leq a, \quad 0 \leq \eta \leq 2\pi \text{ and } \frac{\partial T}{\partial \xi} = 0 \text{ at } \xi = a \]

then finite Mathieu transform is defined as

\[ f(q_{2n,m}) = \int_{0}^{2\pi} \int_{0}^{a} T(\xi, \eta)[\cosh 2\xi - \cos 2\eta] \cdot Ce_{2n}(\xi, q_{2n,m}) \cdot ce_{2n}(\eta, q_{2n,m}) \, d\xi \, d\eta \]  

(2)

Where \( q_{2n,m} \) is a root of the equation

\[ Ce_{2n}(a, q) = 0 \]

Then at any point within the range,

\[ T(\xi, \eta) = \sum_{n=0}^{\infty} C_{2n} Ce_{2n}(\xi, q_{2n,m}) \cdot ce_{2n}(\eta, q_{2n,m}) \]

\[
= \sum_{n=0}^{\infty} \left[ \sum_{m=1}^{\infty} C_{2n} Ce_{2n}(\xi, q_{2n,m}) \cdot ce_{2n}(\eta, q_{2n,m}) \right]
\]

(3)

Where constant \( C_{2n} \) is

\[
C_{2n} = \frac{\int_{0}^{a} \int_{0}^{2\pi} Ce_{2n}(\xi, q_{2n,m}) \cdot ce_{2n}(\eta, q_{2n,m}) \cdot d\xi \, d\eta}{\int_{0}^{a} \int_{0}^{2\pi} Ce_{2n}(\xi, q_{2n,m}) \cdot d\xi \, d\eta}
\]

(4)

\[
\bar{T}(q_{2n,m}) = \frac{\int_{0}^{a} Ce_{2n}^{2}(\xi, q_{2n,m}) \cdot d\xi}{\pi \int_{0}^{a} Ce_{2n}^{2}(\xi, q_{2n,m}) \cdot d\xi}
\]

(5)

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\[ \theta_{2n,m} = \frac{1}{\pi} \int_0^{2\pi} Ce_{2n}(\eta, q_{2n,m}) \cosh 2\eta \, d\eta \]

**INVERSION FORMULA OF MATHIEU TRANSFORM**

(6)

The inversion formula for Mathieu Transform is given by

\[ T(\xi, \eta) = \frac{1}{\pi} \int_0^{2\pi} Ce_{2n}(\xi, q_{2n,m}) Ce_{2n}(\eta, q_{2n,m}) \frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2} \, d\xi \, d\eta \]

(7)

**PROPERTIES OF FINITE MATHIEU TRANSFORM**

\[ a \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \] \(Ce_{2n}(\xi, q_{2n,m}) Ce_{2n}(\eta, q_{2n,m}) \, d\xi \, d\eta \]

Taking

\[ I_1 = \int \int Ce_{2n}(\xi, q_{2n,m}) Ce_{2n}(\eta, q_{2n,m}) \frac{\partial^2 T}{\partial \xi^2} \, d\xi \, d\eta \]

(8)

And

\[ I_2 = \int \int Ce_{2n}(\xi, q_{2n,m}) Ce_{2n}(\eta, q_{2n,m}) \frac{\partial^2 T}{\partial \eta^2} \, d\xi \, d\eta \]

(9)

Now integration by parts we obtain

\[ I_1 = 2\pi \int_0^{2\pi} Ce_{2n}(\eta, q_{2n,m}) \frac{\partial T}{\partial \xi} Ce_{2n}(\xi, q_{2n,m}) \, d\eta \]

\[ + \int \int T \frac{\partial^2}{\partial \xi^2} \{Ce_{2n}(\xi, q_{2n,m}) Ce_{2n}(\eta, q_{2n,m})\} \, d\eta \, d\xi \]

(10)

\[ I_1 = 2\pi A_0^{(2n)} \left[ Ce_{2n}(\xi, q_{2n,m}) \frac{\partial T}{\partial \xi} - T \frac{\partial}{\partial \xi} Ce_{2n}(\xi, q_{2n,m}) \right]_0^\alpha \]

\[ + \int \int T \frac{\partial^2}{\partial \xi^2} \{Ce_{2n}(\xi, q_{2n,m}) Ce_{2n}(\eta, q_{2n,m})\} \, d\eta \, d\xi \]

(11)

Since

\[ Ce_{2n}(\xi, q_{2n,m}) = 0 \text{ and } \frac{\partial T}{\partial \xi} = 0 \text{ at } \xi = 0, \frac{\partial}{\partial \xi} Ce_{2n}(\xi, q_{2n,m}) = 0 \text{ at } \xi = 0 \]

\[ I_1 = -2\pi A_0^{(2n)} T(\xi, \eta,t) Ce_{2n}(a, q_{2n,m}) - \frac{4q_{2n,m}}{h^2} kT - a_n T \]

(12)

\[ I_2 = \int \int Ce_{2n}(\xi, q_{2n,m}) Ce_{2n}(\eta, q_{2n,m}) \frac{\partial^2 T}{\partial \eta^2} \, d\xi \, d\eta \]

\[ + \int \int T \frac{\partial^2}{\partial \eta^2} \{Ce_{2n}(\xi, q_{2n,m}) Ce_{2n}(\eta, q_{2n,m})\} \, d\eta \, d\xi \]

(13)

\[ I_2 = 2\pi A_0^{(2n)} T(\xi, \eta,t) Ce_{2n}(a, q_{2n,m}) + kT - a_n T \]

(14)

Since

\[ Ce_{2n}(\xi, q_{2n,m}) = 0 \text{ and } \frac{\partial^2}{\partial \eta^2} = 0 \text{ at } \eta = 0, \frac{\partial}{\partial \eta} Ce_{2n}(\xi, q_{2n,m}) = 0 \text{ at } \eta = 0 \]

\[ I_2 = -2\pi A_0^{(2n)} T(\xi, \eta,t) Ce_{2n}(a, q_{2n,m}) - \frac{4q_{2n,m}}{h^2} kT - a_n T \]

**3. SOLUTION OF THE PROBLEM**

Multiplying \((\cosh 2\xi - \cos 2\eta) Ce_{2n}(\xi, q_{2n,m}) Ce_{2n}(\eta, q_{2n,m})\) in equation (7) and integrating w.r.t. \(\xi\) from 0 to \(a\) and w.r.t. \(\eta\) from 0 to \(2\pi\) and using properties of Mathieu Transform (6) we get

\[ \frac{d\bar{T}}{dt} = -2\pi A_0^{(2n)} T(\xi, \eta,t) Ce_{2n}(a, q_{2n,m}) - \frac{4q_{2n,m}}{h^2} kT - a_n T \]

or

\[ \frac{d\bar{T}}{dt} = -2\pi A_0^{(2n)} T(\xi, \eta,t) Ce_{2n}(a, q_{2n,m}) - kT - a_n T \]

(15)

Where

\[ \lambda_{2n,m}^2 = \frac{4q_{2n,m}}{h^2} \]

or

\[ \bar{T} = -2\pi A_0^{(2n)} T(\xi, \eta,t) Ce_{2n}(a, q_{2n,m}) + kT - a_n T \]

Which is an ordinary linear diff. Eqn.

\[ I.F. = e^{k(\lambda_{2n,m}^2 + a_n^2) t} = e^{k(\lambda_{2n,m}^2 + a_n^2) t} \]

And solution is

\[ \bar{T} e^{k(\lambda_{2n,m}^2 + a_n^2) t} = -2\pi A_0^{(2n)} Ce_{2n}(a, q_{2n,m}) \int \left[ f(\eta,t) e^{k(\lambda_{2n,m}^2 + a_n^2) t} \right] d\tau + A \]

at \( t = 0, \bar{T} = 0, A = 0 \)

Hence

\[ \bar{T} = -2\pi A_0^{(2n)} Ce_{2n}(a, q_{2n,m}) \int \left[ f(\eta,t) e^{k(\lambda_{2n,m}^2 + a_n^2) t} \right] d\tau \]
Using inversion theorem of Mathieu Transform, and finite Marchi-Fasulo transform, we get

\[ T(\zeta, \eta, z, t) = \sum_{n=0}^{\infty} \frac{P_n(\zeta)}{\lambda_n} \int \frac{e^{i \lambda_n \eta} \delta \cosh \theta_{2n+1}}{2} \int_{-\infty}^{\infty} e^{i \lambda_n \eta} \delta \cosh \theta_{2n+1} \int f(\eta, t) e^{-\lambda_n \eta} d\eta d\tau \]

(14)

4. CONCLUSION

In this paper the temperature distribution of an elliptical cylinder have been determined with the help of finite Mathieu transform and Marchi-Fasulo transform techniques. The expressions are represented graphically. The results that are obtained can be applied to the structures or machines in engineering applications.

Temperature

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References