Generalized derivations of Rings in the center

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Abstract — Ashraf, Rehman, Bell, Martindale and Daif have obtained commutativity of prime and semiprime rings with derivations satisfying certain polynomial constants. Ashraf and Nadeem established that a prime ring R with nonzero ideal A must be commutative if it admits a derivation d satisfying either of the properties d(xy) + xy ∈ U ord(xy) − xy ∈ U for all x, y ∈ R. Hvala initiated the algebraic study of generalized derivation and extended some results concerning derivation to generalized derivation. In 2007 Ashraf, Asma Ali and Shakir Ali studied commutativity of a prime associative ring in which the generalized derivation F satisfies certain properties. In this paper we prove the commutativity of a prime nonassociative ring R satisfying any one of the following properties:

(i) F(xy) − xy ∈ U, (ii) F(xy) + xy ∈ U, (iii) F(xy) + xy ∈ U and (iv) F(x)y + xy ∈ U for all x, y in R, where F is a generalized derivation on R and U is the center of R.

Index Terms — Center, Prime ring, derivation, Generalized derivation.

1 INTRODUCTION

Throughout this paper R denotes a prime nonassociative ring satisfying [xy, z] = x[y, z] + [x, z]y for all x, y, z ∈ R. A ring R is prime if aRb = (0) implies that a = 0 or b = 0. An additive mapping d : R → R is called a derivation if d(xy) = d(x)y + xd(y) holds for all x, y in R. An additive mapping F : R → R is said to be a generalized derivation on R if there exists a derivation d : R → R such that F(xy) = F(x)y + xd(y) holds for all x, y in R.

2 RESULTS

Theorem 1: Let R be a prime nonassociative ring satisfying [x, y, z] = x[y, z] + [x, z]y for all x, y, z ∈ R and A be an associative nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that F(xy) = F(x)y + xd(y) holds for all x, y ∈ R, then R is commutative.

Proof: If F = 0, then xy ∈ U for all x, y in A. In particular [xy, x] = 0 for all x, y ∈ A and hence x[xy, x] = 0. By replacing y with yz, we get x[xy, z] = 0 for all x, y, z ∈ A.

Hence it follows that xRA[z, x] = (0) for all x, z ∈ A.

Thus the primeness of R forces for each x ∈ A, either x = 0 or A[x, z] = (0).

But x = 0 also implies that A[x, z] = (0).

Hence in both cases we find that A[x, z] = (0) for all z ∈ A, that is, AR[x, z] = (0).

Since A ≠ (0) and R is prime, the above expression yields that [x, z] = 0 for all x, z ∈ A. Now by replacing x with x, we get x[r, z] = 0. Again by replacing x with xs, we get x[s, z] = 0 for all x, z ∈ A and r, s ∈ R. That is, xR[r, z] = (0).

The primeness of R forces that either x = 0 or [r, z] = 0, but A ≠ (0), we have [r, z] = 0.

Now by replacing z with zs to get [r, s] = 0 for all z ∈ A and r, s ∈ R, this implies that zR[r, s] = (0).

The primeness of R forces that either z = 0 or [r, s] = 0, but A ≠ (0), we have [r, s] = 0 for all r, s ∈ R. Hence R is commutative.

Now we assume that F ≠ 0. For any x, y ∈ A, we have F(xy) − xy ∈ U. This can be rewritten as F(xy) + xd(y) − xy ∈ U.

Now by replacing y by yz, we obtain F(xyz + xd(y)z + xyd(z) − xy ∈ U for all x, y, z ∈ A.

Thus, in particular

[(F(xy) + xd(y) − xy)z + xyd(z)] = 0, (1)

for all x, y, z ∈ A.

This gives that [xyd(z), z] = 0 for all x, y, z ∈ A and hence

xy[(d(z), z) + x[y, z]d(z)] ∈ U for all x, y, z ∈ A.

For any yi ∈ A, by replacing x by yi in 2 and using 2, we get

[yi, z]xyd(z) = 0 for all x, y, z ∈ A and hence [yi, z]xRA[d(z)] = (0).

Thus, the primeness of R implies that for each z ∈ A, either A[d(z)] = (0) or [yi, z]x = 0. The set z ∈ A for which these two properties hold are additive subgroups of A whose union is A. Therefore either A[d(z)] = (0) or [yi, z]x = 0. So the set A is a prime ring.

Since A ≠ (0) and R is prime, the above expression yields that d(z) = 0 for all z ∈ A. This implies that d(r) = 0 for all r ∈ R.

Hence it follows that d(r) = 0, that is, AR[d(r)] = (0).

Since A ≠ (0), the primeness of R yields that d(r) = 0 for all r ∈ R, a contradiction. On the other hand if [yi, z]x = 0 for all x, y, z ∈ A, then [yi, z]RA = (0) for all yi, z ∈ A. The primeness of R implies that [yi, z] = 0 for all yi, z ∈ A and hence R is commutative.

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Theorem 2: Let R be a prime nonassociative ring satisfying 
\([xy, z] = x[y, z] + [x, z]y\) for all \(x, y, z\) in R and A be an associative nonzero ideal of R. If R admits a generalized derivation \(F\) associated with a nonzero derivation \(d\) such that \(F(xy) = xy \in U\) for all \(x, y \in A\), then R is commutative.

Proof: If \(F\) is a generalized derivation satisfying the property \(F(xy) = xy \in U\) for all \(x, y \in A\), then the generalized derivation \((-F)\) satisfies the condition \((-F)(xy) = xy \in U\) for all \(x, y \in A\). Using the same arguments as used in Theorem 1, we conclude that R is commutative.

Theorem 3: Let R be a prime nonassociative ring satisfying \([xy, z] = x[y, z] + [x, z]y\) for all \(x, y, z\) in R and A be an associative nonzero ideal of R. If R admits a generalized derivation \(F\) associated with a nonzero derivation \(d\) such that \(F(xy) = xy \in U\) for all \(x, y \in A\), then R is commutative.

Proof: By hypothesis \(F(xy) = xy \in U\) for all \(x, y \in A\). If \(F = 0\), then \(xy \in U\) for all \(x, y \in A\). Using the same arguments as we used in Theorem 1, we conclude that R is commutative. Now we assume that \(F \neq 0\). For any \(x, y \in A\), we have \(F(xy) = xy \in U\). By replacing \(y\) with \(yr\), we get
\[
F(xy)(yr + yd(r)) - xy \in U.
\]
That is,
\[
(F(xy)(y) - xy) + F(xy)yd(r) \in U,
\]
for all \(x, y \in A\) and \(r \in R\). This implies that for all \(x, y \in A\) and \(r \in R\),
\[
[F(xy)yd(r), r] = 0,
\]
This can be rewritten as
\[
[F(xy)yd(r), r] + [F(xy), r]yd(r) = 0,
\]
for all \(x, y \in A\) and \(r \in R\). Substituting \(F(xy)yd(r)\) in 5 and using 5, we get
\[
[F(xy), r]F(xy)yd(r) = 0,
\]
for all \(x, y \in A\) and \(r \in R\). That is,
\[
[F(xy), r]F(xy)Ad(r) = 0.
\]
Thus for each \(r \in R\), primeness of R forces that either \([F(xy), r]F(xy) = 0\) or \(Ad(r) = 0\). The set of \(r \in R\) for which these two properties hold form additive subgroups of R whose union is R. Hence either \([F(xy), r]F(xy) = 0\) for all \(x \in A\), \(r \in R\) or \(Ad(r) = 0\) for all \(r \in R\). If \(Ad(r) = 0\) for all \(r \in R\), then \(Ad(r) = 0\) for all \(r \in R\). Since \(A \neq 0\) and R is prime, the above relation yields that \(d = 0\) which is a contradiction. Therefore, we assume the remaining possibility that \([F(xy), r]F(xy) = 0\) for all \(x \in A\), \(r \in R\).

Theorem 4: Let R be a prime nonassociative ring satisfying \([xy, z] = x[y, z] + [x, z]y\) for all \(x, y, z\) in R and A be an associative nonzero ideal of R. If R admits a generalized derivation \(F\) associated with a nonzero derivation \(d\) such that \(F(xy) = xy \in U\) for all \(x, y \in A\), then R is commutative.

Proof: We can prove this theorem by using the same arguments as in Theorem 3. Hence we conclude that R is commutative.

3 REFERENCES