Fuzzy Completely Weakly $e$-irresolute Functions

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Abstract—In this paper, we introduce a new class of functions called fuzzy completely $e$-irresolute functions between fuzzy topological spaces and also in this paper, fuzzy $e'$-open sets and fuzzy $e'$-closed sets are used to define and investigate a new class of functions called fuzzy completely weakly $e$-irresolute. Relationships between the new class and other classes of functions are established.

Key words and phrases: Fuzzy topology, fuzzy $e'$-open sets, fuzzy $e'$-irresolute functions, fuzzy $e'$-open set, fuzzy completely $e'$-irresolute, Fuzzy $e'$-continuous, fuzzy $e'$-connected.


1 INTRODUCTION

Ever since the introduction of fuzzy sets by Zadeh [20], the fuzzy concepts has invaded almost all branches of Mathematics. The concept of fuzzy topological space has introduced by Chang [5] in 1968. Since then many fuzzy topologists have extended various notions in classical topology to fuzzy topological spaces. In this paper, fuzzy $e'$-open sets and fuzzy $e'$-closed sets are used to define and investigate a new class of functions called fuzzy completely weakly $e'$-irresolute. Relationships between the new class and other classes of functions are established. Throughout this paper $X$ and $Y$ are always fuzzy topological spaces. The class of all fuzzy sets on a universe $X$ will be denoted by $X$. Let $A$ be a fuzzy subset of a space $X$. The fuzzy closure of $A$, fuzzy interior of $A$, fuzzy $\delta$-closure of $A$ and the fuzzy $\delta$-interior of $A$ are denoted by $Cl(A)$, $Int(A)$, $Cl_\delta(A)$ and $Int_\delta(A)$ respectively.

A fuzzy subset $A$ of space $X$ is called fuzzy regular open [1](resp. fuzzy regular closed) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$. The fuzzy $\delta$-interior of fuzzy subset $A$ of $X$ is the union of all fuzzy regular open sets contained in $A$. A fuzzy subset $A$ is called fuzzy $\delta$-open [12] if $A = Int_\delta(A)$.

The complement of fuzzy $\delta$-open set is called fuzzy $\delta$-closed (i.e., $A = Cl_\delta(A)$).

2 PRELIMINARIES

Now, we introduce some basic notions and results that are used in the sequel.

Definition 2.1. A fuzzy topology on a nonempty set $X$ is a family $\delta$ of fuzzy subsets of $X$ which satisfies the following three conditions:

(i) $0, 1 \in \delta$,

(ii) If $g, h \in \delta$, their $g \wedge h \in \delta$,

(iii) $\bigvee_{i \in I} f_i \in \delta$.

The pair $(X, \tau)$ is called a fuzzy topological space [5].

Definition 2.2. Members of $\delta$ are called fuzzy open sets [5] and complements of fuzzy open sets are called fuzzy closed sets [5], where the complement of a fuzzy set $A$, denoted by $A^C$, is $1 - A$.

Definition 2.3. [15] The fuzzy subset $x_a$ of a non-empty set $X$, which $x \in X$ and $0 < a \leq 1$ defined by

$$x_a(p) = \begin{cases} a & \text{if} \quad p = x \\ 0 & \text{if} \quad p \neq x \end{cases}$$

is called a fuzzy point in $X$ with support $x$ and value $a$. The fuzzy point $x_a$ is called point.

Definition 2.4. [15] Let $\lambda$ be fuzzy set in $X$ and $x_a$ a fuzzy point in $X$. we say that $X_a \leq \lambda$. 

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Definition 2.5. [9] A fuzzy set \( \lambda \) of a fuzzy topological space \( X \) is said to be fuzzy \( \gamma \)-open if \( \lambda \leq \text{Cl} \)
\((\text{Int} \lambda) \vee \text{Int}(\text{Cl}(\lambda))\) where \( \text{Cl}(\lambda) = \{\mu : \mu \geq \lambda, \mu \}
\) is fuzzy closed in \( X \} \) and \( \text{Int}(\lambda) = \{\mu : \mu \leq \lambda, \mu \}
\) is fuzzy open in \( X \} \). If \( \lambda \) is fuzzy \( \gamma \)-open, then \( 1 - \lambda \) is fuzzy \( \gamma \)-closed.

Definition 2.6. [3] Let \( f : X \rightarrow Y \) be a mapping. Then \( f \) is called a fuzzy \( \gamma \)-irresolute mapping if \( f^{-1}(V) \) is a fuzzy \( \gamma \)-open set in \( X \) for each fuzzy \( \gamma \)-open set in \( Y \).

Definition 2.7. [17] A fuzzy set \( \lambda \) of a fuzzy topological space \( X \) is said to be fuzzy \( e \)-open (resp. regular open [1]) if \( \lambda \leq \text{Cl}(\text{Int}_\lambda \lambda) \vee \text{Int}(\text{Cl}(\lambda)) \) (resp. \( \lambda = \text{Int}(\text{Cl}(\lambda)) \)) where \( \text{Cl}(\lambda) = \{\mu : \mu \geq \lambda, \mu \}
\) is fuzzy closed in \( X \} \) and \( \text{Int}(\lambda) = \{\mu : \mu \leq \lambda, \mu \}
\) is fuzzy open in \( X \} \). If \( \lambda \) is fuzzy \( e \)-open, then \( 1 - \lambda \) is fuzzy \( e \)-closed.

Definition 2.8. [17] Let \( X \) be a fuzzy topological space and \( \lambda \) be any fuzzy set in \( X \). The fuzzy \( e \)-closure of \( \lambda \) in \( X \) is denoted by \( e\text{Cl}(\lambda) \) as follows:
\[ e\text{Cl}(\mu) = \{\lambda : \lambda \geq \mu, \lambda \text{ is a fuzzy } e \text{-closed set of } X \} \]. Similarly we can define \( e\text{Int}(\lambda) \).

Remark 2.9. For a fuzzy set \( \lambda \) of \( X \), \( 1 - e\text{Int}(\lambda) = e\text{Cl} \)
\((1 - \lambda) \).

Remark 2.10. A fuzzy set \( \lambda \) is fuzzy \( e \)-closed if and only if \( e\text{Cl}(\lambda) = \lambda \).

Definition 2.11. [15] A fuzzy set \( A \) in \( X \) is said to be \( q \)-coincident with a fuzzy set \( B \), denoted by \( AqB \), if there exists \( x \in X \) such that \( A(x) + B(x) > 1 \). It is known that \( A \leq B \) if and only if \( A \) and \( 1 - B \) are not \( q \)-coincident, denote \( A \rightarrow q (1 - B) \).

Definition 2.12. [15] A fuzzy set \( B \) is a quasi neighbourhood \((q \text{-neighbourhood, for short}) \) of \( A \) if and only if there exists a fuzzy open set \( U \) such that \( AqU \leq B \).

Definition 2.13. A fuzzy set \( A \) in \( X \) is said to be a \( e-q \) neighbourhood \((e-q\text{-nbd, for short}) \) of \( x_a \) if and only if there exists a fuzzy \( e \)-open set \( V \) in \( X \) such that \( x_a q V \leq A \).

Theorem 2.14. [15] In a fuzzy topological space \( X \), \( \lambda \) be a fuzzy \( e \)-closed (resp. fuzzy \( e \)-open) if and only if \( \lambda = e\text{Cl}(\lambda) \) (resp. \( \lambda = e\text{Int}(\lambda) \)).

Definition 2.15. [3] Let \( X \) and \( Y \) be two fuzzy topological spaces. Let \( \lambda \in I^X, \mu \in I^Y \). Then \( f(\lambda) \) is a fuzzy subset of \( Y \), defined by \( f(\lambda) : Y \rightarrow [0,1] \)
\[ f(\lambda)(y) = \sup_{x \in f^{-1}(\{y\})} \lambda(x) \quad \text{if} \quad f^{-1}(\{y\}) \neq \phi \]
\[ 0 \quad \text{if} \quad f^{-1}(\{y\}) = \phi \]
and \( f^{-1}(\mu) \) is a fuzzy subset of \( X \), defined by \( f^{-1}(\mu)(x) = \mu(f(x)) \).

Lemma 2.16. [1] Let \( f : X \rightarrow Y \) be a function and \( \{\lambda_\alpha\} \) be a family of fuzzy sets of \( Y \), then
\( f^{-1}(\bigcup \lambda_\alpha) = \bigcup f^{-1}(\lambda_\alpha) \),
\( f^{-1}(\bigcap \lambda_\alpha) = \bigcap f^{-1}(\lambda_\alpha) \).

Lemma 2.17. [1] For functions \( f_1 : X_1 \rightarrow Y_1 \), and fuzzy sets \( \lambda_i \) of \( Y_i \), \( i = 1, 2 \); we have \( (f_1 \times f_2)^{-1}(\lambda_1 \times \lambda_2) = f_1^{-1}(\lambda_1) \times f_2^{-1}(\lambda_2) \).

Lemma 2.18. [1] Let \( g : X \times Y \rightarrow X \times Y \) be the graph of a function \( f : X \rightarrow Y \). Then, if \( \lambda \) is a fuzzy set of \( X \) and \( \mu \) is a fuzzy set of \( Y \) \( g^{-1}(\lambda \times \mu) = \lambda \wedge f^{-1}(\mu) \).

Definition 2.19. A function \( f : X \rightarrow Y \) is said to be:
1. fuzzy completely continuous [4] if \( f^{-1}(V) \) is fuzzy regular open in \( X \) for each fuzzy open set \( V \) in \( Y \);
2. fuzzy \( e \)-irresolute [16] if \( f^{-1}(V) \) is fuzzy \( e \)-open in \( X \) for each fuzzy \( e \)-open set \( V \) in \( Y \);
3. fuzzy \( e \)-continuous [17] if \( f^{-1}(V) \) is fuzzy \( e \)-open in \( X \) for each fuzzy open set \( V \) in \( Y \);
4. fuzzy totally continuous [11] if \( f^{-1}(V) \) is fuzzy clopen in \( X \) for each fuzzy subset \( V \) in \( Y \);
5. fuzzy open [19] if \( f(V) \) is fuzzy open set in \( Y \) for each fuzzy open set \( V \) in \( X \);
6. fuzzy almost open [13] if \( f(V) \) is fuzzy regular open set in \( Y \) for each fuzzy regular open set \( V \) in \( X \);
7. fuzzy strongly continuous [2] if \( f^{-1}(V) \) is fuzzy open fuzzy...
closed set in $X$ for every fuzzy set $\lambda$ in $Y$.

**Definition 2.20.** A function $f : X \rightarrow Y$ is called fuzzy $e$-open [16] (resp. fuzzy pre- $e$-open) if the image of each fuzzy open (resp. fuzzy $e$-open) set in $X$ is fuzzy $e$-open in $Y$.

**Definition 2.21.** [2] A function $f : X \rightarrow Y$ is called fuzzy completely continuous if $f^{-1}(V)$ is fuzzy regular open in $X$ for every fuzzy open set $V$ of $Y$.

**Definition 2.22.** [16] A function $f : X \rightarrow Y$ is called fuzzy $e$-irresolute (resp. Fuzzy $e$-continuous) if $f^{-1}(V)$ is fuzzy $e$-open in $X$ for every fuzzy $e$-open (resp. fuzzy open) set $V$ of $Y$.

**Definition 2.23.** A space $(X, \tau)$ is called fuzzy nearly compact [10] (resp. fuzzy $e$-compact) if every fuzzy regular open (resp. fuzzy $e$-open) cover of $X$ has a finite subcover.

**Definition 2.24.** [18] A space $X$ is called fuzzy almost normal if for each fuzzy closed set $A$ and each fuzzy regular closed set $B$ such that $A \cap B = \emptyset$, there exists disjoint fuzzy open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

### 3 Fuzzy Completely $e$-Irresolute Function

**Definition 3.1.** Let $(X, \tau)$ and $(Y, \sigma)$ be a fuzzy topological spaces. A function $f : (X, \tau) \rightarrow (Y, \sigma)$, is said to be a fuzzy completely $e$-irresolute function if $f^{-1}(V)$ is fuzzy regular open in $X$ for every fuzzy $e$-open set $\lambda$ in $Y$.

**Remark 3.2.** Every fuzzy strongly continuous function is fuzzy $e$-irresolute, but the converse is not true.

**Example 3.3** Let $X = Y = \{a, b, c\}$. Define fuzzy sets $\mu_1, \mu_2 : X \rightarrow [0,1]$ such that $\tau = \{0,1\}$ and

$$\sigma = \{0,1, \mu_1, \mu_2\} \quad \text{where} \quad \mu_1 = \frac{0.3}{a} + \frac{0.4}{b} + \frac{0.5}{c}, \quad \mu_2 = \frac{0.7}{a} + \frac{0.5}{b} + \frac{0.5}{c}$$

Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = 1$, $f(b) = 2$, $f(c) = 3$. Then $f$ is fuzzy $e$-irresolute but not fuzzy strongly continuous.

**Remark 3.4.** Every completely $e$-irresolute function is fuzzy $e$-irresolute. But the converse is not true.

**Example 3.5** Let $X = Y = \{a, b, c\}$. Define fuzzy sets $\mu_1, \mu_2 : X \rightarrow [0,1]$ such that $\tau = \{0,1, \mu_1, \mu_2\}$ and

$$\sigma = \{0,1, \mu_1, \mu_2\} \quad \text{where} \quad \mu_1 = \frac{0.3}{a} + \frac{0.4}{b} + \frac{0.5}{c}, \quad \mu_2 = \frac{0.7}{a} + \frac{0.5}{b} + \frac{0.5}{c}$$

Define $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then $f$ is fuzzy $e$-irresolute but not fuzzy completely $e$-irresolute.

**Remark 3.6.** Every $e$-irresolute function is fuzzy $e$-irresolute. But the converse is not true.

**Example 3.7.** Let $X = Y = \{a, b, c\}$. Define fuzzy sets $\mu_1, \mu_2, \mu_3, \mu_4 : X \rightarrow [0,1]$ such that $\tau = \{0,1, \mu_1, \mu_2, \mu_3, \mu_4\}$ and $\sigma = \{0,1, \mu_1, \mu_2\}$ where

$$\mu_1 = \frac{0.3}{a} + \frac{0.4}{b} + \frac{0.5}{c}, \quad \mu_2 = \frac{0.7}{a} + \frac{0.5}{b} + \frac{0.5}{c}$$

Define $f : (X, \lambda) \rightarrow (Y, \sigma)$ be the identity function. Then $f$ is fuzzy $e$-irresolute but not fuzzy completely $e$-irresolute.

**Theorem 3.8.** If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a fuzzy completely $e$-irresolute function $A$ is any fuzzy open subset of $X$, then the restriction $f |_A : A \rightarrow Y$ is fuzzy completely $e$-irresolute.

**Proof.** Let $\lambda$ be a fuzzy $e$-open subset of $Y$. By hypothesis, $f^{-1}(\lambda)$ is fuzzy regular open in $X$. Since $A$ is fuzzy open in $X$, then $(f |_A)^{-1}(\lambda) \cap A$ is fuzzy regular open in $A$. Therefore, $f |_A$ is fuzzy completely $e$-irresolute.

**Theorem 3.9.** The following hold for functions $f : X \rightarrow Y$ and $g : Y \rightarrow Z$.  

1. If $f : X \rightarrow Y$ is fuzzy completely $e$-irresolute and $g : Y \rightarrow Z$ is fuzzy $e$-irresolute, then $g \circ f : X \rightarrow Z$ is fuzzy completely $e$-irresolute.
2. If function $f : X \rightarrow Y$ is fuzzy completely continuous and is fuzzy completely $e$-irresolute, then $g \circ f : X \rightarrow Z$ is fuzzy completely $e$-irresolute.
fuzzy completely $e$-irresolute.

3. If $f : X \to Y$ is fuzzy completely $e$-irresolute and $g : Y \to Z$ is fuzzy $e$-continuous, then $g \circ f : X \to Z$ is fuzzy completely continuous.

**Proof.** Obvious.

**Definition 3.10.** A space $X$ is said to be fuzzy $e$-connected, if $X$ cannot be expressed as the union of two nonempty fuzzy $e$-open sets.

**Theorem 3.11.** If a mapping $f : X \to Y$ is fuzzy completely $e$-irresolute surjection and $X$ is fuzzy almost connected then $Y$ is fuzzy $e$-connected.

**Proof.** Assume that $X$ is fuzzy connected and $Y$ is not fuzzy $e$-connected. Then $Y$ can be written as $Y = U \cup V$ such that $U$ and $V$ are disjoint nonempty fuzzy $e$-open sets. Since $f$ is fuzzy completely $e$-irresolute, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint fuzzy regular open sets and $X = f^{-1}(U) \cup f^{-1}(V)$. This shows that $X$ is not fuzzy connected. This is a contradiction.

**Definition 3.12.** A space $X$ is called fuzzy almost regular [6] (resp. fuzzy strongly $e$-regular) if for every fuzzy regular closed (resp. fuzzy $e$-closed) set $F \subseteq X$ and any point $x \in X - F$, there exists disjoint fuzzy open (resp. fuzzy $e$-open) sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$.

**Definition 3.13.** A function $f : X \to Y$ is called fuzzy pre-$e$-closed if the image of every fuzzy $e$-closed subset of $X$ is fuzzy $e$-closed set in $Y$.

**Theorem 3.14.** If a mapping $f : X \to Y$ is fuzzy pre-$e$-closed, then for each subset $B$ of $Y$ and a fuzzy $e$-open set $U$ of $X$ containing $f^{-1}(B)$ there exists a fuzzy $e$-open set $V$ containing $B$ such that $f^{-1}(V) \subseteq U$.

**Proof.** Obvious.

**Theorem 3.15.** If $f$ is fuzzy completely $e$-irresolute $e$-open from an almost regular space $X$ onto a space $Y$, then $Y$ is fuzzy strongly $f$-regular.

**Proof.** Let $f$ be fuzzy $e$-closed set in $Y$ with $y \notin F$ such that $y = f(x)$. Since $f$ is fuzzy completely $e$-irresolute function, $f^{-1}(F)$ is fuzzy regular closed and so fuzzy closed set in $X$ and hence $x \notin f^{-1}(F)$. By almost regularity of $X$ there exists disjoint fuzzy open sets $U$ and $V$ such that $x \in U$ and $f^{-1}(F) \subseteq V$. We obtain that $y = f(x) \in f(U)$ and $F \subseteq f(V)$ such that $f(U)$ and $f(V)$ are disjoint fuzzy $e$-open sets. Thus $Y$ is fuzzy strongly $e$-regular.

**Definition 3.16.** A space $X$ is called fuzzy strongly $e$-normal if for every pair of disjoint fuzzy $e$-closed subsets $F_1$ and $F_2$ of $X$ there exists disjoint fuzzy $e$-open sets $U$ and $V$ such that $F_1 \subseteq U$ and $F_2 \subseteq V$.

**Theorem 3.17.** If $f$ is fuzzy completely $e$-irresolute injective function from an fuzzy almost normal spaces $X$ onto a space $Y$ then $Y$ is fuzzy strongly $e$-normal.

**Proof.** Let $F_1$ and $F_2$ be disjoint fuzzy $e$-closed sets in $Y$. Since $f$ is fuzzy completely $e$-irresolute function $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are disjoint fuzzy regular closed and so fuzzy closed set in $X$. By fuzzy almost normality of $X$, there exists disjoint fuzzy open sets $U$ and $V$ such that $f^{-1}(F_1) \subseteq U$ and $f^{-1}(F_2) \subseteq V$. We obtain that $F_1 \subseteq U$ and $F_2 \subseteq V$. Thus $Y$ is fuzzy strongly $e$-open.

**Definition 3.18.** A fuzzy topological space $(X, \tau)$ is said to be fuzzy $e$-$T_1$ (resp. fuzzy $r$-$T_1$) if for each pair of distinct points $x$ and $y$ of $X$, there exists fuzzy $e$-open (resp. fuzzy regular open) sets $U_1$ and $U_2$ such that $x \in U_1$ and $y \in U_2$, $x \notin U_2$ and $y \notin U_1$.

**Theorem 3.19.** If $f : (X, \tau) \to (Y, \sigma)$ is fuzzy completely $e$-irresolute injective function and $Y$ is fuzzy $e$-$T_1$ then $X$ is fuzzy $r$-$T_1$.

**Proof.** Suppose that $Y$ is fuzzy $e$-$T_1$. For any two distinct points $x$ and $y$ of $X$, there exists fuzzy $e$-open sets $F_1$ and $F_2$ in $Y$ such that $f(x) \in F_1$, $f(y) \in F_2$, $f(x) \notin F_2$ and $f(y) \notin F_1$. Since $f$ injective fuzzy completely $e$-irresolute function, we have $X$ is fuzzy $r$-$T_1$.

**Definition 3.20.** A fuzzy topological space $(X, \tau)$ is said to be fuzzy $e$-$T_1$ (resp. fuzzy $r$-$T_1$) if for each pair of distinct points $x$ and $y$ of $X$, there exists disjoint fuzzy $e$-open (resp. fuzzy regular open) sets $A$ and $B$ such that $x \in A$ and $y \in B$. 
Theorem 3.21. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is fuzzy completely e-irresolute injective function and \( Y \) is fuzzy e-\( T_2 \) then \( X \) is fuzzy r-\( T_2 \).

Proof. Suppose that \( Y \) is fuzzy e - T_2. For any two distinct points \( x \) and \( y \) of \( X \), there exists fuzzy e-open sets \( F_1 \) and \( F_2 \) in \( Y \) such that \( f(x) \in F_1, f(y) \in F_2, f(x) \notin F_2 \) and \( f(y) \notin F_1 \). Since \( f \) injective fuzzy completely e-irresolute function, we have \( X \) is fuzzy r-T_1.

4 Fuzzy Completely Weakly e-IRRESOLUTE Function

Definition 4.1. A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is fuzzy completely weakly e-irresolute if and only if the inverse image of each fuzzy e-open set \( V \) in \( Y \) is fuzzy open set in \( X \).

It is evident that every fuzzy completely e-irresolute function is fuzzy completely weakly e-irresolute function and every completely weakly e-irresolute function is fuzzy e-irresolute.

However, none of the above implications are not true as shown in the following example.

Example 4.2 Let \( I = [0,1] \) and \( \mu_1 \) and \( \mu_2 \) be fuzzy subsets of \( I \) defined as:

\[
\mu_1(x) = \begin{cases} 
\frac{1}{5} (6x + 1) & \text{if } 0 \leq x \leq \frac{1}{4} \\
\frac{1}{3} (2x + 1) & \text{if } \frac{1}{4} \leq x \leq 1 
\end{cases}
\]

\[
\mu_2(x) = \begin{cases} 
\frac{1}{10} (4x + 1) & \text{if } 0 \leq x \leq \frac{1}{4} \\
\frac{4}{3} (1-x) & \text{if } \frac{1}{4} \leq x \leq 1 
\end{cases}
\]

Clearly \( \tau_1 = \{0,1\} \) and \( \tau_2 = \{0,1, \mu_1, \mu_2, \mu_1 \lor \mu_2, \mu_1 \land \mu_2 \} \) are topologies on \( I \). Let \( f : (I, \tau_1) \rightarrow (I, \tau_2) \) be defined by \( f(x) = x \) for each \( x \in I \). Then \( f \) is fuzzy e-irresolute but not fuzzy completely weakly e-irresolute.

Let \( g : (I, \tau_2) \rightarrow (I, \tau_3) \) be defined by \( g(x) = x \) for each \( x \in I \). Then \( g^{-1} = (I), g^{-1}(\mu_1) = (\mu_1) \) which is fuzzy open but not regular open in \( (I, \tau_3) \). Therefore, \( g \) is fuzzy completely weakly e-irresolute but not fuzzy completely e-irresolute.

Theorem 4.3. For a function \( f : (X, \tau) \rightarrow (Y, \sigma) \), the following statements are equivalent:

(i) \( f \) is fuzzy completely weakly e-irresolute;
(ii) for each fuzzy point \( x_0 \) in \( X \) and each fuzzy e-open \( e-q \)-nbd \( V \) of \( f(x_0) \), there exists a fuzzy open \( q \)-nbd \( U \) of \( x_0 \) such that \( f(U) \subseteq V \);
(iii) \( f(Cl(A)) \subseteq eCl(f(A)) \), for each fuzzy set \( A \) in \( X \);
(iv) \( Cl(f^{-1}(B)) \subseteq f^{-1}(eCl(B)) \), for each fuzzy set \( B \) in \( Y \);
(v) for each fuzzy e-closed set \( V \) in \( Y \), \( f^{-1}(V) \) is fuzzy closed set in \( X \);
(vi) \( f^{-1}(e-Int(B)) \subseteq Int(f^{-1}(B)) \), for each fuzzy set \( B \) in \( Y \).

Proof. (i) \( \Rightarrow \) (ii). Let \( V \) be any fuzzy e-open \( e-q \)-nbd of \( f(x_0) \) in \( Y \). Then \( V(f(x)) + \alpha > 1 \). We choose a positive real number \( \delta > 1 - \alpha \). Then \( V(f(x)) \) is a fuzzy e-open set, \( f(x_0) \in V \). By hypothesis, there exists an open set \( U \), \( x_0 \in U \) such that \( f(U) \subseteq V \), \( U(X) > \delta > 1 - \alpha \). Therefore, \( U \) is a fuzzy open \( q \)-nbd of \( x_0 \).

(ii) \( \Rightarrow \) (iii). Let \( x_0 \in Cl(A) \) then \( UqA \) and \( f(U)f(A) \) implies \( Vqf(A) \), \( f(x_0) \in eCl(f(A)) \) and \( x_0 f^{-1}(eCl) \).

(iii) \( \Rightarrow \) (iv). Clear.

(iv) \( \Rightarrow \) (ii). Let \( x_0 \) be a fuzzy point in \( X \) and \( V \) be a fuzzy e-open \( e-q \)-nbd of \( f(x_0) \) and let \( f(x_0) \notin eCl(1-V) \), otherwise since \( V \) is a fuzzy e-open \( e-q \)-nbd of \( f(x_0) \), we have \( Vq(1-V) \) which is a contradiction. Thus, \( x_0 \notin f^{-1}(eCl(1-V)) \).

Then there exists a fuzzy open set \( U \) of \( x_0 \) such that \( U \subseteq f^{-1}(1-V) \) which implies that \( f(U) \subseteq V \).

(iv) \( \Rightarrow \) (v). Let \( F \) be any e-closed set in \( Y \). By (iv), we have \( Cl(f^{-1}(F)) \subseteq f^{-1}(eCl(F)) = f^{-1}(F) \) and \( f^{-1}(F) \) is fuzzy closed in \( X \).

(iv) \( \Rightarrow \) (vi). Clear.

(i) \( \Rightarrow \) (vi). For any fuzzy set \( V \) in \( Y \), \( e-Int(V) \) is fuzzy e-open set in \( Y \) and so \( f^{-1}(eInt(V)) \) is fuzzy open set in \( X \).
Hence \( f^{-1}(\text{Int}(V)) = \text{Int}(f^{-1}(\text{Int}(V))) \leq \text{Int}(f^{-1}(V)) \).

\( (vi) \Rightarrow (i) \). Obvious.

**Theorem 4.4.** Let \( X \) and \( Y \) be fuzzy topological space such that \( X \) is product related to \( Y \). Then the product \( U \times V \) of a fuzzy \( e \)-open set \( U \) in \( X \) and fuzzy \( e \)-open set \( V \) in \( Y \) is a fuzzy \( e \)-open set in the fuzzy product space.

**Proof.** Similar to the proof of Theorem 3.10 in [1].

**Theorem 4.5.** If \( f_i : X_i \rightarrow Y_i \) (\( i = 1, 2 \)) are fuzzy completely weakly \( e \)-irresolute functions and \( Y_i \) is product related to \( Y_2 \), then \( f_i : X_1 \times X_2 \rightarrow Y_1 \times Y_2 \) is fuzzy completely weakly \( e \)-irresolute.

**Proof.** Consider \( A = \vee(U_i \times V_i) \) where \( U_i \)'s and \( V_i \)'s are fuzzy \( e \)-open sets of \( Y_i \) and \( Y_2 \), respectively. Since \( Y_i \) is product related to \( Y_2 \), then from Theorem 4.2., \( A \) is fuzzy \( e \)-open set of \( Y_1 \times Y_2 \). By Lemma 2.1. and 2.2., \( f^{-1}(A) = \vee(f_1^{-1}(U_i) \times f_2^{-1}(V_i)) \). Since \( f_1 \) and \( f_2 \) are completely weakly \( e \)-irresolute, \( f^{-1}(A) \) is a fuzzy open in \( X_1 \times X_2 \).

**Theorem 4.6.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function and \( X \) is product related to \( Y \). If the graph \( g : X \rightarrow X \times Y \) of \( f \) is fuzzy completely weakly \( e \)-irresolute, then so is \( f \).

**Proof.** Let \( V \) be a fuzzy \( e \)-open set in \( Y \). By Lemma 2.3., we have \( f^{-1}(V) = 1 \wedge f^{-1}(V) = g^{-1}(1 \times V) \). Since \( g \) is fuzzy completely weakly \( e \)-irresolute and \( 1 \times V \) is fuzzy \( e \)-open set in \( X \times Y \), \( f^{-1}(V) \) is fuzzy open set in \( X \) and so, \( f \) is fuzzy completely weakly \( e \)-irresolute.

Next, the composition and preservation of fuzzy topological structure under the fuzzy completely weakly \( e \)-irresolute which other fuzzy functions are studied.

The proof of the following theorem is obvious and hence omitted.

**Theorem 4.7.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) be two functions.

1. If is fuzzy completely weakly \( e \)-irresolute and \( g \) fuzzy completely \( e \)-irresolute, then \( g \circ f \) is fuzzy completely \( e \)-irresolute.
2. If \( f \) is fuzzy completely weakly \( e \)-irresolute and \( g \) is fuzzy \( e \)-irresolute, then \( g \circ f \) is fuzzy completely weakly \( g \circ f \)-irresolute.
3. If \( f \) is fuzzy completely continuous and \( g \) is fuzzy completely weakly \( e \)-irresolute, then \( g \circ f \) is fuzzy completely \( e \)-irresolute.
4. If \( f \) is fuzzy completely \( e \)-irresolute and \( g \) is fuzzy completely weakly \( e \)-irresolute, then \( g \circ f \) is fuzzy completely \( e \)-irresolute.
5. If \( f \) is fuzzy totally continuous and \( g \) is fuzzy completely weakly \( e \)-irresolute, then \( g \circ f \) is fuzzy completely \( e \)-irresolute.
6. If \( f \) is fuzzy completely weakly \( e \)-irresolute and \( g \) is fuzzy \( e \)-continuous, then \( g \circ f \) is fuzzy continuous.
7. If \( f \) is fuzzy \( e \)-continuous and \( g \) is fuzzy completely weakly \( e \)-irresolute, then \( g \circ f \) is fuzzy \( e \)-irresolute.
8. If \( f \) is fuzzy continuous and \( g \) is fuzzy completely weakly \( e \)-irresolute, then \( g \circ f \) is fuzzy completely weakly \( e \)-irresolute.

**Proof.** Obvious.

**Theorem 4.8.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) is fuzzy almost open surjective function and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) is function such that \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is fuzzy completely \( e \)-irresolute, then \( g \) is fuzzy completely weakly \( e \)-irresolute.

**Proof.** Let \( V \) be a fuzzy \( e \)-open set in \( Z \). Since \( g \circ f \) is fuzzy completely \( e \)-irresolute, \( (g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \) is fuzzy regular open in \( X \). Since \( f \) is fuzzy almost open surjective, \( f(f^{-1}(g^{-1}(V))) = g^{-1}(V) \) is fuzzy open in \( Y \). Therefore, \( g \) is fuzzy completely weakly \( e \)-irresolute.

**Theorem 4.9.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) is fuzzy open surjective function and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) is function such that \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is fuzzy completely weakly \( e \)-irresolute, then \( g \) is fuzzy completely weakly \( e \)-irresolute.

**Proof.** Similar to the proof of Theorem 4.6.

**Theorem 4.10.** Let \( P_i \) be the projection function from \( \prod X_i \) onto \( X_i \). If \( f : X \rightarrow \prod X_i \) is fuzzy completely weakly \( e \)-irresolute function, then \( P_i \circ f \) is also fuzzy completely weakly \( e \)-irresolute.

**Proof.** Obvious.

**Definition 4.11.** A collection \( \mu \) of fuzzy sets in a fuzzy space \( X \) is said to be cover [8] of a fuzzy set \( \eta \) of \( X \) if and only if
A fuzzy cover \( \eta \) of a fuzzy set \( \eta \) in a fuzzy space \( X \) is said to have a finite subcover if and only if there exists a finite subcollection 
\[
\rho = \{A_1, A_2, ..., A_n\} \text{ of } \mu \text{ such that } \left( \bigvee_{A \in \rho} A \right)(x) \geq \eta(x),
\]
for every \( x \in s(\eta) \), where \( s(\eta) \) denotes the support of a fuzzy set \( \eta \).

**Definition 4.12.** A fuzzy topological space \( X \) is called:
1. fuzzy compact [7] if every fuzzy open cover \( \lambda \) of \( X \) has a finite subcover.
2. fuzzy \( e \)-compact [16] if every fuzzy \( e \)-open cover \( \lambda \) of \( X \) has a finite subcover.
3. fuzzy \( e \)-closed [16] if every fuzzy \( e \)-open cover \( \lambda \) of \( X \) has a finite subfamily \( V \) of \( \lambda \) such that \( \left( \bigvee_{u \in V} e\text{Cl}(u) \right)(x) = 1 \) for each \( x \in X \).

**Theorem 4.13.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a fuzzy completely \( e \)-irresolute surjective function and \( X \) is fuzzy compact space, then \( Y \) is fuzzy \( e \)-compact.

**Proof.** Let \( \{V_x : x \in X\} \) be any fuzzy \( e \)-open cover of \( Y \). Then \( \{f^{-1}(V_x) : x \in X\} \) is a fuzzy open cover of \( X \). Since \( X \) is fuzzy compact there exists a finite subfamily \( f^{-1}(V_i) : i = 1, 2, ..., n \) of \( \{f^{-1}(V_x) : x \in X\} \) which covers \( X \). Hence, \( Y \) is fuzzy \( e \)-compact.

**Corollary 4.14.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a fuzzy completely \( e \)-irresolute surjective function and \( X \) is fuzzy compact space, then \( Y \) is fuzzy \( e \)-closed.

**Definition 4.15.** Two non-zero fuzzy sets \( A \) and \( B \) in \( X \) are said to be separated [13] (resp. fuzzy \( e \)-separated) if 
\[
\overline{A}q \text{Cl}(B) \text{ and } \overline{B}q \text{Cl}(A) \text{ (resp. } \overline{A}q e\text{Cl}(B) \text{ and } \overline{B}q e\text{Cl}(A)).
\]

**Definition 4.16.** A fuzzy topological space \( X \) is said to be fuzzy connected [14] (resp. fuzzy \( e \)-connected) if it cannot be expressed as the union of two fuzzy separated (resp. fuzzy \( e \)-separated) sets.

**Lemma 4.17.** Two non-zero fuzzy sets \( A \) and \( B \) are fuzzy \( e \)-separated if and only if there exist two fuzzy \( e \)-open sets \( U \) and \( V \) such that \( A \subseteq U, B \subseteq V, AqV \) and \( BqU \).

**Theorem 4.18.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a fuzzy completely weakly \( e \)-irresolute surjective function. If \( U \) is a fuzzy connected subset in \( X \), then \( f(U) \) is fuzzy \( e \)-connected in \( Y \).

**Proof.** Suppose that \( f(U) \) is not \( e \)-connected in \( Y \). Then there exist fuzzy \( e \)-separated subsets \( G \) and \( H \) in \( Y \), such that \( f(U) = G \lor H \). By Lemma 4.1, there exist fuzzy \( e \)-open subsets \( V \) and \( W \) such that \( G \subseteq V, H \subseteq W, GqV \) and \( HqW \). Since \( f \) is fuzzy completely weakly \( e \)-irresolute, \( f^{-1}(G) \) and \( f^{-1}(H) \) are fuzzy open in \( X \) and \( U = f^{-1}(f(U)) = f^{-1}(G \lor H) = f^{-1}(G) \lor f^{-1}(H) \). It is clear that \( f^{-1}(G) \) and \( f^{-1}(H) \) are fuzzy separated in \( X \). Therefore, \( U \) is not fuzzy connected in \( X \).

**Corollary 4.19.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a fuzzy completely weakly \( e \)-irresolute surjective function. If \( U \) is a fuzzy connected subset in \( X \), then \( f(U) \) is also fuzzy \( e \)-connected.

**Theorem 4.20.** A function \( f : X \to Y \) is fuzzy completely weakly \( e \)-irresolute if the graph function \( g : X \to X \times X \), defined by \( g(x) = (x, g(x)) \) for each \( x \in X \) is fuzzy completely weakly \( e \)-irresolute.

**Proof.** Let \( V \) be any fuzzy \( e \)-open set of \( Y \). Then \( 1 \times V \) is a fuzzy \( e \)-open set of \( X \times Y \). Since \( g \) is fuzzy completely \( e \)-irresolute, \( f^{-1}(V) = g^{-1}(1 \times V) \) is fuzzy regular open in \( X \). Thus \( f \) is fuzzy completely weakly \( e \)-irresolute.

**Theorem 4.21.** If \( f : (X, \tau) \to (Y, \sigma) \) is fuzzy completely weakly \( e \)-irresolute function and \( Y \) is fuzzy \( e \)-connected, then \( X \) is fuzzy Hausdorff.

**Proof.** Let \( x, y \) be any two distinct points of \( X \). Since \( f \) is injective, we have \( f(x) \neq f(y) \). Since \( Y \) is fuzzy \( e \)-connected, there exists \( V \) and \( W \) are \( e \)-open sets in \( Y \) such that \( V \equiv W = 0 \). Since \( f \) is fuzzy completely weakly \( e \)-irresolute, there exists fuzzy open sets \( G \) and \( H \) in \( X \) such that \( f(G) \equiv V \) and \( f(H) \equiv W \). Hence we obtain \( G \equiv H = 0 \). This shows that \( X \) is fuzzy Hausdorff.

**Theorem 4.22.** If a function \( f : X \to Y \) is a fuzzy completely weakly \( e \)-irresolute surjection and \( X \) is fuzzy connected, then \( Y \) is fuzzy \( e \)-connected.

**Proof.** Suppose that \( Y \) is not fuzzy \( e \)-connected. There exists non empty fuzzy \( e \)-open sets \( V \) and \( W \) of \( Y \) such that \( Y = V \lor W \). Since \( f \) is fuzzy completely weakly \( e \)-irresolute
\[ f^{-1}(V) \text{ and } f^{-1}(W) \] are fuzzy open sets and \[ X = f^{-1}(V) \lor f^{-1}(W). \] This shows that \( X \) is not fuzzy connected. This is a contradiction.

**CONCLUSION**

We have defined and proved basic properties of Fuzzy Completely \( e \)-Irresolute Functions and Fuzzy Completely Weakly \( e \)-Irresolute Function. Many results have been established to show how far topological structures are preserved by these \( e \)-Irresolute Functions. We also have provided examples where such properties fail to be preserved.

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