Birkhoff and New Orthogonality in Normed Linear Spaces Via 2-HH Norm

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Abstract

The p-HH norms were introduced by Kikianty and Dragomir on the Cartesian square of normed spaces. P-norms and p-HH norms induces the same topology, so they are equivalent, but geometrically they are different. Besides that, E. Kikianty and S.S. Dragimor introduced HH-P orthogonality and HH-I orthogonality by using 2-HH norm and discussed main properties of these orthogonalities. In this paper, we test the concept of 2-HH norm to Birkhoff and a new orthogonality in normed spaces and discuss some properties of these orthogonalities.

Keywords: Birkhoff orthogonality, Hermite-Hadamard’s inequality, Pythagorean orthogonality, p-HH norm, Logarithmic mean

1 Introduction

An inner-product on X defines a norm on X by \( \|x\|^2 = \langle x, x \rangle \). Every innerproduct spaces are normed spaces, but the converse may not be true. A best example of normed space which is not an inner-product space is \( l^p = \{(x_n), x_n \in \mathbb{R} : \sum |x_n| < \infty \} \) for \( p \neq 2 \).

Definition. The \( p-HH \) norm on \( X^2 = X \times X \) is defined by

\[
\|(x, y)\|_{p-HH} = \left( \int_0^1 \|(1-t)x + ty\|^p \, dt \right)^{\frac{1}{p}}
\]

for any \( x, y \in X^2 \) and \( 1 \leq p < \infty \).

The 2-HH norm is defined as follows:

\[
\|(x, y)\|_{2-HH}^2 = \int_0^1 \|(1-t)x + ty\|^2 \, dt = \frac{1}{3} \left( \|x\|^2 + \langle x, y \rangle + \|y\|^2 \right)
\]

The p-HH norms are equivalent to p-norms on \( X^2 \), as they induce the same topology, but geometrically they are different. The p-HH norm is an extension of the generalized logarithmic mean which is connected by the Hermite-Hadamards inequality to p-norm. The definition of the generalized logarithmic mean and Hermite-Hadamards inequality are as follows:
**Definition.** [12] For any convex function \( f : [a, b] \to \mathbb{R} \), the Hermite-Hadamard’s inequality is defined as
\[
(b - a) f\left(\frac{a + b}{2}\right) \leq \int_a^b f(t) dt \leq (b - a) \left[ \frac{f(a) + f(b)}{2} \right].
\]
This inequality has been extended (see [12]) for convex function \( f : [x, y] \to \mathbb{R} \), where \([x, y] = \{ (1 - t)x + ty, t \in [0, 1] \}\). In that case Hermites-Hadamards integral inequality becomes
\[
f\left(\frac{x + y}{2}\right) \leq \int_0^1 f[(1 - t)x + ty] dt \leq \frac{f(x) + f(y)}{2} \tag{1}
\]
Using the convexity of \( f(x) = \|x\|^p \quad (x \in X, p \geq 1) \) and relation (1) we have
\[
\left\|\frac{x + y}{2}\right\|^p \leq \left[ \int_0^1 \| (1 - t)x + ty \|^p dt \right]^\frac{1}{p} \leq \frac{1}{2^\frac{p}{2}} (\|x\|^p + \|y\|^p)^\frac{1}{p}.
\]

### 1.1 HH-P Orthogonality

**Definition.** [3, 4] A vector \( x \) is said to be orthogonal to \( y \) in the sense of Pythagorean if \( \|x - y\|^2 = \|x\|^2 + \|y\|^2 \).

[8] Let \((X, \|\cdot\|)\) be a normed space. Then \( x \perp_{\text{HH-P}} y \iff \int_0^1 \|(1 - t)x + ty\|^2 dt = \frac{1}{2}(\|x\|^2 + \|y\|^2) \).

**1.1.1 Properties of HH-P orthogonality**

1. HH-P orthogonality satisfies non-degeneracy, simplification, continuity and symmetry.
2. HH-P orthogonality is existent.
3. HH-P orthogonality is unique.
4. HH-P orthogonality is homogeneous if and only if the space is inner-product space.
5. HH-P orthogonality is additive if the space is an inner-product space.

### 1.2 HH-I orthogonality

**Definition.** [5] A vector \( x \) is said to be isosceles orthogonal to \( y \) if \( \|x - y\| = \|x + y\| \).

[8] Let \( x, y \in X \) such that \( \|(1 - t)x + ty\| = \|(1 - t)x - ty\| \) a.e. on \([0, 1]\). Then \( x \) is said to be HH-I orthogonal to \( y \) iff
\[
\int_0^1 \|(1 - t)x + ty\| dt = \int_0^1 \|(1 - t)x - ty\| dt.
\]
1.2.1 Properties of HH-I Orthogonality

1. The HH-I orthogonality satisfies non-degeneracy, simplification, continuity and symmetry properties.

2. HH-I orthogonality is existent.

3. If HH-I orthogonality is homogeneous in a normed space X, then X is an inner-product space.

4. If HH-I orthogonality is additive, then the space is an inner-product space.

5. HH-I orthogonality is neither right additive nor homogeneous.

Definition. [2] In a normed linear space \( X \),

\[ x \perp y \iff \sum_{k=1}^{m} a_k \|b_k x + c_k y\|^2 = 0, \]

where \( m \geq 2 \) and \( a_k, b_k, c_k \) are real numbers such that

\[ \sum_{k=1}^{m} a_k b_k c_k \neq 0, \quad \sum_{k=1}^{m} a_k b_k^2 = \sum_{k=1}^{m} a_k c_k^2 = 0. \]

1.3 HH-C Orthogonality

[8] Let \((X, \|\|)\) be a normed space and \( t \in [0, 1] \). then \( x \in X \) is said to be HH-C orthogonal to \( y \in X \) if and only if

\[ \sum_{j=1}^{m} \alpha_j \int_{0}^{1} \|(1 - t)\beta_j x + t\gamma_j y\|^2 dt = 0 \]

satisfying the conditions

\[ \sum_{j=1}^{m} \alpha_j \beta_j \gamma_j \neq 0 \quad \text{and} \quad \sum_{j=1}^{m} \alpha_j \beta_j \gamma_j^2 = 0. \]

HH-P orthogonality is a particular case of HH-C orthogonality

Let us take

\[ \sum_{j=1}^{3} \alpha_j \int_{0}^{1} \|(1 - t)\beta_j x + t\gamma_j y\|^2 dt = 0 \]

\[ \Rightarrow \alpha_1 \int_{0}^{1} \|(1 - t)\beta_1 x + t\gamma_1 y\|^2 dt + \alpha_2 \int_{0}^{1} \|(1 - t)\beta_2 x + t\gamma_2 y\|^2 dt + \alpha_3 \int_{0}^{1} \|(1 - t)\beta_3 x + t\gamma_3 y\|^2 dt = 0 \]
Taking $\alpha_1 = -1$, $\alpha_2 = \alpha_3 = 1$, $\beta_1 = \beta_2 = 1$, $\beta_3 = 0$, $\gamma_1 = \gamma_3 = 1$ and $\gamma_2 = 0$, we get

$$
- \int_0^1 \|(1 - t)x + ty\|^2 dt + \int_0^1 \|(1 - t)x\|^2 dt + \int_0^1 \|ty\|^2 dt = 0
$$

$$
\Rightarrow - \int_0^1 \|(1 - t)x + ty\|^2 dt + \frac{1}{3}(\|x\|^2 + \|y\|^2) = 0
$$

$$
\therefore \int_0^1 \|(1 - t)x + ty\|^2 dt = \frac{1}{3}(\|x\|^2 + \|y\|^2)
$$

Now

$$
\sum_{k=1}^3 \alpha_j \beta_j \gamma_j = \alpha_1 \beta_1 \gamma_1 + \alpha_2 \beta_2 \gamma_2 + \alpha_3 \beta_3 \gamma_3 = -1, \quad \sum_{j=1}^m \alpha_j \beta_j \gamma_j^2 = \alpha_1 \beta_1^2 + \alpha_2 \beta_2^2 + \alpha_3 \beta_3^2 = 0
$$

and

$$
\sum_{j=1}^m \alpha_j \gamma_j = \alpha_1 \gamma_1 + \alpha_2 \gamma_2 + \alpha_3 \gamma_3 = 0
$$

Which shows that HH-P orthogonality is a particular case of HH-C orthogonality.

**HH-I orthogonality is a particular case of HH-C orthogonality**

Let us take

$$
\sum_{j=1}^2 \alpha_j \int_0^1 \|(1 - t)\beta_j x + t\gamma_j y\|^2 dt = 0
$$

$$
\Rightarrow \alpha_1 \int_0^1 \|(1 - t)\beta_1 x + t\gamma_1 y\|^2 dt + \alpha_2 \int_0^1 \|(1 - t)\beta_2 x + t\gamma_2 y\|^2 dt = 0
$$

Taking $\alpha_1 = \frac{1}{2}$, $\alpha_2 = -\frac{1}{2}$, $\beta_1 = \beta_2 = 1$, $\gamma_1 = 1$, $\gamma_2 = -1$, we get

$$
\frac{1}{2} \int_0^1 \|(1 - t)x + ty\|^2 dt - \frac{1}{2} \int_0^1 \|(1 - t)x - ty\|^2 dt = 0
$$

$$
\Rightarrow \int_0^1 \|(1 - t)x + ty\|^2 dt = \int_0^1 \|(1 - t)x - ty\|^2 dt
$$

Now

$$
\sum_{k=1}^2 \alpha_j \beta_j \gamma_j = \alpha_1 \beta_1 \gamma_1 + \alpha_2 \beta_2 \gamma_2 = 1, \quad \sum_{k=1}^2 \alpha_j \beta_j^2 = \alpha_1 \beta_1^2 + \alpha_2 \beta_2^2 = 0
$$

and

$$
\sum_{k=1}^2 \alpha_j \gamma_j^2 = \alpha_1 \gamma_1^2 + \alpha_2 \gamma_2^2 = 0
$$

**1.3.1 Properties of HH-C orthogonality**

1. HH-C orthogonality satisfies non-degeneracy, simplification, and continuity property.

2. HH-C orthogonality is not symmetric.

3. HH-C orthogonality is neither additive nor homogeneous.
2 Main Result

Definition. [11] A vector $x$ is orthogonal to $y$ if

$$\left\| x + \frac{1}{2}y \right\|^2 + \left\| x - \frac{1}{2}y \right\|^2 = \frac{1}{2} \left\| \sqrt{2}x + y \right\|^2 + \left\| x \right\|^2$$

Lemma 2.1. For an abstract Euclidean Space $X$, orthogonality relation $\left\| x + \frac{1}{2}y \right\|^2 + \left\| x - \frac{1}{2}y \right\|^2 = \frac{1}{2} \left\| \sqrt{2}x + y \right\|^2 + \left\| x \right\|^2$ implies Birkhoff orthogonality if $y = \frac{x}{1-\alpha}$.

Proof. Suppose $x \perp y$. Then by definition,

$$\left\| x + \frac{1}{2}y \right\|^2 + \left\| x - \frac{1}{2}y \right\|^2 = \frac{1}{2} \left\| \sqrt{2}x + y \right\|^2 + \left\| x \right\|^2$$

$$\Rightarrow \left\| x + \frac{1}{2}y \right\|^2 + \left\| x - \frac{1}{2}y \right\|^2 \geq \left\| x \right\|^2$$

$$\Rightarrow \left\| x + \frac{1}{2}y - x + \frac{1}{2}y \right\|^2 \geq \left\| x \right\|^2$$

$$\Rightarrow \left\| y \right\|^2 \geq \left\| x \right\|^2 \ldots \ldots \ldots \ (1)$$

Since $y = \frac{x}{1-\alpha}$ so that $y = x + \alpha y$. Therefore form the relation (1)

$$\left\| x + \alpha y \right\|^2 \geq \left\| x \right\|^2$$

$$\Rightarrow \left\| x + \alpha y \right\| \geq \left\| x \right\|$$

$$\Rightarrow x \perp_B y.$$ 

But the converse of above lemma may not be true. Consider $X = (\mathbb{R}^2, \left\| . \right\|_1)$, where $\left\| . \right\|_1 = \sum_{k=1}^{2} |x_k|$ for some $x = (x_1, x_2) \in X$. Let $x = (-2, 1), y = (2, 2)$, and $\alpha \in \mathbb{R}$ we have

$$\left\| x + \alpha y \right\|_1 = \left\| (2, 1) + \alpha (2, 2) \right\|_1 = -2 + 2\alpha, 1 + 2\alpha \right\|_1 = | -2 + 2\alpha | + | 1 + 2\alpha | \geq 3 = \left\| x \right\|_1$$

But

$$\left\| x + \frac{1}{2}y \right\|^2 + \left\| x - \frac{1}{2}y \right\|^2 = \left\| (-2, 1) + \frac{1}{2}(2, 2) \right\|^2 + \left\| (-2, 1) - \frac{1}{2}(2, 2) \right\|^2$$

$$= \left\| (-2, 1) + (1, 1) \right\|^2 + \left\| (-2, 1) - (1, 1) \right\|^2$$

$$= 18$$

$$\frac{1}{2} \left\| \sqrt{2}x + y \right\|^2 + \left\| x \right\|^2 = \frac{1}{2} \left\| \sqrt{2}(-2, 1) + (2, 2) \right\|^2 + \left\| (-2, 1) \right\|^2$$

$$= \frac{1}{2} \left\| (-2\sqrt{2} + 2, \sqrt{2} + 2) \right\|^2 + 9$$

$$= \frac{1}{2} (0.828 + 3.4142)^2 + 9$$

$$= 17.99$$

which shows that $x$ is not orthogonal to $y$ in the sense of above orthogonality.
3  Birkhoff Orthogonality Via 2-HH norm

Definition. [6, 9] A vector $x$ is said to be orthogonal to $y$ in the sense of Birkhoff if $\|x\| \leq \|x + \alpha y\|$ for all $\alpha \in \mathbb{R}$.

In the case of $2-HH$ norm,

$$
\int_0^1 \|(1-t)x + \lambda ty\|^2 = \int_0^1 \langle (1-t)x + \lambda ty, (1-t)x + \lambda ty \rangle dt
= \|x\|^2 \int_0^1 (1-t)^2 dt + 2\lambda \langle x, y \rangle \int_0^1 (1-t)dt + \lambda^2 \|y\|^2 \int_0^1 t^2 dt.
$$

If $x \perp$, then

$$
\int_0^1 \|(1-t)x + \lambda ty\|^2 = \|x\|^2 \int_0^1 (1-t)^2 dt + \lambda^2 \|y\|^2 \int_0^1 t^2 dt
= \frac{1}{3}(\|x\|^2 + \|\lambda y\|^2) \quad ... \ (1)
$$

But $\int_0^1 \|x\|^2 dt = \|x\|^2 \int_0^1 (1-t)^2 dt = \frac{1}{3} \|x\|^2. \quad ... \ (2)$

Since $\|\lambda y\|^2$ is a non-negative quantity, so from relation (1) and (2), we conclude that

$$
\int_0^1 \|(1-t)x + \lambda ty\|^2 \geq \int_0^1 \|x\|^2 dt. \quad ... \ (3)
$$

Keeping the above result in our mind, we can conclude that $x \perp_{2-HH} B(y)$ if the relation (3) is satisfied.

4  New Orthogonality Via 2-HH Norm

[11] A vector $x \in X$ is said to be orthogonal to the vector $y \in Y$ if and only if

$$
\left\| x + \frac{1}{2} y \right\|^2 + \left\| x - \frac{1}{2} y \right\|^2 = \frac{1}{2} \left\| \sqrt{2}x + y \right\|^2 + \|x\|^2.
$$

Using the concept of $2-HH$ norm,

$$
\left\| x + \frac{1}{2} y \right\|^2 + \left\| x - \frac{1}{2} y \right\|^2 = \frac{1}{2} \left\| \sqrt{2}x + y \right\|^2 + \|x\|^2 \ a.e \ \text{on} \ [0,1]
$$

and we obtain a definition of new orthogonality by using 2-HH norm is as follows: $x \perp y$ iff

$$
\int_0^1 \left\| (1-t)x + \frac{1}{2} ty \right\|^2 dt + \int_0^1 \left\| (1-t)x - \frac{1}{2} ty \right\|^2 dt = \frac{1}{2} \int_0^1 \left\| \sqrt{2}(1-t)x + ty \right\|^2 dt + \int_0^1 \| (1-t)x \|^2 dt.
$$

..........(1)
To verify the above definition, the left hand side of relation (1)

$$\int_0^1 \left\| (1-t)x + \frac{1}{2}ty \right\|^2 dt + \int_0^1 \left\| (1-t)x - \frac{1}{2}ty \right\|^2 dt = \int_0^1 \langle (1-t)x + \frac{1}{2}ty, (1-t)x + \frac{1}{2}ty \rangle dt$$

$$+ \int_0^1 \langle (1-t)x - \frac{1}{2}ty, (1-t)x - \frac{1}{2}ty \rangle dt$$

$$= \frac{1}{3} \|x\|^2 + \frac{1}{12} \|y\|^2 + \frac{1}{3} \|x\|^2 + \frac{1}{12} \|y\|^2$$

$$= \frac{2}{3} \|x\|^2 + \frac{1}{6} \|y\|^2.$$

Again the right hand side of relation (1)

$$\frac{1}{2} \int_0^1 \left\| \sqrt{2}(1-t)x + ty \right\|^2 dt + \int_0^1 \| (1-t)x \|^2 dt = \frac{1}{2} \int_0^1 \langle \sqrt{2}(1-t)x + ty, \sqrt{2}(1-t)x + ty \rangle dt + \frac{1}{3} \|x\|^2$$

$$= \frac{1}{2} \left( \frac{2}{3} \|x\|^2 + \frac{1}{3} \|y\|^2 \right) + \frac{1}{3} \|x\|^2$$

$$= \frac{2}{3} \|x\|^2 + \frac{1}{6} \|y\|^2.$$

**Data Availability**

There is no use of any data for the completion of this study.

**Conflict of Interest**

We authors do no have a conflict of interest for the publication of article.

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**References**


