Approximated Properties of the Modified Periodogram for Stationary Time Series of Two Vector Valued With Missed Data

A.I. EL-Desokey

Abstract - The asymptotic properties of the periodogram for stationary two vector valued time series with missed data is presented, and the dispersion properties are investigated.

Keywords: Data Window, Discrete Time Series, Modified Periodogram, Stability Process, Time Series.

1 INTRODUCTION

This paper is interested to investigate the asymptotic moments of the modified Periodogram based on Data window properties. Many authors as, D.R. Brillinger, [1], EA Farag, MA Ghazal [2], [3], M. A. Ghazal, G.S. Mokaddis, and A. El-Desokey [4], M. A. Ghazal, E. A. El-Desokey,., and Alargt, M.A [5], [6], Ghazal, M. A., A. I. El-Desokey, and A. M. Ben Aros [7], studied the statistical analysis of function of time series with missed observations of discrete and continuous cases. The paper is organized as follow, Section 1, Introduction, Section 2 we will study the approximated properties of the modified periodogram for two vector valued stationary time series with missed observations, section 3 will study the dispersion of the modified periodogram for two vector valued stationary time series with missed observations.

2 THE ASYMPTOTIC PROPERTIES OF THE MODIFIED PERIODOGRAM FOR TWO VECTOR VALUED STATIONARY TIME SERIES WITH MISSED OBSERVATIONS

Let an \((i+j)\) two vector-valued stationary time series
\[
\tau(t) = [X(t) \quad \psi(t)]^\top
\]
t = 0,±1,±2,... with \(X(t)\), \(i\)-vector-valued and \(\psi(t)\) \(j\)-vector-valued. Let the series (2.1) is a strictly stationary \((i+j)\) vector-valued time series with \([X_i(t) \quad \psi_r(t)]^\top\), \(s = 1,2,...,i, r = 1,2,...,j\), with existing moments as follows,
\[
EX(t) = R_x, \quad E\psi(t) = R_{\psi},\quad (2.2)
\]
and the covariances
\[
E\{X(t) - R_x\} [X(t) - R_x]^\top = R_{xx}(v)
\]
\[
E\{X(t) + v - R_x\} [X(t) + v - R_x]^\top = R_{xx}(v), \quad (2.3)
\]
\[
E\{\psi(t) - R_{\psi}\} [\psi(t) - R_{\psi}]^\top = R_{\psi\psi}(v), \quad (2.4)
\]
for \(-\infty < h < \infty\).

From the previous we consider the following Assumption,

Assumption 1. Let \(X_i(t)\), \(t \in R, a = \{1\} i\) be bounded, and vanishes for \(t > T - 1, t < 0\), is called data window. Then consider
\[
\gamma_{a_1,...,a_l} (h) = \sum_{t=0}^{T-1} \prod_{n=1}^{l} X_{a_n}^\top (t) \exp\{-iht\},
\]
for \(-\infty < h < \infty\), and \(a_1,...,a_l = 1,2,...,i\).

Consider,
\[
d_{a} (h)\) is the discrete expanded finite Fourier transform which is defined by

Lecturer in Higher Future Institute for Specialized Technological Studies.

Egypt
Bernoulli sequence of random variables which satisfies assumption II with mean zero, and let \( \tilde{\lambda}_a(t), -\infty < t < \infty \), satisfies

\[
I^{(T)}_{ab}(h) = \left[ I^{(T)}_{ab}(h) \right] = \left[ 2\pi \gamma^{(T)}_{ab}(0) \right]^{-1} C^{(T)}_a(h)C^{(T)}_b(h),
\]

where

\[
E[I^{(T)}_{ab}(h)] = P_{ab} \left[ f_{a_{ab}}(h) f_{b_{ab}}(h) \right] + O(T^{-1}) + O(T^{-1})
\]

where, \( O(T^{-1}) \) is uniform in \( h \).

\[
\gamma^{(T)}_{ab}(h) = \left[ \left( 0 \right) \right]^{-1} \times \left[ P^{4} \gamma^{(T)}_{ab}(h-h) \gamma^{(T)}_{ab}(h-h) \right]^{\eta} + T^{-2} N^{(T)}_{a_{ab}b_{ab}}(h, h, h) + O(T^{-1}),
\]

where,

\[
\eta = \begin{bmatrix} f_{a_{ab}}(h) & f_{a_{ab}}(h) \\ f_{b_{ab}}(h) & f_{b_{ab}}(h) \end{bmatrix}
\]

\[
A = \begin{bmatrix} f_{a_{ab}}(-h) & f_{a_{ab}}(-h) \\ f_{b_{ab}}(-h) & f_{b_{ab}}(-h) \end{bmatrix}
\]

Proof. The proof is omitted.

Assumption II. Let \( X(t) \) is a strictly stationary time series whose moments exist. For each \( s = 1, 2, ..., k-1 \) and any k-tuple \( a_1, a_2, ..., a_k \) we have

\[
\sum_{t=0}^{T-1} \sum_{i=0}^{T-1} \lambda^{(T)}_{ab}(t) \lambda^{(T)}_{ab}(t+1) \leq C^{(T)}_a(h)C^{(T)}_b(h),
\]

where

\[
\lambda^{(T)}_{ab}(t) = \left[ \left( 0 \right) \right]^{-1} \times \left[ P^{4} \lambda^{(T)}_{ab}(t) \lambda^{(T)}_{ab}(t) \right]^{\gamma^{(T)}_{ab}(0)}
\]

\[
= \left[ 2\pi \gamma^{(T)}_{ab}(0) \right]^{-1} \times \left[ P^{4} \gamma^{(T)}_{ab}(h-h) \gamma^{(T)}_{ab}(h-h) \right]^{\eta} + T^{-2} N^{(T)}_{a_{ab}b_{ab}}(h, h, h) + O(T^{-1}),
\]

using assumption I. then we have, \( \gamma^{(T)}_{ab}(0) = O(T) \), and

\[
E[I^{(T)}_{ab}(h)] = P_{ab} \left[ f_{a_{ab}}(h) f_{b_{ab}}(h) B(h) f_{a_{ab}}(h) B(h)^T \right]
\]

Theorem 2.2. Let \( \alpha_a(t) = \ell_a(t) \tau_a(t) \), \( a = 1, 2, ..., \min(i, j) \) and \( \ell_a(t) \) is
\[
\begin{align*}
\text{IJSER} & \quad \text{and using} \\
& \quad \text{hen we have} \\
& = \sum_{t_1, t_2 = 0}^{T-1} \lambda^{(T)}_{a_1}(t_1) \alpha^{(t_2)}_{a_1}(t_2) \exp(iht_1), \\
= & \sum_{t_1, t_2 = 0}^{T-1} \lambda^{(T)}_{a_2}(t_1) \alpha^{(t_2)}_{a_2}(t_2) \exp(iht_2), \\
& = \sum_{t_1, t_2 = 0}^{T-1} \lambda^{(T)}_{b_1}(t_1) \lambda^{(T)}_{b_2}(t_2) \exp(iht_1 - iht_2) \times \\
& \times \left[ p_{h_{b_1}, h_{b_2}} R_{xx}(t_1 - t_2) \quad p_{h_{b_2}, R_{xy}}(t_1 - t_2) \right] \\
& \times \left[ p_{h_{b_2}, R_{yy}}(t_1 - t_2) \quad p_{h_{b_1}, h_{b_2}} B(h) R_{xx}(t_1 - t_2) B(h)^T \right], \\
\end{align*}
\]

putting \( t_1 - t_2 = v_1, t_2 = t \Rightarrow t_1 = t + v_1 \), and using assumption I. Then we have

\[
V_1 = 2\pi p_{a_{b_1}, a_{b_2}}(h - \delta) \left[ \begin{array}{c}
f_{a_{b_1}}(h) \\
\frac{f_{a_{b_2}}(h)}{B(h) f_{a_{b_2}}(h) B(h)^T} \\
\end{array} \right] + \\
\left[ O(1) \quad O(1) \right] + \\
\left[ O(1) \quad O(1) \right], \\
\text{(2.15)}
\]

\[
V_2 = \text{Cov}\left\{ C^{(T)}(h), C^{(T)}(h) \right\}
\]

\[
= \text{Cov}\left\{ \sum_{t_1 = 0}^{T-1} \lambda^{(T)}_{b_1}(t_1) \alpha^{(t_2)}_{b_1}(t_1) \exp(iht_1), \right\} \\
= \text{Cov}\left\{ \sum_{t_2 = 0}^{T-1} \lambda^{(T)}_{b_2}(t_2) \alpha^{(t_2)}_{b_2}(t_2) \exp(iht_2), \right\} \\
= \sum_{t_1, t_2 = 0}^{T-1} \lambda^{(T)}_{b_1}(t_1) \lambda^{(T)}_{b_2}(t_2) \exp(iht_1 - iht_2) \times \\
\times \left[ p_{h_{b_1}, h_{b_2}} R_{xx}(t_1 - t_2) \quad p_{h_{b_2}, R_{xy}}(t_1 - t_2) \right] \\
\times \left[ p_{h_{b_2}, R_{yy}}(t_1 - t_2) \quad p_{h_{b_1}, h_{b_2}} B(h) R_{xx}(t_1 - t_2) B(h)^T \right], \\
\]
\[ N_2 = 2\pi p_{h,h} \gamma_{h,h} (-h-h) \left[ f_{a,b_2}(-h) f_{a,b_2}(-h) \right] + \left[ O(1) \quad O(1) \right] + \left[ O(1) \quad O(1) \right] \]  
\[ S = \text{cum} \left\{ C_{a_i}^{(h)}(h), C_{b_i}^{(h)}(-h), C_{a_2}^{(h)}(h), C_{b_2}^{(h)}(-h) \right\} \]
\[ = \sum_{t_{1,2,3,4}^{(h)}} \lambda^{(h)}(t_1) \lambda^{(h)}(t_2) \lambda(t_3) \lambda(t_4) \exp(-iht_1) \times \exp(iht_2) \exp(-iht_3) \exp(iht_4) \times \text{Cov} \left\{ \alpha_{a_1}(t_1), \alpha_{b_1}(t_2), \alpha_{a_2}(t_3), \alpha_{b_2}(t_4) \right\} \]
\[ = p_{a,b_1,b_2} (2\pi)^3 f_{a_1a_1a_1a_2a_2} (h,-h,h) \gamma_{a,b_1b_2}^{(h)}(0) + O(1) \]
where,
\[ \gamma_{a,b_1b_2}^{(h)}(0) = \sum_{i=0}^{T-1} \lambda^{(h)}(t_i + v_i) \lambda^{(h)}(t_i + v_2) \lambda^{(h)}(t_i + v_3) \lambda^{(h)}(t_i), \]
now,
\[ \text{Cov} \left\{ I_{a,b_1}^{(h)}(h), I_{a,b_2}^{(h)}(h) \right\} = D(V + N + S) \]
\[ = D \left\{ (V_1 \times V_2) + (N_1 \times N_2) + S \right\} \]
\[ = (2\pi)^{-2} \left\{ \gamma_{a,b_1}^{(h)}(0) \gamma_{a,b_2}^{(h)}(0) \right\}^{-1} \times \]
\[ \times \left\{ (2\pi) p_{a,b_1} \gamma_{a,b_1} (h-h) \times \eta + O \right\} \times \]
\[ \times (2\pi) p_{b,b_2} \gamma_{b,b_2} (-h-h) \times A + O \] + \[ (2\pi) p_{b,b_2} \gamma_{a,b_2} (h-h) \times \eta + O \] + \[ (2\pi) p_{b,b_2} \gamma_{b,b_2} (-h-h) \times A + O \] + \[ p_{a,b_1,b_2} (2\pi)^3 f_{a_1a_1a_1a_2a_2} (h,-h,h) \gamma_{a,b_1b_2}^{(h)}(0) + O(1) \right\}, \]
where,
\[ \eta = \left[ f_{a,a_1}(h) \quad f_{a,b_2}(h) \right] \quad \left[ f_{b,a_2}(h) \quad B(h) f_{a,a_2}(h) B(h)^T \right] \],
\[ A = \left[ f_{a,a_2}(-h) \quad f_{a,b_2}(-h) \right] \quad \left[ f_{b,a_2}(-h) \quad B(h) f_{a,a_2}(-h) B(h)^T \right] \],
\[ O = \left[ O(1) \quad O(1) \right] \quad \left[ O(1) \quad O(1) \right] \],
then,
\[ \text{Cov} \left\{ I_{a,b_1}^{(h)}(h), I_{a,b_2}^{(h)}(h) \right\} = (2\pi)^{-2} \left\{ \gamma_{a,b_1}^{(h)}(0) \gamma_{a,b_2}^{(h)}(0) \right\}^{-1} \times \]
\[ \times \left\{ (2\pi)^2 p_{a,b_1} p_{b,b_2} \gamma_{a,a_2} (h-h) \gamma_{b,b_2} (-h-h) \eta A + \right\] + \[ (2\pi)^2 p_{a,b_2} p_{b,a_2} \gamma_{a,b_2} (h+h) \gamma_{b,a_2} (-h-h) \eta A \] + \[ (2\pi)^{-2} \left\{ \gamma_{a,b_1}^{(h)}(0) \gamma_{a,b_2}^{(h)}(0) \right\}^{-1} \times \]
\[ \times (2\pi) p_{a,b_1} \gamma_{a,b_1} (h-h) A + (2\pi) p_{b,b_2} \gamma_{b,b_2} (-h-h) A + \] + \[ (2\pi)^{-2} \left\{ \gamma_{a,b_1}^{(h)}(0) \gamma_{a,b_2}^{(h)}(0) \right\}^{-1} \times \]
\[ \times (2\pi) p_{a,b_1} \gamma_{a,b_1} (h-h) A + (2\pi) p_{b,b_2} \gamma_{b,b_2} (-h-h) A + \] + \[ K_1 + K_2 + K_3 + K_4 \]
From the bounded of \( \lambda^{(h)}(t) \), we have \( \gamma_{a,b}^{(h)}(0) = O(T) \) and \( \gamma_{a,b}^{(h)}(0) = O(T) \), 1, ..., \( \min(i, j) \), and
\[ f_{a_1a_1a_1a_2a_2} (h,-h,h) \] is bounded by a constant \( H \),
\[ a_1, b_1 = 1, ..., \min(i, j), i = 1, ..., k, h, h \in R \), then,
\[ K_3 = O(T^{-1}) + O(T^{-2}) = O(T^{-1}) \]
Also,
\[ K_4 = (2\pi)^{-2} \left\{ \gamma_{a,b_1}^{(h)}(0) \gamma_{a,b_2}^{(h)}(0) \right\}^{-1} \times \]
\[ O^2 + O^2 \] + \[ (2\pi)^{-2} \left\{ \gamma_{a,b_1}^{(h)}(0) \gamma_{a,b_2}^{(h)}(0) \right\}^{-1} \times O \] + \[ (2\pi)^{-2} \left\{ O(T) O(T) \right\}^{-1} \times O \]
\[ = O(T^{-2}) \times O = O(T^{-2}) \left[ O(1) \quad O(1) \right] \quad \left[ O(1) \quad O(1) \right] = \]
using \( K_3, K_4 \) into (2.19). Then,
\[ \text{Cov} \left\{ I_{a,b_1}^{(h)}(h), I_{b,b_2}^{(h)}(h) \right\} = \left\{ \gamma_{a,b_1}^{(h)}(0) \gamma_{a,b_2}^{(h)}(0) \right\}^{-1} \times \]
\[ \times \left[ (p_{a,b_1} p_{b,b_2} \gamma_{a,b_1} (h-h) \gamma_{b,b_2} (h-h) \eta A \right] \]
+ \[ p_{a,b_2} p_{b,a_2} \gamma_{a,b_2} (h+h) \gamma_{b,a_2} (-h+h) \eta A \] + \[ T^{-2} N_{a,a_2a_2b_2}^{(h)} (h,h) + O(T^{-1}) \],
where there exists any constant \( N \) such that,
\[ T^{-2} N_{a,b:2}^{(T)}(h,h) \leq \]
\[ W \{ \gamma_{a,b_1}(h-h) + \gamma_{b_2}(h-h) + \gamma_{a,b_2}(h+h) + \gamma_{b_1}(h+h) \} \]

Then the proof is obtained.

3 THE DISPERSION FOR THE MODIFIED PERIODOGRAM FOR TWO VECTOR VALUED STATIONARY TIME SERIES WITH MISSED OBSERVATIONS.

In this section we will study the dispersion for the modified periodogram for two vector valued stationary time series with missed observations by the following corollaries.

**Corollary 3.1.** Suppose that \( \alpha_a(t) = \ell_a(t) r_a(t) \), \( a = 1, 2, \ldots \), \( \min(i, j) \) are missed observations on the strictly stationary discrete stochastic processes which satisfies assumption II with mean zero, let \( \hat{\lambda}_a(t) \), \(-\infty < t < \infty \), and let

\[ I^{(T)}_{aa}(h) = \left[ I^{(T)}_{ab}(h) \right] = \left[ 2\pi \gamma_{aa}^{(T)}(0) \right]^{-1} C_a^{(T)}(h) \overline{C_b^{(T)}(h)} , \]

then

\[ E[I^{(T)}_{ab}(h)] = p_{a,b_1}(h) f_{a,b_2}(h) f_{b_1}(h) B(h) f_{a,b_2}(h) B(h)^T \]

as \( T \to \infty \), \( a, b = 1, \ldots, \min(i, j), h \in \mathbb{R} \).

**Proof.**

From (2.13) and by taking the limits for both sides then the proof comes directly by using the given constraints.

In Corollary (2.2) below we use of the Kroncker delta function which is given by

\[ \Theta(h) = \begin{cases} 1, & \text{if } h = 0 \\ 0, & \text{otherwise} \end{cases} \]

(3.1)

which is dependence of \( I^{(T)}_{ab}(h) \) and \( I^{(T)}_{a,b_2}(h) \), \( a, b_1 = 1, 2, \ldots, \min(i, j) \), \( i = 1, \ldots, k, h,h \in \mathbb{R} \) is seen to fall off as the function \( \gamma_{ab}^{(T)}(h) \), \( a, b = 1, 2, \ldots, \min(r, s), h \in \mathbb{R} \) fall off. By using the limit, theorem (2.2) becomes:

Now, when \( h \pm h \neq 0 \), \( h, h \in \mathbb{R} \), for some constants \( F \), and the bounded of \( f_{ab}(h) \), \( a, b = 1, 2, \ldots, \min(i, j), h \in \mathbb{R} \) then taking the modulus for both sides of (2.19) and using theorem (2.1), we have \( A(h \pm h = 0) \), using assumption (I), then we get from (2.14), and using the limit, then,

\[ \text{Corollary 3.2.} \text{ For all } h, h \in \mathbb{R} \text{ and under the constraints of theorem (2.2) then,} \]

\[ \text{Cov}[I_{a,b_1}^{(T)}(h), I_{a,b_2}^{(T)}(h)] = P^4 \Psi(h-h) \eta A + P^4 \Psi(h+h) B \eta A + O(T^{-1}), \]

\[ \text{Lim} \text{ Cov}[I_{a,b_1}^{(T)}(h), I_{a,b_2}^{(T)}(h)] = \]

\[ = \begin{cases} P^4 \Psi(h-h) \eta A + P^4 \Psi(h+h) \eta A & \text{if } h \pm h = 0 \\ 0 & \text{if } h \pm h \neq 0 \end{cases} \]

**Proof.**

\[ \text{Cov}[I_{a,b_1}^{(T)}(h), I_{a,b_2}^{(T)}(h)] = P^4 \Psi(h-h) \eta A + P^4 \Psi(h+h) \eta A \]

\[ \text{Cov}[I_{a,b_1}^{(T)}(h), I_{a,b_2}^{(T)}(h)] \leq \begin{cases} 1 \end{cases} \gamma_{a,b_1}^{(T)}(0) \gamma_{a,b_2}^{(T)}(0)^{-1} \times \]

\[ \times \left[ \frac{2 \Gamma_1 u_1}{\sin(h-h)/2} \right]^2 F^2 + \left[ \frac{2 \Gamma_2 u_2}{\sin(h-h)/2} \right]^2 F^2 \]

\[ + T^{-2} T^{-2} N_{a,b_2}(h,h) + (T^{-1}) \]

by using corollary (2.1) we get \( \text{Cov}[I_{a,b_1}^{(T)}(h), I_{a,b_2}^{(T)}(h)] \to 0 \) as \( T \to \infty \), the proof is obtained.

In the case of \( h \pm h = 0 \) then the previous corollary indicates the following one.

**Corollary 3.3** Using theorem (2.2) and corollary (3.2) then we have,

\[ \text{Lim} D[I_{a,b}^{(T)}(h)] = \]

\[ = \begin{cases} P^4 \Psi(h-h) \eta A & \text{if } h = h = \zeta \neq 0 \\ P^4 \Psi(h-h) \eta A + P^4 \Psi(h+h) \eta A & \text{if } h = h = \zeta = 0 \end{cases} \]

**Proof.**

Let \( h = h = \zeta \), \( \zeta \in \mathbb{R}, a_1 = a_2 = a, b_1 = b_2 = b, a, b = 1, 2, \ldots, \min(i, j) \) into corollary (3.2), we get

\[ \text{Lim} D[I_{a,b}^{(T)}(h)] = P^4 \Theta(\zeta - \zeta) \eta A + P^4 \Theta(\zeta + \zeta) \eta A \]

when \( \zeta = 0 \), then, \( \text{Lim} D[I_{a,b}^{(T)}(h)] = P^4 \eta A + P^4 \eta A \).

Hence the proof is obtained.

4 CONCLUSION

It is clear from the study that the properties of the modified periodogram for stationary time series of two vector valued with missed observations are approximately the same properties to the classical one, which will lead to apply in many important fields such as economy, astronomy, and medicine.
REFERENCES


