A Fixed Point Theorem in Modified Intuitionistic Fuzzy Metric Spaces

Syed Shahnawaz Ali [1], Dr. Jainendra Jain [2], Dr. Anil Rajput [3]

Abstract— Fuzzy Mathematics has seen an enormous growth since the introduction of notion of fuzzy sets by Zadeh in 1965. Kramosil and Michalek introduced the notion of fuzzy metric spaces which was later modified by George and Veeramani and others. The notion of intuitionistic fuzzy metric spaces was introduced by Park in 2004. Many authors have studied fixed point and common fixed theorems for mappings on fuzzy metric spaces and intuitionistic fuzzy metric spaces. In this paper we prove a common fixed point theorem for a sequence of mappings in an intuitionistic fuzzy metric space.

Index Terms— Fixed Points, Fuzzy sets, Fuzzy Metric Spaces, Intuitionistic Fuzzy Sets, Intuitionistic Fuzzy Metric Spaces, Triangular Norm, Triangular Co norm.

1 INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh [22] in 1965. Since then, with a view to utilize this concept in topology and analysis, many authors have extensively developed the theory of fuzzy sets. In 1975, Kramosil and Michalek [7] introduced the concept of a fuzzy metric space by generalizing the concept of a probabilistic metric space to the fuzzy situation. The concept of Kramosil and Michalek [7] of a fuzzy metric space was later modified by George and Veeramani [1] in 1994. In 1988, Grabeic [11], following the concept of Kramosil and Michalek [7], obtained the fuzzy version of Banach's fixed point theorem. Jungck [5] introduced the notion of compatible mappings in metric spaces and utilized the same as a tool to improve commutativity conditions in common fixed point theorems. This concept has frequently been employed to prove existence theorems on common fixed points. In recent past, several authors proved various fixed point theorems employing relatively more general contractive conditions. However, the study of common fixed points of non-compatible mappings is also equally interesting which was initiated by Pant [18]. The notion of Intuitionistic Fuzzy Sets was put forward by Atanassov [10] in 1986 and notion of Intuitionistic Fuzzy Metric Spaces was given by Park [9] in 2004 employing the notions of continuous $t-$norm and continuous $t-$ conorm. Fixed point theory is one of the most fruitful and effective tools in mathematics which has enormous applications in several branches of science especially in chaos theory, game theory, theory of differential equation, etc. Intuitionistic fuzzy metric notion is also useful in modeling some physical problems wherein it is necessary to study the relationship between two probability functions as noticed by Gregori et al. [21]. For instance, it has a concrete physical visualization in the context of two slit experiment as the foundation of $E-$infinity theory of high energy physics whose details are available in El Naschie in [12], [13], [14]. The topology induced by intuitionistic fuzzy metric coincides with the topology induced by fuzzy metric as noticed by Gregori et al. [21]. Following this, Saadati et al. [17] reframed the idea of intuitionistic fuzzy metric spaces and proposed a new notion under the name of modified intuitionistic fuzzy metric spaces by introducing the notion of continuous $t-$representable norm.

Fixed point and common fixed point properties for mappings defined on fuzzy metric spaces, intuitionistic fuzzy metric spaces, and $\mathcal{L}-$fuzzy metric spaces have been studied by many authors like H. Adibi et al. [6], S. Sharma [19], J. Goguen [8], V. Gregori and A. Sapena [20], C. Alaca et al. [2], Saadati et al. [15], [16]. Most of the properties which provide the existence of fixed points and common fixed points are of linear contractive type conditions.

In this paper we prove a common fixed point theorem for a sequence of mappings in intuitionistic fuzzy metric spaces introduced by Park [9] and modified by Saadati et al. [17]. For the sake of completeness we recall some definitions and results in the next section.

2 PRELIMINARIES

Definition 1: Let $\mathcal{L} = (L^*, \leq_{L^*})$ be a complete lattice, and U a non empty set called a universe. An $\mathcal{L}-$fuzzy set $\mathcal{A}$ on U is defined as amapping $\mathcal{A}: U \to L^*$. For each u in $U, \mathcal{A}(u)$ represents the degree (in L^*) to which u satisfies $\mathcal{A}$.

Lemma 2: *Consider the set L^* and operation $\leq_{L^*}$ defined by :*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \ \&\ x_1 + x_2 \leq 1\},$$
$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$$

and $x_2 \geq y_2$, *for every* $(x_1, x_2), (y_1, y_2) \in L^*$.
Then, $(L^*, \leq_{L^*})$ *is a complete lattice.*

Definition 3: An intuitionistic fuzzy set $\mathcal{A}_{\zeta,\eta}$ in a universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) | u \in U\}$, where, for all $u \in U$, $\zeta_{\mathcal{A}}(u) \in [0,1]$ and $\eta_{\mathcal{A}}(u) \in [0,1]$ are called the

1. Syed Shahnawaz Ali is working as Assisstant Professor in Mathematics at Sagar Institute of Research, Technology & Science, Bhopal, India. E-mail: shahnawaz_hunk@yahoo.com
2. Dr. Jainendra Jain is working as Professor in Mathematics at Sagar Institute of Research, Technology & Science, Bhopal, India. E-mail: jj.28481@gmail.com
3. Dr. Anil Rajput is working as Professor in Mathematics at Chandra Shekhar Azad P. G. College, Sehore, India. E-mail: dranilrajput@hotmail.com

membership degree and the non-membership degree, respectively of u in $\mathcal{A}_{\zeta,\eta}$ and furthermore they satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

Definition 4: For every $z_\alpha = (x_\alpha, y_\alpha) \in L^*$ we define

$$\vee (z_\alpha) = \big(sup(x_\alpha), inf(y_\alpha)\big).$$

Since $z_\alpha \in L^*$ then $x_\alpha + y_\alpha \leq 1$ so $sup(x_\alpha) + inf(y_\alpha) \leq sup(x_\alpha + y_\alpha) \leq 1$, that is $\vee(z_\alpha) \in L^*$. We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$.
Classically, a triangular norm $* = T$ on $([0,1], \leq)$ is defined as an increasing, commutative, associative mapping $T: [0,1]^2 \to [0,1]$ satisfying $T(1,x) = 1 * x = x$, for all $x \in [0,1]$. A triangular conorm $S = \lozenge$ is defined as an increasing, commutative, associative mapping $S: [0,1]^2 \to [0,1]$ satisfying $S(0,x) = 0 \lozenge x = x$, for all $x \in [0,1]$. Using the lattice $(L^*, \leq_{L^*})$ these definitions can be straightforwardly extended.

Definition 5: [3, 4] A triangular norm (t–norm) on L^* is a mapping $\mathcal{T} : (L^*)^2 \to L^*$ satisfying the following conditions:

1) $(\forall x \in L^*)(\mathcal{T}(x, 1_{L^*}) = x)$, (boundary condition)
2) $(\forall (x,y) \in (L^*)^2)\big(\mathcal{T}(x,y) = \mathcal{T}(y,x)\big)$, (commutativity)
3) $(\forall (x,y,z) \in (L^*)^3)\Big(\mathcal{T}\big(x, \mathcal{T}(y,z)\big) = \mathcal{T}\big(\mathcal{T}(x,y), z\big)\Big)$, (associativity),
4) $(\forall (x,x',y,y') \in (L^*)^4)(x \leq_{L^*} x')$ and $\big(y \leq_{L^*} y' \Longrightarrow \mathcal{T}(x,y) \leq_{L^*} \mathcal{T}(x',y')\big)$. (monotonicity).

Definition 6: [3, 4] A continuous t–norm $\mathcal{T}$ on L^* is called continuous t– representable if and only if there exist a continuous t–norm $*$ and a continuous t–conorm $\lozenge$ on $[0,1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,

$$\mathcal{T}(x,y) = (x_1 * y_1, x_2 \lozenge y_2).$$

Now define a sequence $\mathcal{T}^n$ recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n\big(x^{(1)}, \cdots, x^{(n+1)}\big) = \mathcal{T}\big(\mathcal{T}^{n-1}\big(x^{(1)}, \cdots, x^{(n)}\big), x^{(n+1)}\big)$$

for $n \geq 2$ and $x^{(i)} \in L^*$.
We say the continuous t–representable norm is natural and write $\mathcal{T}_n$ whenever $\mathcal{T}_n(a,b) = \mathcal{T}_n(c,d)$ and $a \leq_{L^*} c$ implies $b \leq_{L^*} d$.

Definition 7: A negator on L^* is any decreasing mapping $\mathcal{N} : L^* \to L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^*$, then $\mathcal{N}$ is called an involutive negator. A negator on $[0,1]$ is a decreasing mapping $N: [0,1] \to [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_S denotes the standard negator on $[0,1]$ defined as $N_S(x) = 1 - x$ for all $x \in [0,1]$.

Definition 8: Let M, N are fuzzy sets from $X^2 \times (0, +\infty)$ to $[0,1]$ such that $M(x,y,t) + N(x,y,t) \leq 1$ for all $x, y \in X$ and $t > 0$. The 3 –tuple $\big(X, \mathcal{M}_{M,N}, \mathcal{T}\big)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t –representable norm and $\mathcal{M}_{M,N}$ is a mapping $X^2 \times (0, +\infty) \to L^*$ (an intuitionistic fuzzy set, see Definition 3) satisfying the following conditions for every $x, y \in X$ and $t, s > 0$:

1) $\mathcal{M}_{M,N}(x,y,t) >_{L^*} 0_{L^*}$;
2) $\mathcal{M}_{M,N}(x,y,t) = 1_{L^*}$ if and only if $x = y$;
3) $\mathcal{M}_{M,N}(x,y,t) = \mathcal{M}_{M,N}(y,x,t)$;
4) $\mathcal{M}_{M,N}(x,y,t+s) \geq_{L^*} \mathcal{T}\Big(\mathcal{M}_{M,N}(x,z,t), \mathcal{M}_{M,N}(z,y,s)\Big)$;
5) $\mathcal{M}_{M,N}(x,y,\cdot): (0,\infty) \to L^*$ is continuous.

In this case $\mathcal{M}_{M,N}$ is called an intuitionistic fuzzy metric space. Here, $\mathcal{M}_{M,N}(x,y,t) = (M(x,y,t), N(x,y,t))$.

Example 9: Let (X,d) be a metric space. Define $\mathcal{T}(a,b) = \{a_1 b_1, \min(a_2 + b_2, 1)\}$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$ and let M and N be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x,y,t) = \big(M(x,y,t), N(x,y,t)\big)$$

$$= \left(\frac{ht^n}{ht^n + md(x,y)}, \frac{md(x,y)}{ht^n + md(x,y)}\right),$$

for all $h, m, n, t \in \mathbb{R}^+$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Example 10: Let $X = N$. Define $\mathcal{T}(a,b) = \{(\max(0, a_1 + b_1 - 1), a_2 + b_2 - a_2 b_2\}$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$ and let M and N be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x,y,t) = \big(M(x,y,t), N(x,y,t)\big) = \begin{cases} \left(\frac{x}{y}, \frac{y-x}{y}\right) & if\ x \leq y \\ \left(\frac{y}{x}, \frac{x-y}{x}\right) & if\ y \leq x \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Definition 11: Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitionistic fuzzy metric space and $\{x_n\}$ be a sequence in X.

1) A sequence $\{x_n\}$ is said to be convergent to $x \in X$ in the intuitionistic fuzzy metric space $\big(X, \mathcal{M}_{M,N}, \mathcal{T}\big)$ and denoted by $x_n \xrightarrow{\mathcal{M}_{M,N}} x$ if $\mathcal{M}_{M,N}(x_n, x, t) \to 1_{L^*}$ as $n \to \infty$ for every $t > 0$.
2) A sequence $\{x_n\}$ in an intuitionistic fuzzy metric space $\big(X, \mathcal{M}_{M,N}, \mathcal{T}\big)$ is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}_{M,N}(x_n, x_m, t) >_{L^*} (N_s(\varepsilon), \varepsilon)$, and for each $n, m \geq n_0$; here N_s is the standard negator.
3) An intuitionistic fuzzy metric space is said to be complete if and only if every Cauchy sequence in this space is convergent. Henceforth, we assume that $\mathcal{T}$ is a continuous t –norm on the lattice $\mathcal{L}$ such that for every $\mu \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$, there exists $\lambda \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$ such that $\mathcal{T}^{n-1}(\mathcal{N}(\lambda), \cdots, \mathcal{N}(\lambda)) \geq_{L^*} \mathcal{N}(\mu)$.

Lemma 12: *Let $\mathcal{M}_{M,N}$ be an intuitionistic fuzzy metric. Then for any $t > 0$, $\mathcal{M}_{M,N}(x,y,t)$ is nondecreasing with respect to t in $(L^*, \leq_{L^*})$ for all $x, y \in X$.*

Definition 13: Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitionistic fuzzy metric space. For $t > 0$, we define the open ball $B(x,r,t)$ with center $x \in X$ and $0 < r < 1$ by

$$B(x,r,t) = \{y \in X : \mathcal{M}_{M,N}(x,y,t) >_{L^*} (N_s(r), r)\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x,r,t) \subset A$. Let $\mathcal{T}_{\mathcal{M}_{M,N}}$ denote the family of all open subset of X. $\mathcal{T}_{\mathcal{M}_{M,N}}$ is called the topology induced by the intuitionistic fuzzy metric space.

Definition 14: Let $(X,\mathcal{M}_{M,N},\mathcal{T})$ be an intuitionistic fuzzy metric space. A subset A of X is said to be $IF-$bounded if there exist $t>0$ and $0<r<1$ such that $\mathcal{M}_{M,N}(x,y,t) >_{L^*} (N_s(r),r)$ for each $x,y\in A$.

Definition 15: Let $(X,\mathcal{M}_{M,N},\mathcal{T})$ be an intuitionistic fuzzy metric space. $\mathcal{M}$ is said to be continuous on $X\times X\times]0,\infty[$ if
$$\lim_{n\to\infty}\mathcal{M}_{M,N}(x_n,y_n,t_n)=\mathcal{M}_{M,N}(x,y,t)$$
whenever $\{(x_n,y_n,t_n)\}$ is a sequence in $X\times X\times]0,\infty[$ which converges to a point
$$(x,y,t)\in X\times X\times]0,\infty[\ i.e.,\ \lim_{n\to\infty}\mathcal{M}_{M,N}(x_n,x,t)=\lim_{n\to\infty}\mathcal{M}_{M,N}(y_n,y,t)=1_{L^*}$$
and $\lim_{n\to\infty}\mathcal{M}_{M,N}(x,y,t_n)=\mathcal{M}_{M,N}(x,y,t)$.

Lemma 16: *Let $(X,\mathcal{M}_{M,N},\mathcal{T})$ be an intuitionistic fuzzy metric space and define $E_{\lambda,\mathcal{M}_{M,N}}: X^2\to\mathbb{R}^+\cup\{0\}$ by*

$$E_{\lambda,\mathcal{M}_{M,N}}(x,y)=inf\{t>0:\mathcal{M}_{M,N}(x,y,t)>_{L^*}\mathcal{N}(\lambda)\}$$

for each $\lambda\in L^*\setminus\{0_{L^*},1_{L^*}\}$ and $x,y\in X$ here, $\mathcal{N}$ is an involutive negator. Then we have
(i) For any $\mu\in L^*\setminus\{0_{L^*},1_{L^*}\}$, there exists $\lambda\in L^*\setminus\{0_{L^*},1_{L^*}\}$ such that $E_{\mu,\mathcal{M}_{M,N}}(x_1,x_n)\le E_{\lambda,\mathcal{M}_{M,N}}(x_1,x_2)+E_{\lambda,\mathcal{M}_{M,N}}(x_2,x_3)+\cdots+E_{\lambda,\mathcal{M}_{M,N}}(x_{n-1},x_n)$ for any $x_1,x_2,x_3,\cdots x_n\in X$.

(ii) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent to x with respect to intuitionistic fuzzy metric $\mathcal{M}_{M,N}$ if and only if $E_{\lambda,\mathcal{M}_{M,N}}(x_n,x)\to 0$. Also, the sequence $\{x_n\}$ is a Cauchy sequence with respect to intuitionistic fuzzy metric $\mathcal{M}_{M,N}$ if and only if it is a Cauchy sequence with $E_{\lambda,\mathcal{M}_{M,N}}$.

Lemma 17: *Let $(X,\mathcal{M}_{M,N},\mathcal{T})$ be an intuitionistic fuzzy metric space. If*

$$\mathcal{M}_{M,N}(x_n,x_{n+1},t)\ge_{L^*}\mathcal{M}_{M,N}\left(x_0,x_1,\frac{t}{k^n}\right)$$

for some $k<1$ and $n\in\mathbb{N}$ then $\{x_n\}$ is a Cauchy sequence.
We now extend the above definitions and results.

Definition 18: Let M,N are fuzzy sets from $X^3\times(0,+\infty)$ to $[0,1]$such that $M(x,y,z,t)+N(x,y,z,t)\le 1$ for all $x,y,z\in X$ and $t>0$. The $3-$tuple $(X,\mathcal{M}_{M,N},\mathcal{T})$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous $t-$representable and $\mathcal{M}_{M,N}$ is a mapping $X^3\times(0,+\infty)\to L^*$ (an intuitionistic fuzzy set, see Definition 3) satisfying the following conditions for every $x,y,z,w\in X$ and $t,s>0$:
1) $\mathcal{M}_{M,N}(x,y,z,t)>_{L^*}0_{L^*}$;
2) $\mathcal{M}_{M,N}(x,y,z,t)=1_{L^*}$ if and only if $x=y=z$;
3) $\mathcal{M}_{M,N}(x,y,z,t)=\mathcal{M}_{M,N}(x,z,y,t)=\mathcal{M}_{M,N}(y,z,x,t)$;
4) $\mathcal{M}_{M,N}(x,y,z,t+s)\ge_{L^*}\mathcal{T}\left(\mathcal{M}_{M,N}(x,y,w,t),\mathcal{M}_{M,N}(w,z,z,s)\right)$;
5) $\mathcal{M}_{M,N}(x,y,z,\cdot):(0,\infty)\to L^*$ is continuous.

In this case $\mathcal{M}_{M,N}$ is called an intuitionistic fuzzy metric. Here, $\mathcal{M}_{M,N}(x,y,z,t)=(M(x,y,z,t),N(x,y,z,t))$.

Example 19: Let (X,d) be a metric space. Define $\mathcal{T}(a,b)=(a_1b_1,a_2b_2,\min\{a_3+b_3,1\})$ for all $a=(a_1,a_2,a_3)$ and $b=(b_1,b_2,b_3)\in L^*$ and let M and N be fuzzy sets on $X^3\times(0,\infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x,y,z,t)=\left(M(x,y,z\,t),N(x,y,z,t)\right)$$

$$=\left(\frac{ht^n}{ht^n+md(x,y,z)},\frac{md(x,y,z)}{ht^n+md(x,y,z)}\right),$$

for all $h,m,n,t\in\mathbb{R}^+$. Then $(X,\mathcal{M}_{M,N},\mathcal{T})$ is an intuitionistic fuzzy metric space.

Definition 20: Let $(X,\mathcal{M}_{M,N},\mathcal{T})$ be an intuitionistic fuzzy metric space and $\{x_n\}$ be a sequence in X.
1) A sequence $\{x_n\}$ is said to be convergent to $x\in X$ in the intuitionistic fuzzy metric space $(X,\mathcal{M}_{M,N},\mathcal{T})$ and denoted by $x_n\xrightarrow{\mathcal{M}_{M,N}}x$ if $\mathcal{M}_{M,N}(x_n,x,x,t)\to 1_{L^*}$ as $n\to\infty$ for every $t>0$.
2) A sequence $\{x_n\}$ in an intuitionistic fuzzy metric space $(X,\mathcal{M}_{M,N},\mathcal{T})$ is called a Cauchy sequence if for each $0<\varepsilon<1$ and $t>0$, there exists $n_0\in\mathbb{N}$ such that $\mathcal{M}_{M,N}(x_n,x_n,x_m,t)>_{L^*}(N_s(\varepsilon),\varepsilon)$, and for each $n,m\ge n_0$; here N_s is the standard negator.
3) An intuitionistic fuzzy metric space is said to be complete if and only if every Cauchy sequence in this space is convergent.

Lemma 21: Let $\mathcal{M}_{M,N}$ be an intuitionistic fuzzy metric. Then for any $t>0$, $\mathcal{M}_{M,N}(x,y,z,t)$ is nondecreasing with respect to t in $(L^*,\le_{L^*})$ for all $x,y,z\in X$.

Definition 22: Let $(X,\mathcal{M}_{M,N},\mathcal{T})$ be an intuitionistic fuzzy metric space. For $t>0$, we define the open ball $B(x,r,t)$ with center $x\in X$ and $0<r<1$ by
$B(x,r,t)=\{y\in X:\ \mathcal{M}_{M,N}(x,y,y,t)>_{L^*}(N_s(r),r)\}$
A subset $A\subset X$ is called open if for each $x\in A$, there exist $t>0$ and $0<r<1$ such that $B(x,r,t)\subset A$. Let $\mathcal{T}_{\mathcal{M}_{M,N}}$ denote the family of all open subset of X. $\mathcal{T}_{\mathcal{M}_{M,N}}$ is called the topology induced by the intuitionistic fuzzy metric space.

Definition 23: Let $(X,\mathcal{M}_{M,N},\mathcal{T})$ be an intuitionistic fuzzy metric space. $\mathcal{M}$ is said to be continuous on $X^3\times(0,\infty)$ if
$$\lim_{n\to\infty}\mathcal{M}_{M,N}(x_n,y_n,z_n,t_n)=\mathcal{M}_{M,N}(x,y,z,t)$$
Whenever $\{(x_n,y_n,z_n,t_n)\}$ is a sequence in $X^3\times(0,\infty)$ which converges to a point $(x,y,z,t)\in X^3\times(0,\infty)$, $i.e.$,
$\lim_{n\to\infty}x_n=x$, $\lim_{n\to\infty}y_n=y$, $\lim_{n\to\infty}z_n=z$, and
$$\lim_{n\to\infty}\mathcal{M}_{M,N}(x,y,z,t_n)=\mathcal{M}_{M,N}(x,y,z,t).$$

Lemma 24: *Let $(X,\mathcal{M}_{M,N},\mathcal{T})$ be an intuitionistic fuzzy metric space and define $E_{\lambda,\mathcal{M}_{M,N}}: X^3\to\mathbb{R}^+\cup\{0\}$ by*

$$E_{\lambda,\mathcal{M}_{M,N}}(x,y,z)=inf\{t>0:\mathcal{M}_{M,N}(x,y,z,t)>_{L^*}\mathcal{N}(\lambda)\}$$

for each $\lambda\in L^*\setminus\{0_{L^*},1_{L^*}\}$ and $x,y,z\in X$ here, $\mathcal{N}$ is an involutive negator. Then we have

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(i) For any $\mu \in L^*$ $\{0_{L^*}, 1_{L^*}\}$ there exists $\lambda \in L^*$ $\{0_{L^*}, 1_{L^*}\}$ such that $E_{\mu,\mathcal{M}_{M,N}}(x_1, x_2, x_n) \le E_{\lambda,\mathcal{M}_{M,N}}(x_1, x_2, x_3) + E_{\lambda,\mathcal{M}_{M,N}}(x_2, x_3, x_4) +$

$\cdots + E_{\lambda,\mathcal{M}_{M,N}}(x_{n-2}, x_{n-1}, x_n)$ *for any* $x_1, x_2, x_3, \cdots x_n \in X$.

(ii) *The sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent with respect to intuitionistic fuzzy metric $\mathcal{M}_{M,N}$ if and only if $E_{\lambda,\mathcal{M}_{M,N}}(x_n, x, x) \to 0$. Also the sequence $\{x_n\}$ is a Cauchy sequence with respect to intuitionistic fuzzy metric $\mathcal{M}_{M,N}$ if and only if it is a Cauchy sequence with $E_{\lambda,\mathcal{M}_{M,N}}$.*

Proof: For (i), by the continuity of $t-$norms, for every $\mu \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$, we can find a $\lambda \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$ such that $\mathcal{T}(\mathcal{N}(\lambda), \mathcal{N}(\lambda)) \ge_{L^*} \mathcal{N}(\mu)$. By definition 18 , we have

$$\mathcal{M}_{M,N}(x, y, z, E_{\lambda,\mathcal{M}_{M,N}}(x_1, x_2, x_3) + E_{\lambda,\mathcal{M}_{M,N}}(x_2, x_3, x_4) + \cdots + E_{\lambda,\mathcal{M}_{M,N}}(x_{n-2}, x_{n-1}, x_n) + n\delta)$$

$$\ge_{L^*} \mathcal{T}\,(\mathcal{M}_{M,N}(x, y, w, E_{\lambda,\mathcal{M}_{M,N}}(x_1, x_2, x_3) + E_{\lambda,\mathcal{M}_{M,N}}(x_2, x_3, x_4) + \cdots + E_{\lambda,\mathcal{M}_{M,N}}(x_{n-4}, x_{n-3}, x_{n-2}) + \frac{n\delta}{2}),$$

$$\mathcal{M}_{M,N}(w, z, z, E_{\lambda,\mathcal{M}_{M,N}}(x_{n-3}, x_{n-2}, x_{n-1}) + E_{\lambda,\mathcal{M}_{M,N}}(x_{n-2}, x_{n-1}, x_n) + \frac{n\delta}{2}))$$

$$\ge_{L^*} \mathcal{T}\,(\mathcal{N}(\lambda), \mathcal{N}(\lambda)) \ge_{L^*} \mathcal{N}(\mu)$$

for every $\delta > 0$, which implies that

$$E_{\mu,\mathcal{M}_{M,N}}(x_1, x_2, x_n) \le E_{\lambda,\mathcal{M}_{M,N}}(x_1, x_2, x_3) + E_{\lambda,\mathcal{M}_{M,N}}(x_2, x_3, x_4) + \cdots + E_{\lambda,\mathcal{M}_{M,N}}(x_{n-2}, x_{n-1}, x_n) + n\delta$$

Since $\delta > 0$ was arbitrary, we have

$$E_{\mu,\mathcal{M}_{M,N}}(x_1, x_2, x_n) \le E_{\lambda,\mathcal{M}_{M,N}}(x_1, x_2, x_3) + E_{\lambda,\mathcal{M}_{M,N}}(x_2, x_3, x_4) + \cdots + E_{\lambda,\mathcal{M}_{M,N}}(x_{n-2}, x_{n-1}, x_n).$$

For (ii), we have

$$\mathcal{M}_{M,N}(x_n, x, x, \eta) >_{L^*} \mathcal{N}(\lambda) \Leftrightarrow E_{\lambda,\mathcal{M}_{M,N}}(x_n, x, x) < \eta$$

for every $\eta > 0$.

Lemma 25: *Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitionistic fuzzy metric space. If*

$$\mathcal{M}_{M,N}(x_n, x_{n+1}, x_{n+2}, t) \ge_{L^*} \mathcal{M}_{M,N}\left(x_0, x_1, x_2, \frac{t}{k^n}\right)$$

for some $k < 1$ and $n \in \mathbb{N}$ then $\{x_n\}$ is a Cauchy sequence.

Proof: For every $\lambda \in L \setminus \{0_{L^*}, 1_{L^*}\}$ and $x_n \in X$, we have

$$E_{\lambda,\mathcal{M}_{M,N}}(x_n, x_{n+1}, x_{n+2}, t)$$

$$= inf\{t > 0 : \mathcal{M}_{M,N}(x_n, x_{n+1}, x_{n+2}, t) >_{L^*} \mathcal{N}(\lambda)\}$$

$$\le inf\{t > 0 : \mathcal{M}_{M,N}\left(x_0, x_1, x_2, \frac{t}{k^n}\right) >_{L^*} \mathcal{N}(\lambda)\}$$

$$= inf\{ k^n t : \mathcal{M}_{M,N}(x_0, x_1, x_2, t) >_{L^*} \mathcal{N}(\lambda)\}$$
$$= k^n\, inf\{t > 0 : \mathcal{M}_{M,N}(x_0, x_1, x_2, t) >_{L^*} \mathcal{N}(\lambda)\}$$
$$= k^n E_{\lambda,\mathcal{M}_{M,N}}(x_n, x_{n+1}, x_{n+2}, t)$$

From lemma (24), for every $\mu \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$ there exists $\lambda \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$, such that

$$E_{\mu,\mathcal{M}_{M,N}}(x_n, x_{n+1}, x_m)$$

$$\le E_{\gamma,\mathcal{M}_{M,N}}(x_n, x_{n+1}, x_{n+2}) + E_{\gamma,\mathcal{M}_{M,N}}(x_{n+1}, x_{n+2}, x_{n+3}) + \cdots + E_{\gamma,\mathcal{M}_{M,N}}(x_{m-2}, x_{m-1}, x_m)$$

$$\le k^n E_{\gamma,\mathcal{M}_{M,N}}(x_0, x_1, x_2) + k^{n+1} E_{\gamma,\mathcal{M}_{M,N}}(x_0, x_1, x_2) + \cdots + k^{m-2} E_{\gamma,\mathcal{M}_{M,N}}(x_0, x_1, x_2)$$

$$= E_{\gamma,\mathcal{M}_{M,N}}(x_0, x_1, x_2) \sum_{j=n}^{m-2} k^j \longrightarrow 0.$$

Hence sequence $\{x_n\}$ is a Cauchy sequence.

3 THE MAIN RESULT

Theorem 1: *Let $\{A_n\}$ be a sequence of mappings A_i of a complete intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ into itself such that, for any three mappings A_i, A_j, A_l*

$$\mathcal{M}_{M,N}(A_i^m(x), A_j^m(y), A_l^m(z), \alpha_{i,j,l} t) \ge_{L^*} \mathcal{M}_{M,N}(x, y, z, t)$$

for some m; here $0 < \alpha_{i,j,l} < k < 1$ for $i, j, l = 1, 2, \cdots, x, y, z \in X$ and $t > 0$. Then the sequence $\{A_n\}$ has a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X and define a sequence $\{x_n\}$ in X by $x_1 = A_1^m(x_0), x_2 = A_2^m(x_1), x_3 = A_3^m(x_2), \cdots$. Then we have

$$\mathcal{M}_{M,N}(x_1, x_2, x_3, t) = \mathcal{M}_{M,N}(A_1^m(x_0), A_2^m(x_1), A_3^m(x_2), t) \ge_{L^*} \mathcal{M}_{M,N}\left(x_0, x_1, x_2, \frac{t}{\alpha_{1,2,3}}\right),$$

$$\mathcal{M}_{M,N}(x_2, x_3, x_4, t) = \mathcal{M}_{M,N}(A_2^m(x_1), A_3^m(x_2), A_4^m(x_3), t)$$

$$\ge_{L^*} \mathcal{M}_{M,N}(x_1, x_2, x_3, \frac{t}{\alpha_{2,3,4}})$$

$$\ge_{L^*} \mathcal{M}_{M,N}\left(x_0, x_1, x_2, \frac{t}{\alpha_{1,2,3}\, \alpha_{2,3,4}}\right)$$

and so on. By induction, we have

$$\mathcal{M}_{M,N}(x_n, x_{n+1}, x_{n+2}, t) \ge_{L^*} \mathcal{M}_{M,N}\left(x_0, x_1, x_2, \frac{t}{\prod_{i=1}^n \alpha_{i,i+1,i+2}}\right)$$

for $n = 1,2,3 \cdots$, which implies

$E_{\lambda,\mathcal{M}_{M,N}}(x_n, x_{n+1}, x_{n+2}, t)$

$= \inf\{t > 0 : \mathcal{M}_{M,N}(x_n, x_{n+1}, x_{n+2}, t) >_{L^*} \mathcal{N}(\lambda)\}$

$\leq \inf\{t > 0 : \mathcal{M}_{M,N}\left(x_0, x_1, x_2, \frac{t}{\prod_{i=1}^{n} \alpha_{i,i+1,i+2}}\right) >_{L^*} \mathcal{N}(\lambda)\}$

$= \inf\{\prod_{i=1}^{n} \alpha_{i,i+1,i+2}\, t > 0 : \mathcal{M}_{M,N}(x_0, x_1, x_2, t) >_{L^*} \mathcal{N}(\lambda)\}$

$= \prod_{i=1}^{n} \alpha_{i,i+1,i+2} \inf\{t > 0 : \mathcal{M}_{M,N}(x_0, x_1, x_2, t) >_{L^*} \mathcal{N}(\lambda)\}$

$= \prod_{i=1}^{n} \alpha_{i,i+1,i+2}\, E_{\lambda,\mathcal{M}_{M,N}}(x_0, x_1, x_2,) \leq k^n E_{\lambda,\mathcal{M}_{M,N}}(x_0, x_1, x_2,)$

for every $\lambda \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$.

Now, we show that $\{x_n\}$ is a Cauchy sequence. For every $\mu \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$, there exists $\gamma \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$ such that

$E_{\mu,\mathcal{M}_{M,N}}(x_n, x_{n+1}, x_m)$

$\leq E_{\gamma,\mathcal{M}_{M,N}}(x_n, x_{n+1}, x_{n+2}) + E_{\gamma,\mathcal{M}_{M,N}}(x_{n+1}, x_{n+2}, x_{n+3}) + \cdots$

$+ E_{\gamma,\mathcal{M}_{M,N}}(x_{m-2}, x_{m-1}, x_m)$

$\leq k^n E_{\gamma,\mathcal{M}_{M,N}}(x_0, x_1, x_2) + k^{n+1} E_{\gamma,\mathcal{M}_{M,N}}(x_0, x_1, x_2) + \cdots$

$+ k^{m-2} E_{\gamma,\mathcal{M}_{M,N}}(x_0, x_1, x_2)$

$= E_{\gamma,\mathcal{M}_{M,N}}(x_0, x_1, x_2) \sum_{j=n}^{m-2} k^j \longrightarrow 0.$

As $m, n \to \infty$. Since X is left complete, there is $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now we prove that x is a periodic point of A_i for any $i = 1, 2, \cdots$, we have

$\mathcal{M}_{M,N}(x, x, A_i^m(x), t)$
$\geq_{L^*} \mathcal{T}\left(\mathcal{M}_{M,N}(x, x, x_n, t - kt), \mathcal{M}_{M,N}(x_n, A_i^m(x), A_i^m(x), kt)\right)$
$= \mathcal{T}(\mathcal{M}_{M,N}(x, x, x_n, t(1-k)),$
$\mathcal{M}_{M,N}(A_n^m(x_{n-1}), A_i^m(x), A_i^m(x), kt))$
$\geq_{L^*} \mathcal{T}(\mathcal{M}_{M,N}(x, x, x_n, t(1-k)),$
$\mathcal{M}_{M,N}(A_n^m(x_{n-1}), A_i^m(x), A_i^m(x), \alpha_{n,i,i}\, t))$
$\geq_{L^*} \mathcal{T}(\mathcal{M}_{M,N}(x, x, x_n, t(1-k)), \mathcal{M}_{M,N}(x_{n-1}, x, x, t))$
$\to \mathcal{T}(1_{L^*}, 1_{L^*}) = 1_{L^*}$

as $n \to \infty$. Thus $\mathcal{M}_{M,N}(x, x, A_i^m(x), t) = 1_{L^*}$ and we get $A_i^m(x) = x$.

To show uniqueness, assume that $y \neq x$ is another periodic point of A_i. Then we have

$1_{L^*} \geq_{L^*} \mathcal{M}_{M,N}(x, x, y, t) = \mathcal{M}_{M,N}(A_i^m(x), A_i^m(x), A_j^m(y), t)$

$\geq_{L^*} \mathcal{M}_{M,N}(x, x, y, \frac{t}{\alpha_{i,j,l}})$

$\geq_{L^*} \mathcal{M}_{M,N}(x, x, y, \frac{t}{k}) = \mathcal{M}_{M,N}(A_i^m(x), A_i^m(x), A_j^m(y), t/k)$

$\geq_{L^*} \mathcal{M}_{M,N}(x, x, y, \frac{t}{k^2}) \geq_{L^*} \cdots \geq_{L^*} \mathcal{M}_{M,N}(x, x, y, \frac{t}{k^n}) \to 1_{L^*}$

as $n \to \infty$. Therefore, for every $t > 0$, we have $\mathcal{M}(x, x, y, t) = 1_{L^*}$, *i.e.*, $x = y$. Also $A_i(x) = A_i(A_i^m(x)) = A_i^m(A_i(x))$, *i.e.*, $A_i(x)$ is also a periodic point of A_i. Therefore, $x = A_i(x)$, *i.e.*, x is a unique common fixed periodic point of the mappings A_n for $n = 1, 2, \cdots$. This completes the proof.

4 CONCLUSIONS

In this paper we have proved a common fixed point theorem for a sequence of mappings for the modified intuitionistic fuzzy metric spaces defined using the notion of continuous t – representable norms. The result can be extended for more general conditions.

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