STUDIES ON COSMOLOGICAL MODELS WITH INFLATION AND COUPLED OF SCALAR FIELD

A Dissertation submitted to the

Department of Mathematics, University of Chittagong

for the partial fulfillment of the requirement for the degree of

Masters of Science in Mathematics (Applied Group)

Author

RAJIB KARMAKER

Examination Roll No.: 09203031

Registration No.: 09203031

Session: 2012-2013

(Held In 2015)

Chittagong

Department of Mathematics
University of Chittagong
Chittagong-4331
Bangladesh

Dedicated to

My Respected and Beloved Parents

Ruhidas Karmaker

&

Chaya Rani Karmaker

Certification

This is to certify that the M.Sc. thesis titled **“STUDIES ON COSMOLOGICAL MODELS WITH INFLATION AND COUPLED OF SCALAR FIELD”** *submitted by* ***Rajib Karmaker*** *is a record of bonafide research work done under my supervision. It is further certified that the thesis represents independent work by the candidate and collaboration was necessitated by the nature and scope of the problems dealt with.*

Supervisor

(Dr. Abul Kalam Azad)

Professor
Department of Mathematics
University of Chittagong
Chittagong-4331, Bangladesh

Acknowledgement

First of all, I express my deep gratitude to Almighty God ,the most merciful and the most beneficial who has enable me to perform this research work and submit this.

I would like to express my most recondite sense of gratitude, innumerous thanks, deepest sense of respect to my esteemed teacher and research supervisor **Dr. Abul Kalam Azad**, a well known and distinguished Cosmologist of Bangladesh and Professor, Department of Mathematics, University of Chittagong for his deep affections, heartiest encouragement, unparalleled inspiration, Valuable suggestions, untried supervision and provided constant guidance during the whole period of my research work.

I am extremely grateful to Professor Dr. Md. Ashraful Islam, Chairman, Department of Mathematics, University of Chittagong for providing necessary facilities of his Department.

I express my deepest gratitude to my respected teachers Professor Dr. Nil Ratan Bhattacherjee, Professor Dr. Talky Bhattacherjee, Professor Dr. Nurun Nahar Begum, Professor Dr. Mosle Uddin Ahmed, Professor Md. Zahed, Professor Md. Shafiul Alam Tarafaer, Professor Dr. Munshi Nazrul Islam, Professor Dr. Ganesh Chandra Roy, Professor Dr.Motiur Rahman, Professor Dr.Mahiuddin, Professor Dr. Milon Kanti Dhar, Associate Professor Dr. Md. Aman Ullah for their valuable suggestions and inspirations.

I also express my great respect to honorable Assistant Professor G.M Waliullah, Assistant Professor A.S.M Moinuddion, Assistant Professor Md. Abdul Alim, Assistant Professor Md. Amzad Hossain, Assistant Professor H.S Faruque Alam, Assistant Professor Md. Forkan, Assistant Professor Md. Maksud Alam, Assistant Professor Md. Abdullah Murad & Assistant Professor Md. Mohiul Islam for their cordial love &

inspiration. They always looked after my progress and motivated me to accomplish this thesis.

I would like to thank all officers and staffs of Mathematics department for their continuous help during the research period.

Cordial thanks are also extended to all my classmates, friends and well wishers specially Asif, Kholil, Sharif, Shujit, Tutul & Shantunu for their constant encouragement and co-operation.

Last but not least, my heart self gratitude goes to my parents for giving me the best they could.

And finally, I express my gratitude to all of my family members and related for their blessing, unbounded love, continuous inspiration in every step, during the accomplishment of my thesis, which led all of my success in this thesis.

December, 2015 The Author

Contents

Chapters ***Page No.***

❖ **Chapter -6** **(204-229)**

❒ **Name of Chapter : Scalar field cosmology in phase space.**

❖ **Chapter -7** **(230-248)**

❖ **Name of the chapter: Inflation in homogenous & isotropic space-time.**

❖ **Chapter -8** **(249-296)**

❒ **Name of the chapter : Inflation in non-minimally coupled Theories & Inflation via modified gravity**

Abstract

This thesis is mainly expository and all the chapters contains little bit original calculations. The title of this thesis is "**STUDIES ON COSMOLOGICAL MODELS WITH INFLATION AND COUPLED OF SCALAR FIELD**". This thesis deals with -*The Fundamental of General Relativity, On the cosmological models, Classical & Quantization problem, An exact scalar field inflationary cosmological model which solves Cosmological constant problem, The scalar field potential, Scalar field cosmology in phase space, Inflation in homogenous & isotropic space-time, Inflation in non-minimally coupled theories & inflation via modified gravity.*

We have seen that the Friedmann models, if they are regarded as physically valid, predict that the density of mass-energy must have been very high in the early epoch of the universe. We have studied about the inflationary universe which is the modification of the standard hot Big-Bang model and also the solution of the problems which arise in the FRW model. We find that over a broad range of initial conditions, the predicted value of the inflation driven by a scalar field, which must be coupled to the curvature if the Einstein equivalence principle has to be satisfied.

In recent years, there have been some interests in studying the mathematical and physical interpretations of different models and theories of cosmology. The purpose of this research is firstly to study the physical and mathematical properties of the known solutions and secondly to attempt to find out new physically inserting solutions, with particular references.

Symbols

☞ *Redshift*	z
☞ *Hubble constant*	H_0
☞ *Physical distance*	r
☞ *Velocity*	v
☞ *Frequency*	f
☞ *Temperature*	T
☞ *Boltzmann constant*	k_B
☞ *Energy density*	ε
☞ *Radiation constant*	α
☞ *Newton's gravitational constant*	G
☞ *Mass density*	ρ
☞ *Scale factor*	R
☞ *Comoving distance*	x
☞ *Curvature*	k
☞ *Pressure*	p
☞ *Hubble parameter*	H
☞ *Number*	n,N
☞ *Hubble/ Planck's constant*	h
☞ *Present density parameter*	Ω_0
☞ *Critical density*	ρ_c
☞ *Density parameter*	Ω
☞ *Curvature 'density parameter'*	Ω_k
☞ *Deceleration parameter*	
☞ *Cosmological constant*	Λ
☞ *Cosmological constant density*	Ω_{Λ}
☞ *Time*	t
☞ *Present age*	t_o
☞ *Baryon density parameter*	Ω_B
☞ *Helium abundance*	Y_4
☞ *Luminosity distance*	d_{1um}
☞ *Angular diameter distance*	d_{diam}

~ Some conversion factors ~

1 pc = 3.261 light years = 3.086 x 10^{16} m

1 year = 3.156 x 10^{7} sec

1 eV = 2.602 x 10^{-19} J

1 M_{Θ} = 2.989 X 10^{30} kg

1 J = l kgm^{2} sec-^{2}

1 Hz = 1 sec^{-1}

~Some fundamental constants~

Newton's constant	G	6.672 X 10^{-11} $m^3kg^{-1}sec^{-2}$
Speed of light	C	2.998 X 10^{8} m sec^{-1}
		Or 3.076 x 10^{-7}M pc yr^{-1}
Reduced Planck constant	$\hbar=h/2\pi$	2.055 x 10^{-34} m^2 kg sec^{-1}
Boltzmann constant	k_B	2.381 x 10-23 J K^{-1}
		Or 8.619 x 1O-5 eV K^{-1}
Radiation constant	$\alpha=\pi^2k_B^4/15\hbar^3C^3$	8.565 x 10^{-16} J m^{-3} K^{-4}
Electron mass—energy	m_ec^2	0.511 MeV
Proton mass—energy	m_pc^2	938.3 MeV
Neutron mass—energy	m_nc^2	939.6 MeV
Thomson cross-section	σ_e	6.652 x 10^{-29} m^2
Free neutron half-life	t_{half}	614 sec

General Introduction

This thesis is mainly expository and the entire chapters contain an elaborate calculation. As the title implies that the thesis is concerned with the "**STUDIES ON COSMOLOGICAL MODELS WITH INFLATION AND COUPLED OF SCALAR FIELD**". As is so, we must preview on FRW model and inflationary model. The FRW model is meant the Friedman-Robertson-Walker model. The Robertson-Walker model (Robertson 1933) provides a major application of Einstein's general theory of relativity in cosmology. The discovery of *Friedman* solution (Friedman, 1922) within the framework of homogenous & isotropic universe models allowed the cosmological considerations to be treated in a mathematical manner which was a subject so far dominated by largely speculative ideas about the overall structure of the universe. The major assumptions used in arrival at the Robertson-Walker geometry are the large scale homogeneity and isotropy of the universe. The homogeneity in space means that the universe is roughly the same at all spatial points and that the matter is uniformly distributed all over the space. This is an assumption difficult to check, even though the universe is clearly inhomogeneous at the local scales of stars and star clusters, it is generally argued that an overall homogeneity will be achieved only at a large enough scale in a statistical sense only. It is possible to have observational tests on the assumption of isotropy, that is, the universe must be the same in all directions. One could check the distribution of galaxies in the different directions together with their apparent magnitudes and red-shifts and also the distribution of radio forces similarly. Such observations are again interpreted frequently as providing an evidence for isotropic distributions of matter in the universe from our vantage point.

Cosmology has entered one of the most exciting episodes in its history, with an unprecedented increase in the quantity and quality of observational data being collected.

The study of the cosmos in the form of classical scientific astronomy using mathematical descriptions is traced back to early seventh century AD. From seventh to fourteenth century, Persian mathematicians and astronomers *Kharazmi* (780–850), *Biruni* (973–1048), *Khayyam* (1048–1131), *Tusi* (1201–1274), and *Kashani* (1380–1429) each contributed a lot to the field of astronomy. Their contributions were further developed and improved by European astronomers *Copernicus* (1473–1543), *Galileo* (1564–1642), and *Kepler* (1571–1630) over the next three centuries. *Sir Isaac Newton's* theory of gravitation revolutionized astronomical calculations by late sixteenth century (in 1687, Newton published his Principia). Newtonian mechanics made it possible to formulate the motion of all celestial bodies in the solar system and beyond.

New discoveries and theories within the last century have drastically changed our understanding of the cosmos. With the advent of *Einstein's* General Theory of Relativity and the observational discovery on the expansion of the Universe by *Slipher*, as well as *Hubble's* discovery of Hubble's law (indicating that far galaxies are receding from us) as early as 1920s, cosmology became a much more distinct science than astronomy. In 1922, *Alexander Friedmann's* solutions to Einstein's equations formulated the evolution of a relativistic expanding or contracting dynamic Universe. The more advanced models are now known as *Friedmann–Lemaitre–Robertson–Walker (FLRW)* models of cosmology due to many enhancements and contributions from other cosmologists.

From 1930s onward, the **Big Bang** theory formed the basis for explaining the expansion of the Universe. However, the original Big Bang theory endured three problems, namely, the smoothness problem, the horizon problem, and the flatness problem. The first problem asks why the matter is uniformly distributed in the Universe. The second problem concerns the large-scale uniformity of the observable Universe. Finally, the third problem asks why the Universe is close to being spatially flat.

With the introduction of the **Inflationary Model** of cosmology in 1980s by *Alan Guth*, the three problems of the Big Bang cosmological model were solved. According to inflationary cosmology, the size of the Universe expanded exponentially to an extremely huge number (10^{60}) of its original size. This happened in a very short time from 10^{-35} to 10^{-32} s after the Big Bang. Collectively, the Big Bang model and Inflation Models of cosmology described the origin and expansion of the Universe. The modern form of inflationary cosmology is due to *A.Linde, A. Albrecht and P. Steinhardt*. In Guth's original model the inflation field was assumed to be trapped in a false vacuum and assumed a local value which is minimum. The inflation field comes out from the local minimum value by quantum tunneling and as universe inflates, tunneling takes place. However, these ideas when pursued gave empty universe and therefore rejected. *Guth* further tried to improve the idea but they led to others difficulties. In modern cosmology inflation is one of the essential ingredient for building cosmological models of the early universe. It is now understood that it can solve satisfactorily some of the outstanding problems of the standard big bang cosmology. The basic idea of inflation is that there was an epoch in the early universe when the vacuum energy density of the universe dominated leading to an accelerated expansion of the universe.

By the mid-1990s, new observations led to new models of cosmology. The modern Standard Model of Cosmology, which is generally accepted among cosmologists, integrates the following theories, models, and concepts: a fixed background space-time, the General Theory of Relativity, Dark Matter, Dark Energy, initial conditions at Big Bang (best described by Inflationary Models), and the Standard Model of particle physics. Although the Standard Model of Cosmology has its own outstanding problems such as Dark Matter and Dark Energy, and issues with inflation, yet it explains all the observations.

Scalar fields play a fundamental role in many scientific disciplines and applications. The increasing computational power offers scientists and digital artists' novel opportunities for complex simulations, measurements, and models that generate large amounts of data. One of the most studied issues in Cosmology is the dynamics of cosmological scalar fields, mostly because of their usefulness in providing models for different needed processes in the evolution of the universe.

Scalar fields are the most widely used dynamical dark energy models where late time acceleration can be obtained by adjusting the slope of the scalar field potential around suitable epoch. But the cosmological evolution of these models is severely constrained by very accurate cosmological observations. From the measurements of temperature anisotropy in the cosmic microwave background (CMB) radiation, the distance to last scattering is very well determined. This restricts the equation of state for the scalar field to be very close to $\omega = -1$ at present.

The work presented here has been largely derived from the books by ***Jamal Nazrul Islam :*** *"The Introduction to Mathematical Cosmology"*, Cambridge University press 2002 ***;*** *"Global Aspects in Gravitation and*

Cosmology" by ***Pankoz S. Joshi***, Oxford University Press,Inc.,New York ***; "Gravitation Gauge Theories and the Early Universe"*** by ***B.R, Iyer ,N. Mukunda*** and ***C.V Vishveshwara ;*** *"Principles and Applications of the General Theory Relativty"* by ***Steven Weinberg ;*** *"An Introduction to Cosmology"* by ***Jayanta Vishnu Narlikar*** forward by ***Sir Fred Hoyle***, Cambridge University Press 2002, *General Relativity and Cosmology* for Undergraduate by ***Professor John W. Norbury***, Physics Department, University of Wisconsin-Milwaukee.**;** "*Particle Physics and Inflationary Cosmology*", by ***Andrei. Linde***, Department of Physics,Stampford University, Stamford CA 94305-4060, USA **;** "*Cosmological Inflation and Large Scale Structure"*, by ***A. Liddle*** ,University of Sussex :***David H. Lyth*** , University of Lancaster: Cambridge University press ***; "The New Physics"*** edited by ***Paul Davies,*** the professor of Theoretical physics, The University of Adelaide: Cambridge University press ***; "Cosmology"***: *"The origin and Evolution of Cosmic Structure"* by ***Peter Coles*** , Astronomy Unit Queen Mary and Westfield College: University of London, United Kingdom and Francesco LUCCHIN Department of Astromina, University di Padova ,Italy.

This thesis is mainly a review work of established ideas. The entire thesis consists of eight chapters except the general introduction. This introductory chapter does not contain any mathematical work. It is almost ornamental. Every chapter has got an introduction of its own. The various chapters are organized as follows:

- **Chapter-1** of this thesis deals with **"The Fundamental of General Relativity"**

In this chapter we shall assume familiarity with the special theory of relativity. Two inertial observers *ie.* two observers who move uniformly in a straight line relative to each other, describe nature in identical term.

Certainly, aesthetic demands, that is, nature would not show preference between two observers with any type of relative motion. This implies that we must search for a more general principle of relativity, demanding invariance, not merely under more general transformations arising out of non-uniform for relative motion of two observers. This was one motivation for going over from Special Relativity to General Theory of Relativity

- ❖ **Chapter-2** of this thesis deals with **"On the Cosmological models"**

In this chapter we discuss about the cosmology & different types of cosmological models that gives an overall ideas of the universe.

Cosmology is concerned with the extragalactic world. It is the study of the large-scale structure of the universe extending to distances of light years, a study of the overall dynamical and physical behavior of billions of galaxies spread across vast distances and of the evolution of this enormous system over several billions years.

In this chapter we have derived the Robertson-Walker metric with the energy-momentum tensor as that of a perfect fluid in which in which the matter is at rest in the local frame. While the Robertson-Walker metric incorporates the symmetry properties and the kinematics of space-time, the Einstein equations provide the dynamic that is the manners in which the matter and the space-time in turn, are affected by the forces present the universe. Here we try to discuss briefly about why we need to study the microwave background, the origin of the cosmic microwave background. Then we transform the terms from scalar to ratio.

- ❖ **Chapter-3** of this thesis deals with **"Classical & Quantization problem"**

In this chapter we discuss about the Classical Klein-Gordon Field, Equation of states, Velocity & acceleration equation Cosmological constant and its alternative derivation by using the Lagrangian equation. Limiting solution are solved here with derivation also for the condition of the kinetic and potential energy. We also discuss & calculate exactly solvable model of Inflation, Cosmological constant & scalar field, Density fluctuation, Equation of state for variable cosmological constant, Wheeler-DeWitt equation, Quantization.

- ❖ **Chapter-4** of this thesis deals with **“An exact scalar field inflationary cosmological model which solves Cosmological constant problem”**

In this chapter we will show a method to construct an infinite number of exact scalar field inflationary cosmological models. This model predicts existence of dark matter/energy and gives an extremely accurate estimate of present energy density of dark matter and energy. Along with explanations of graceful exit, radiation era, matter domination, this model also indicates the reason for present accelerated state of the universe.

- ❖ **Chapter-5** of this thesis deals with **“The Scalar field potential”**

In this chapter we will study cosmological scaling solutions in spatially flat isotropic model. We assume that scaling solutions describe a perfect fluid with equation of state $p_M = (\gamma - 1)\rho_M, (w_M = \gamma - 1)$ and scalar field φ with the potential V(φ). Then we derive exact form of the scalar field potential. We introduce a scaling solution which was derived in the previous chapter. We construct a field potential assuming that a perfect fluid dominates. We will show this simple fact in case of a power-law expansion for a general cosmological background governed by the Friedmann equation. Form of the resulting general solution has informative features.

- ❖ **Chapter-6** of this thesis deals with **“Scalar field cosmology in phase space”**

In this chapter we approach the spatially homogeneous and isotropic cosmology of scalar fields minimally coupled to gravity from the phase space point of view. Although dynamical system methods have been widely used in cosmology since the 1960s and this type of analysis has been performed for non-minimally coupled scalar fields and general scalar-tensor or f(R) gravity
The purpose of this paper is to discuss these general features, specifically the geometry of the phase space, the existence, nature, and stability of the fixed points, and the late-time behavior of the solutions, without specifying the form of the scalar field potential energy density, and

instead making some generic assumptions on its behavior (boundedness, presence of asymptotes, etc.).

Then we choosing the Hubble radius $L = H^{-1}$ as system's IR cutoff, we implement the connection between the holographic dark energy and scalar fields models. We review interacting HDE with Hubble radius as systems' IR cutoff. In this section we reconstruct the analytical form of the potentials as a function of scalar field, and the dynamics of the scalar fields as a function of time, by suggesting a correspondence between holographic energy density and scalar field models old dark energy.

- **Chapter-7** of this thesis deals with **"Inflation in homogenous & isotropic space-time"**

In this chapter we introduce the inflationary paradigm as a solution to the flatness and horizon problem of standard (pre-inflationary) Big Bang cosmology. We describe the simple scenario in which inflation is modeled by means of a single scalar field which rolls slowly on its potential.

- **Chapter-8** of this thesis deals with **"Inflation in non-minimally coupled theories & inflation via modified gravity"**

In this chapter we introduce a non-minimal coupling between the inflation field and gravity. This leads to several interesting consequences which we explore. In particular it leads to a lowering of the tensor-to-scalar ratio r, as compared to minimally coupled models in general. The models we present throughout the remainder of the thesis all feature a non-minimal coupling term and the results will be derive.

Next we consider The Chaotic inflation in slow-roll approximation, Cosmological Constant associated with chaotic inflation & inflation within f(R)-theories of gravity. In particular we consider the Starobinsky model of inflation, and find that it is connected to matter scalar field models with a non-minimal coupling to gravity. We then consider quantum induced marginal deformations of the Starobinsky action, and find that such deformations significantly shift the predicted tensor-to scalar towards higher values. At last we discuss sources for these corrections.

CHAPTER -1

THE FUNDAMENTAL OF GENERAL RELATIVITY

1.1 Introduction:

General Relativity is one of the most important theoretical developments of the 20^{th} century. It is a theory about the structure and dynamics of space time itself, and its interaction with matter. Einstein's extraordinary intuition led him, in 1915, ten years after the development of special Relativity, to suggest-something which was verified soon after by a number of important experimental measurements-that the gravitational 'forces' as perceived by the Newtonian approach was incorrect, and that the correct approach was to assume that this 'forces' was the result of nonzero curvature of space time, which itself was the consequence of a non-trivial mass distribution. This is the main idea behind Einstein's theory of gravitation, the so-called General Relativity **[42]**.

There are important differences from Newtonian Gravitation. For instance, a satellite orbiting around a massive body in Einstein's theory of General Relativity is floating freely, without the influence of any force, following a geodesic curve in the curved space time induced by the presence of the massive body. This is in sharp contrast to the Newtonian approach, where the inverse-square law gravitational force characterizes the satellite motion. Moreover, General relativity, being a relativistic theory ,i.e. a natural extension of special Relativity for non flat space times, shares all the novel ingredients of the latter, such as the lack of objective simultaneity of events, the existence of a limiting velocity, that of light in vacuum etc, which were absent in the Newtonian approach. Nevertheless, for consistency, there is a limit in which Einstein's theory reproduces partially some of the results of Newton's theory (e.g. for large distances away from the gravitational centers of attraction the orbits resemble those predicted by Newton).

Despite the 85 years that passed since Einstein put forward his famous equations, the theory of General Relativity remains a Classical theory of the gravitational field, whose quantum version is still an elusive object of intense and exciting theoretical debate. This should be contrasted with the rest of the fundamental interactions in Nature (electromagnetic, weak and strong) whose quantum field theories are sufficiently developed and confirmed by Experiment to a great extent. Nevertheless, the classical theory of Einstein's gravity has been verified by experiment to a point that no one doubts today about its validity, at least for sufficiently low energy scales that describe a big portion of the observable universe to date. It should be noted, though, that there are still some predictions of this classical theory, namely gravitational waves, whose experimental confirmation is still lacking, and for this purpose important satellite and terrestrial experiments are currently under construction or design.

1.2 The Principle of Equivalence:

The Principle of equivalence of Equivalence rests on the equality of gravitational and inertial mass, demonstrated by Galileo, Huygens, Newton, Bessel and Eotvos. Einstein reflected that as a consequence, no external static homogenous gravitational field could be detected in a freely falling elevator, for the observers, their test bodies, and the elevator itself would respond to the field with the same acceleration. This can be easily proved for a system of particles N, moving with no relativistic velocities under the influence of forces $F(x_N - x_M)$ (e.g. electrostatic or gravitational forces) and an external gravitational field g .

The equations of motion are

$$m_N \frac{d^2 x_N}{dt^2} = m_N g + \sum_M F(x_N - x_M) \quad \dots \dots \dots (1.1)$$

Suppose that we perform a non-Galilean space-time co-ordinate transformation

$$x' = x - \frac{1}{2}gt^2 \quad , \quad t' = t \qquad \dots \dots \dots (1.2)$$

Then g will be canceled by an inertial 'force', and the equation of motion will becomes

$$m_N \frac{d^2 x'_N}{dt'^2} = \sum_M F(x'_N - x'_M) \quad \dots \dots \dots (1.3)$$

Hence the original observer O who uses coordinates xt, and his freely falling friend O' who uses $x't'$ will detect no difference in the laws of mechanics, except that O will say that he feels a gravitational field and O' will say that he does not. The equivalence principle says that this cancellation of gravitational by inertial force (and hence their equivalence) will obtain for all freely falling systems, whether or not they can be described by simple equations such as (1.1).

We are not yet ready to state the principle of equivalence in its final forms, because the preceding remark dealt only with a static homogeneous gravitational field. Had g depended on x or t, we should not have been able to eliminate it from the equations of motions by the acceleration (1.2). For example, the earth is in free all about the sun, and the most part we on earth do not feel the sun's gravitational field, but the slight in homogeneity in this field (about1 part in 6000 from noon to midnight) is enough to raise impressive tides in our oceans. Even the observers in Einstein's freely falling elevator would by falling radially towards the centre of the earth, and hence would approach each other as the elevator descended.

Although inertial forces do not exactly cancel gravitational forces freely falling systems in an inhomogeneous or time-dependent gravitational field , we can still expect an approximate cancellation if we restrict our attention to such a small region of space and time that the field changes very little over the region. Therefore we formulate the equivalence principle as the statement that -

At every space-time point in an arbitrary gravitational field it is possible to choose a "locally inertial coordinate system" such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in uncelebrated Cartesian coordinate systems in the absence of gravitation.

There is a little vagueness here about what we mean by "the same form as in unaccelerated Cartesian coordinate systems", so to avoid any possible ambiguity we can specify that by this we mean the form given to the laws of nature by special relativity. There is also a question of how small is "sufficiently small". Roughly speaking, we mean that the region must be small enough so that the gravitational field is sensibly constant throughout it, but we cannot be more precise until we learn how to represent the gravitational field mathematically

Occasionally one finds references to a "weak principle of Equivalence" and a "Strong Principle of Equivalence." The strong Principle of equivalence is just what I have already stated, with "laws of nature" meaning all the laws of nature. The weak principle is the same, but with "laws of nature" replaced by "laws of motion of freely falling particles." That is the weak principle is nothing but a restatement of the observed equality of gravitational and inertial mass, whereas the strong principle is a generalization of these observations that governs the effects of gravitation on all physical systems.

The experiments of Eotvos, Dicke and their predecessors provide direct verification only of the weak Principle of Equivalence, but they provide some indirect evidence for the strong principle. The mass of different substances arises in different proportions from the masses of the neutrons and protons plus electrons of which they are composed and from the strong and electromagnetic forces that bind these particles together, so the ratio of gravitational to inertial mass will be equal for all these substances only if it is equal for their constituents. Wapstra & Nijgh have shown that the limits set by Eotvos on any possible inequality in the ratio of gravitational to inertial mass for glass, cork, antimonite, and brass imply that this ratio is equal for neutrons and protons plus electrons to 1 part in 6×10^5 and equal for neutrons and binding energies to 1 part in 1.2×10^4 . To this accuracy an observer in a freely falling coordinates system will detect no gravitational forces on neutrons, Hydrogen, or their binding energies. It would be difficult to conceive of a theory that satisfies this requirement and does not go all the way to the strong principle (that no gravitational effects of any sort can be felt in a locally inertial frame.)

We might, however distinguish two versions of the strong principle of equivalence, a “Very strong principle,” which applies to all phenomena except gravitation itself. Certainly the experiments of Eotvos & Dicke are not accurate enough to say whether gravitational binding energies affect inertial and gravitational masses in the same way. This equation might be settled by studying the motion of a small body in orbit about a large body that is itself in free fall in a gravitational field. For instance the gravitational binding energy of the earth contributes a fraction -8.4×10^{-10} of its total mass, whereas the gravitational binding energy of an artificial satellite contributes a very much smaller fraction of its mass. Thus if (to take an extreme case) the (negative) gravitational bindings energy contributes fully to the inertial mass but not at all to the gravitational mass, then that for the earth by a fraction 8.4 $\times 10^{-10}$. The earth is in free fall, with the gravitational attraction of the sun balanced by the inertial forces owing to the earth's revolution. The gravitational and inertial forces on the satellite owing to the presence of the sun and the earth's revolution are equal (neglecting for a moment the distance between the satellite and the earth's center of mass) to the gravitational and inertial forces on the earth times the ratio of gravitational or inertial masses, so these two forces are not in balance for the satellite, the gravitational forces being greater than the inertial force by a fraction 8.4 $\times 10^{-10}$. The acceleration owing to the sun's gravity is at the orbit of the earth about 6 $\times 10^{-4}$ of the acceleration owing to the earth's gravity at the surface of the earth. So we conclude that if the gravitational binding energy of the earth contributed fully to its inertial mass but not at all to its gravitational mass, then an artificial satellite in a low orbit about the earth would feel an effective attraction toward the sun equal to about 5.4 $\times 10^{-13}$ times its gravitational attraction toward the earth. This tiny effect would be entirely masked by "tidal" forces because the satellite is far from the center of mass of the earth and there is no prospect of its being measured. This is a pity, because it is precisely the very strong assumption that the Principle of equivalence applies to gravitational fields**[63]**.

1.3 Gravitational Forces:

Consider a particle freely under the influence of purely gravitational forces. According to the Principle of equivalences, there is a freely falling coordinate system ξ^{α} in which its equation of motion is that of a straight line in space-time, that is,

$$\frac{d^2\xi^{\alpha}}{d\tau^2} = 0 \quad \dots\dots\dots (1.4)$$

With $d\tau$ the proper time

$$d\tau^2 = -\eta_{\alpha\beta} d\xi^\alpha d\xi^\beta \quad \ldots \ldots \ldots (1.5)$$

Now suppose that we use any other coordinate system x^μ, which may be a Cartesian coordinate system at rest in the laboratory , but also may be curvilinear, accelerated, rotating, or what we will. The freely falling coordinates ξα are functions of the x^μ and equation (1.4) becomes **[63],**

$$\frac{d}{d\tau}\left(\frac{\delta\xi^\alpha}{\delta x^\mu}\frac{dx^\mu}{d\tau}\right) = 0$$

$$\Rightarrow \frac{\delta\xi^\alpha}{\delta x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\delta^2\xi^\alpha}{\delta x^\mu \delta x^\vartheta}\frac{dx^\mu}{d\tau}\frac{dx^\vartheta}{d\tau} = 0$$

Multiply this by $\frac{\delta x^\lambda}{\delta\xi^\alpha}$ and use the familiar product rule

$$\frac{\delta\xi^\alpha}{\delta x^\mu}\frac{\delta x^\lambda}{\delta\xi^\alpha} = \delta^\lambda_\mu$$

This gives the equation of motion

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\vartheta}\frac{dx^\mu}{d\tau}\frac{dx^\vartheta}{d\tau} = 0 \quad \ldots \ldots \ldots (1.6)$$

Where $\Gamma^\lambda_{\mu\vartheta}$ is the affine connection, defined by

$$\Gamma^\lambda_{\mu\vartheta} \equiv \frac{\delta x^\lambda}{\delta\xi^\alpha}\frac{\delta^2\xi^\alpha}{\delta x^\mu \delta x^\vartheta} \quad \ldots \ldots \ldots (1.7)$$

The proper time (1.5) may also be expressed in an arbitrary coordinate system,

$$d\tau^2 = -\eta_{\alpha\beta} \frac{\delta\xi^\alpha}{\delta x^\mu} dx^\mu \frac{\delta\xi^\beta}{\delta x^\vartheta} dx^\vartheta \quad \ldots\ldots\ldots (1.8)$$

$$\Rightarrow d\tau^2 = -g_{\mu\vartheta} dx^\mu dx^\vartheta \quad \ldots\ldots\ldots (1.9)$$

Where $g_{\mu\vartheta}$ is the metric tensor, defined by

$$g_{\mu\vartheta} \equiv \frac{\delta\xi^\alpha}{\delta x^\mu} \frac{\delta\xi^\beta}{\delta x^\vartheta} \eta_{\alpha\beta} \quad \ldots\ldots\ldots (1.10)$$

For a photon or a neutrino the equation of motion in a freely falling system is the same as (1.4), except that the independent variable cannot be taken as the proper time (1.5), because for massless particles the right-hand side of (1.5) vanishes. Instead of τ we can use $\sigma = \xi^\alpha$, so that (1.4) & (1.5) become

$$\frac{d^2\xi^\alpha}{d\sigma^2} = 0$$

$$\Rightarrow -\eta_{\alpha\beta} \frac{d\xi^\alpha}{d\sigma} \frac{d\xi^\beta}{d\sigma} = 0$$

Following the same reasoning as before, we find that the equation of motion in an arbitrary gravitational field and an arbitrary coordinate system is

$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\lambda\vartheta} \frac{dx^\vartheta}{d\sigma} \frac{dx^\lambda}{d\sigma} = 0 \quad \ldots\ldots\ldots (1.11)$$

$$-g_{\mu\vartheta} \frac{dx^\vartheta}{d\sigma} \frac{dx^\lambda}{d\sigma} = 0 \quad \ldots\ldots\ldots (1.12)$$

With $\Gamma^{\mu}_{\lambda\vartheta}$ and $g_{\mu\vartheta}$ given as before by (1.7) & (1.10).

Incidentally, in both (1.6) and (1.11) we do not need to know what τ and σ are in order to find the motion of our particle,, for these equations when solved give $x^{\mu}(\tau)$

$x^{\mu}(\sigma)$, and τ or σ can be eliminated to give x(t). The purpose of (1.9), is tell us how to compute the proper time, whereas the purpose of (1.12) is to impose initial conditions appropriate to a massless particle. In particular, Equation (1.12) tells us that the time *dt* for a photon to travel a distance d*x* is determined by the quadric equation

$$g_{00}dt^2 + 2g_{i0}dx^i dt + g_{ij}dx^i dx^j = 0$$

With *i* and *j assumed over the values 1, 2, and 3.* The solution is

$$dt = \frac{1}{g_{00}}\left[-g_{i0}dx^i - \{(g_{i0}g_{j0} - g_{ij}g_{00})dx^i dx^j\}^{\frac{1}{2}} \right] \quad \dots\dots\dots (1.13)$$

And time required for light to travel along any path may be calculated by integrating *dt* along the path.

The values of the metric tensor $g_{\mu\vartheta}$ and the affine connection $\Gamma^{\lambda}_{\mu\vartheta}$ at a point X in an arbitrary coordinate system x^{μ} provide enough information to determine the locally inertial coordinate's $\xi^{\alpha}(x)$ in a neighborhood of X. First, we multiply equation (1.7) by $\frac{\delta\xi^{\alpha}}{\delta x^{\lambda}}$. And use the product rule

$$\frac{\delta\xi^{\beta}}{\delta x^{\vartheta}}\frac{\delta x^{\lambda}}{\delta\xi^{\alpha}} = \delta^{\beta}_{\alpha}$$

Thereby obtaining the differential equations for ξ^{α}

$$\frac{\delta^2 \xi^\alpha}{\delta x^\mu \delta x^\vartheta} = \Gamma^\lambda_{\mu\vartheta} \frac{\delta \xi^\alpha}{\delta x^\lambda} \qquad \ldots \ldots \ldots (1.14)$$

The solution is

$$\xi^\alpha(x) = a^\alpha + b^\alpha_\mu (x^\mu - X^\mu) + \frac{1}{2} b^\alpha_\lambda \Gamma^\lambda_{\mu\vartheta} (x^\mu - X^\mu)(x^\vartheta - X^\vartheta) + \ldots.. \qquad \ldots \ldots \ldots (1.15)$$

Where

$$a^\alpha = \xi^\alpha(x) \ , \quad b^\alpha_\lambda = \frac{\delta \xi^\alpha(X)}{\delta X^\lambda} \qquad \ldots \ldots \ldots (1.16)$$

From equation (1.10) we also learn that

$$\eta_{\alpha\beta} b^\alpha_\mu b^\beta_\vartheta = g_{\mu\vartheta}(X) \qquad \ldots \ldots \ldots (1.17)$$

Thus given $\Gamma^\lambda_{\mu\vartheta}$ and $g_{\mu\vartheta}$ at X, the locally inertial coordinates ξ^α are determined to order $(x - X)^2$, except for the ambiguity in the constants a^α and b^α_λ. The b^α_λ are determined by equation (1.16) up to a Lorentz Transformation $b^\alpha_\mu \to \Lambda^\alpha_\beta b^\beta_\mu$, so the ambiguity in the solution for ξ^α (x) just reflects the fact that if ξ^α are locally inertial coordinates, then so are $\Lambda^\alpha_\beta \xi^\beta + c^\alpha$. Hence, since $\Gamma^\lambda_{\mu\vartheta}$ and $g_{\mu\vartheta}$ determine the locally inertial coordinates up to an inhomogeneous Lorentz transformation and since the gravitational field can have no effects in a locally inertial coordinates system, we should not be surprised to find that all effects of gravitation are comprised in $\Gamma^\lambda_{\mu\vartheta}$ & $g_{\mu\vartheta}$.

1.4 The Principle of General Covariance:

The Principle of General Covariance states that a physical equation holds in a general gravitational field, if two conditions are met:

1. The equation holds in the absence of gravitation; that is it agrees with the laws of special relativity when the metric tensor $g_{\alpha\beta}$ equals the Minkowski tensor $\eta_{\alpha\beta}$ and when the affine connection $\Gamma^{\alpha}_{\beta\gamma}$ vanishes.
2. The equation is generally covariant; that is, it preserves its form under a general coordinate transformation $x \to x'$.

To see that the Principle of General Covariance follows from the Principle of Equivalence, let us suppose that we are in an arbitrary gravitational field, and consider any equation that satisfies the two above conditions. From condition-2 we learn that the equation will be true in all coordinate systems if it is true in anyone coordinate system. But at any given point there is a class of coordinate systems, the locally inertial systems, in which the effects of gravitation are absent. Condition-1 then tells us that our equations holds in these systems and hence in all other coordinate systems.

It should be stressed that general covariance by itself is empty of physical content. Any equation can be made generally covariant by writing it in anyone coordinate system , and then working out what it looks like in order arbitrary coordinate system .Indeed from childhood we have become familiar with the appearance of physical equations in non-Cartesian systems, such as polar coordinates. The significance of the Principle of general covariance lies in its statement about the effects of gravitation, that a physical equation by its general covariance will be true in a gravitational field if it is true in the absence of gravitation.

The meaning of general covariance can be brought forward by comparing it with Lorentz invariance. Just as any equation can be made generally covariant, so any equation can be made Lorentz –invariant, by writing it in one coordinate system and then working out what it looks like after a Lorentz transformation. However, if we do this with a non-relativistic equation like Newton's second law, we find after making it Lorentz –invariant that a new quantity has entered the equation, which of course is the velocity of the coordinate frame with respect the original reference frame. The requirement that this velocity not appear in the transformed equation is what we call the Principle of Special Relativity, or "Lorentz invariance" for short, and this requirement places very powerful restrictions on the original equation. Similarly, when we make an equation generally covariant, new ingredients will enter, that is the metric tensor $g_{\mu\vartheta}$and the affine connection $\Gamma^{\lambda}_{\mu\vartheta}$. The difference is that we do not require that these quantities drop out at the end, and hence we do not obtain any restrictions on the equation we start with; rather we explot the presence of $g_{\mu\vartheta}$ and $\Gamma^{\lambda}_{\mu\vartheta}$ to represent gravitational fields. To put this briefly; The Principle of General Covariance is not an invariance principle, like the Principle of Galilean or Special Relativity, but is instead a statement about the effects of gravitation and about nothing else. In particular, general covariance does not imply Lorentz invariance - there are generally covariant theories of gravitation that allow the construction of inertial frames at any point in a gravitational field, but that satisfy Galilean relativity rather than Special Relativity in these frame.

The Principle of General Covariance can only be applied on a scale that is small compared with the space-time distances typical of the gravitational field, for it is only on this small scale that we are assured by the Principle of Equivalence of being able to construct a coordinate system in which the effects of gravitation are absent. For instance, the radius of the moon is not so very much smaller than the earth –moon separation, so we cannot accurately calculate the motion of the moon by finding generally covariant equations that reduce to the correct equations for a freely moving moon in the absence of gravitation. We can, however, treat the moon as a ball of rock and calculate its motion by applying the Principle of General Covariance to determine the gravitational forces on each infinitesimal element of the lunar mass **[63].**

1.5 Metric in a Gravitational Field:

Suppose that space-station in the shape of a wheel has been constructed in a region of space far from other attracting bodies and that it is set rotating in its plane about its centre with angular velocity ω .An observer O, wearing a space-suit is located outside the station and does not participate in the rotary motion; his frame of reference is therefore inertial. *O* watches C, a member of station's crew, measuring the dimensions of the station using a metric rule. C first measures the radius of the station from its centre to its outer wall by laying his rule is moving laterally throughout the measuring process, but this motion does not affect its length in his frame and he will accordingly agree with the radius *r* recorded by C. C next lays his rule around the outer wall of the station and records a perimeter p .During this process, however, O sees the rule a factor $\sqrt{1-\frac{\omega^2 r^2}{c^2}}$. He will accordingly correct the length of the perimeter found by C to the value $p\sqrt{1-\frac{\omega^2 r^2}{c^2}}$. Since O's frame is inertial, Euclidean geometry is valid for all space measurements referred to the frame and he must find that

$$p\sqrt{1-\frac{\omega^2 r^2}{c^2}} = 2\pi r \qquad \dots \dots \dots (1.18)$$

Thus

$$p = 2\pi r\left(1-\frac{\omega^2 r^2}{c^2}\right)^{-\frac{1}{2}} \qquad \dots \dots \dots (1.19)$$

This last equation indicates that C will discover that the Euclidean formula $p = 2\pi r$ is not valid for measurements made in rotating frame of the space-station. But C is entitled to regard the station frame as being at rest, provided he accepts the existence of a gravitational field which will account for the centrifugal and Coriolis forces he experiences. We conclude that relative to a frame at rest in such a gravitational afield, spatial measurements will not be in conformity with Euclidean geometry **[31]**.

By the principle of equivalence the conclusion which has been reached concerning the non-Euclidean nature of space in which there is present a gravitational field of the centrifugal-Coriolis type, must be extended to all gravitational field. However, in the case of a field such as that which surrounds the earth, it will not be possible (as it is for the centrifugal-Coriolis field) to find an inertial frame of reference relative to which the field vanishes and for which the spatial geometry is Euclidean. Such afield will be termed irreducible. Even in an irreducible field, however , a frame can always be found which is inertial for a sufficiently small region of space and a sufficiently small time duration. Thus, within a space-ship which is not rotating relative to the extragalactic nebulae and which is falling freely in the earth's gravitational field, free particles will follow straight –line paths at constant speed for considerable periods of time and the condition will be inertial. A coordinate frame fixes in the ship will accordingly simulate an inertial frame over a restricted region of space and time and its geometry will be approximately Euclidean.

Since a rectangular Cartesian coordinate frame can be set up only in a space possessing a Euclidean metric, this method of specifying the relative positions of events must be abandoned in an irreducible gravitational field (except over small regions as has just been explained). Instead, the positions and times of all events will be specified by references to a very general type of frame which we can suppose constructed as follows : Imagine the whole of the cosmos is filled by a fluid whose motion is arbitrary but non-turbulent (i.e. particles of the fluid which are initially close together , remain in proximity to one another). Let each molecule of the fluid be a clock which runs smoothly, but not necessarily at a constant rate as judged by a standard atomic clock. No attempt will be made to synchronize clocks which are separated by a finite distance, but it will be assumed that, as this distance tends to zero, the reading of the clocks will always approach one another. Each clock will be allocated three spatial coordinate's ξ^1, ξ^2, ξ^3 according to any scheme which ensures that the coordinates of adjacent clocks only differ infinitesimally. The coordinate's ξ^α of a clock will be supposed never to change. Any event taking place anywhere in the cosmos can now allocated unique space-time coordinates ξ^i(i=1,2,3,4) as follows: (ξ^1, ξ^2, ξ^3) are the spatial coordinates belonging to the clock which happens to be adjacent to the event when it occurs, and ξ^4 is the time shown on this clock at this instant.

We shall now further generalize the coordinates allocated to an event. Let x^i(i=1,2,3,4) be any functions of the ξ^i such that, to each set of values of the ξ^i there corresponds one set of values of the x^i and conversely. We shall write

$$x^i = x^i(\xi^1, \xi^2, \xi^3, \xi^4) \quad \dots \dots \dots (1.20)$$

Then the x^i also, will be accepted as coordinates, with respect to a new frame of reference, of the event whose Coordinates were previously taken to be the ξ^i. It should be noted that, in general each of the new coordinate x^i will depend upon both the time and the position of the event, i.e. it will not necessarily be the case that three of the coordinate's x^i are spatial in nature and one is temporal. All possible events will now be mapped upon a space δ_4 , So that each event is represented by a point of the space and the x^i will be the coordinates of this point with respect to a coordinate frame δ_4 will be referred to as the space-time continuum.

It has been remarked that, in any gravitational field, it is always possible to define a frame relative to which the field vanishes over a restricted region and which behaves as an inertial frame for events occurring in this region and extend ending over a small interval of time .Such a frame will be falling freely in the gravitational field and will accordingly be referred to as a local free-fall frame. Suppose, then that such an inertial frame S is found for two contiguous events. Any other frame in uniform motion relative to S will be inertial for these events. Observers at rest in all such frames will be able to construct rectangular Cartesian axes and measure the proper time interval $d\tau$ between the events. If, for one such observer, the events at the points having rectangular Cartesian coordinates $(x,y,z),(x+dx , y+dy , z+dz)$ occur at the times t, $t + dt$ respectively, then

$$d\tau^2 = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2) \quad \dots \dots \dots (1.21)$$

The interval between the events ds will be defined by

$$ds^2 = -c^2 d\tau^2 = dx^2 + dy^2 + dz^2 - c^2 d\tau^2 \quad \ldots\ldots\ldots (1.22)$$

The coordinates (*x*,*y*,*z*,*t*) of an event in this quasi-inertial frame will be related to the coordinates x^i defined earlier, by equations

$$x = x\,(x^1, x^2, x^3, x^4) \quad \text{etc.} \quad \ldots\ldots\ldots (1.23)$$

And hence

$$dx = \frac{\delta x}{\delta x^i} dx^i \quad \ldots\ldots\ldots (1.24)$$

Substituting for *dx* ,*dy* , *dz*, *dt* in equation (1.24), we obtain the result

$$ds^2 = g_{ij} dx^i dx^j \quad \ldots\ldots\ldots (1.25)$$

Determining the interval *ds* between two events contiguous in space-time, relative to a general coordinates frames valid for the whole of space-time. The space-time continuum can accordingly be treated as a Riemannian space with metric given by equation (1.25).

1.6 Motion of a Free Particle in a Gravitational field:

In a region of space which is at a great distance from material bodies, rectangular Cartesian axes *Oxyz* can be found constituting an inertial frame. If time is measure by clocks synchronized within this frame and moving with it, the motion of a freely moving test particle relative to the frame will be uniform. Thus if (*x*,*y*,*z*) is the position of such a particle at time t, its equation of motion can be written

$$\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} = 0 \quad \ldots\ldots\ldots (1.26)$$

Let *ds* the interval between the events of the particle arriving at the point (*x*,*y*,*z*) at time t and the contiguous event of the particle arriving at (*x*+*dx*

,*y*+*dy* ,*z*+*dz*) at $t + dt$ .Then *ds* is given by equation (1.22) and if υ is the speed of the particle, it follows from this equation that

$$ds = \left(\nu^2 - c^2\right)^{\frac{1}{2}} dt \quad \ldots\ldots\ldots (1.27)$$

Since υ is constant, it now follows that equations (1.26) can be expressed in the form

$$\frac{d^2x}{ds^2} = \frac{d^2y}{ds^2} = \frac{d^2z}{ds^2} = 0 \quad \ldots\ldots\ldots (1.28)$$

Also from equation (1.27) it may be deduced that

$$\frac{d^2t}{ds^2} = 0 \quad \ldots\ldots\ldots (1.29)$$

Equation (1.28) & (1.29) determine the family of world-lines of free particles in space-time relative to an inertial frame **[31]**.

Now suppose that other reference frame and procedure for measuring time is adopted in this region of space, e.g. a frame which is in uniform rotation with respect to an event in this frame might be employed (x^1, x^2, x^3, x^4) be the coordinates of an event in this frame. The interval between two contiguous events will then be given by equation (1.25). If an observing using this frame releases a test particle and observers its motion relative to the frame, he will denote that it is not uniform or even rectilinear and will be able to account for this fact by assuming the presence of a gravitational field. He will find that the particle's equations of motion are

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \quad \ldots\ldots\ldots (1.30)$$

This is a tensor equation defining a geodesic and valid in every frame if it is valid in one. But in the *xyzt*-frame, the g_{ij} are all constant and the three index symbols vanish. Hence, in this frame, the equations (1.30) reduce to the equations (1.28) & (1.29) and these are known to be true for the particles motion. We have shown, therefore, that the effect of a gravitational field of the reducible variety upon the motion of a test particle can be allowed for when the form taken by the metric tensor g_{ij} of the space-time manifold is known relative to the frame being employed. This means that the g_{ij} determine, and are determined by, the gravitational field.

The ideas of the previous paragraph will now be extended to regions of space where irreducible gravitational fields are present. It has been pointed out that, for any sufficiently small region of such space and interval of time, an inertial frame can be found and consequently the paths of freely moving particles will be governed in such a small region by equations (1.30). It will now be assumed that these are the equations motion of free particles without any restrictions, i.e. that the world-line of a free particle is a geodesic for the space-time manifold or that the world-line of a free particle has constant direction. This appears to be the natural generalization of the Galilean law of inertia whereby, even in an even in an irreducible gravitational field, a particle's trajectory through space-time is the straightest possible after consideration has been given to the intrinsic curvature of the continuum. It will then follow that the motions of particle falling freely in any gravitational field can be determined relative to any frame when the components g_{ij} of the metric tensor for this frame are known. Thus the g_{ij} will always specify the gravitational field observed to be present in a frame and the only distinction between irreducible and reducible fields will be that, for the latter will be possible to find a coordinate frame in space-time for which the metric tensor has all its components zero except

$$g_{11} = g_{22} = g_{33} = 1 \quad \& \quad g_{44} = -c^2 \qquad \ldots \ldots \ldots (1.31)$$

Whereas for the former this will not be possible.

Since the Christoffel symbols vanish in a frame which is geodesic at some point of space-time, in such a frame equation (1.30) reduce to $\frac{d^2x^i}{ds^2} = 0$ over a small neighborhood of the point. If, in addition, the frame is chosen to be quasi-Euclidean with metric (1.22), equation (1.26) will be valid over the neighborhood and a freely falling body will have very nearly uniform motion such a frame can therefore be identified with a local freely falling frame.

1.7 Derivation of the Field Equations:

The field equations for gravitation are inevitably going to be more complicated than those for electromagnetism. Maxwell's equations are linear because the electromagnetic field does not itself carry charge, whereas gravitational fields do carry energy and momentum and must therefore contribute to their own source. That is, the gravitational field equations will have to be nonlinear partial different equations, the nonlinearity representing the effect of gravitational itself **[63].**

In dealing with these nonlinear effects we are guided once again by the Principle of Equivalence. At any point X in an arbitrarily strong gravitational field. We can define a locally inertial coordinate system such that

$$g_{\alpha\beta}(X) = \eta_{\alpha\beta} \ldots \ldots \ldots (1.32)$$

$$\left(\frac{\partial g_{\alpha\beta}(x)}{\partial x^{\gamma}}\right)_{x = X} = 0 \ldots \ldots \ldots (1.33)$$

Hence for x near X, the metric tensor $g_{\alpha\beta}$ can differ from $\eta_{\alpha\beta}$ only by terms quadratic in x – X. In this coordinate system the gravitational field is weak near X, and we can hope to describe the field by linear partial differential equations. And once we know what these weak-field equations are, we can find the general field equations by reversing the coordinate transformation that made the field weak.

Unfortunately, we have very little empirical information about the weak-field equations. This is not for any fundamental reason, but rather because gravitational radiation is so weakly generated and absorbed by matter, that it has not yet certainly been detected.

First let us recall that in a weak static field produced by a non relativistic mass densityρ, the time -time component of the metric tensor is approximately given by

$$g_{00} > \simeq -(1 + 2\phi)$$

Here ϕ is the Newtonian potential, determined by Poisson's equation

$$\nabla^2 \emptyset = 4\pi G\rho$$

Where G is Newton's constant, equal to 6.670×10^{-8} in c.g.s. units. Furthermore, the energy density T_{00} for non relativistic matter is just equal to its mass density

$$T_{00} > \simeq \rho$$

Combining the above, we have then

$$\nabla^2 g_{00} = -8\pi G T_{00} \quad \dots \dots \dots (1.34)$$

This field equation is only supposed to hold for weak static fields generated by non relativistic matter, and is not even Lorentz invariant as it stands. However, (1.34) leads us to guess that the weak-field equations for general distribution $T_{\alpha\beta}$ of energy and momentum take the form

$$G_{\alpha\beta} = -8\pi G T_{\alpha\beta} \quad \dots\dots\dots (1.35)$$

Where $G_{\alpha\beta}$ is a linear combination of the metric and its first and second derivatives. It follows then from the Principle of Equivalence that the equations which govern gravitational fields of arbitrary strength must take the form

$$G_{\mu\vartheta} = -8\pi G T_{\mu\vartheta} \quad \dots\dots\dots (1.36)$$

Where $G_{\mu\vartheta}$ is a tensor which reduces to $G_{\alpha\beta}$ for weak fields.

In general, there will be a variety of tensor $G_{\mu\vartheta}$ that can be formed from the metric tensor and its derivatives, and that reduce in the weak-field limit to a given$G_{\alpha\beta}$. Let us imagine $G_{\mu\vartheta}$ to be expanded in a sum of products of derivatives of the metric, and classify each term according to the total number N of derivatives of the metric components. (For example, a term with N = 3 could be linear in the third derivatives of the metric, or a product of a first derivative with a second derivative, or a product of three first derivatives.) The whole of $G_{\mu\vartheta}$ must have to dimensions of a second derivative, so each term of type $N \neq 2$apperas multiplied with a constant having the dimensions of length to the power N – 2; such terms will become negligible for gravitational fields of sufficiently large or small space –time scale if $N > 2$ or $N < 2$, respectively. In order to remove the ambiguity in $G_{\mu\vartheta}$, we shall assume that the gravitational field equations are uniform in scale, so that only terms with N = 2 are allowed.

Let us review what we know about the left-hand- side of the field equation (1.36):

(A) By definition, $G_{\mu\vartheta}$ is a tensor.

(B) By assumption, $G_{\mu\vartheta}$ consists only terms those are either linear in the second derivatives or quadratic in the first derivatives of the metric.

(C) Since $T_{\mu\vartheta}$ is symmetric, so is $G_{\mu\vartheta}$.

(D) Since $T_{\mu\vartheta}$ is conserved (in the sense of covariant differentiation) so is $G_{\mu\vartheta}$:

$$G^{\mu}_{\vartheta;\mu} = 0 \qquad \ldots\ldots\ldots (1.37)$$

(E) For a weak stationary field produced by non relativistic matter the 00 component of (1.36) must reduce to (1.34), so in this limit

$$G_{00} \cong \nabla^2 g_{oo} \qquad \ldots\ldots\ldots (1.38)$$

These properties are all we will need to find $G_{\mu\vartheta}$.

We saw that the most general way of constructing a field satisfying (A) and (B) is by contraction of the curvature tensor $R^{\lambda}_{\mu\lambda x}$. The antisymmetry property of $R^{\lambda}_{\mu v k}$ shows that there are only two tensors that can be formed by contracting $R_{\lambda\mu\upsilon\kappa}$; that is, the Ricci tensor $R_{\mu k} \equiv R^{\lambda}_{\mu\lambda k}$ and the curvature scalar $R = R^{\mu}{}_{\mu}$. Hence (A) and (B) require $G_{\mu\vartheta}$ to take the form

$$G_{\mu\vartheta} = C_1 R_{\mu\vartheta} + C_2\, g_{\mu\vartheta} R \ldots\ldots\ldots (1.39)$$

where C_1 and C_2 are constants. This is automatically symmetric, so (C) tells us nothing new. Using the Bianchi identity gives the covariant divergence of $G_{\mu\vartheta}$ as

$$G^{\mu}_{\vartheta;\mu} = \left(\frac{C_1}{2} + C_2\right) R_{;\upsilon}$$

So (D) allows two possibilities: either $C_2 = -C_1/2$, or $R_{;\upsilon}$ vanishes everywhere. We can reject the second possibility, because (1.39) and (1.36) give

$$G^{\mu}{}_{\mu} = (C_1 + 4C_2)R = -8\pi G T^{\mu}{}_{\mu}$$

Thus if $R_{;\upsilon} \equiv \partial R/\partial x^{\upsilon}$ vanishes, then so must $\partial T^{\mu}{}_{\mu}/\partial x^{\upsilon}$, and this is not the case in the presence of inhomogeneous non relativistic matter. We conclude then that $C_2 = C_1/2$, so (1.39) becomes

$$G_{\mu\vartheta} = C_1\left(R_{\mu\vartheta} - \frac{1}{2}g_{\mu\vartheta}R\right) \quad \dots\dots\dots (1.40)$$

Finally we use the property (E) to the constant C_1. A non-relativistic system always has $|T_{ij}| << |T_{00}|$, so we are concerned here with a case where $|G_{ij}| << |G_{00}|$ or using (1.40),

$$R_{ij} \cong \frac{1}{2} g_{ij} R$$

Furthermore, we deal here with a weak field, so $g_{\alpha\beta} \cong \eta_{\alpha\beta}$, The curvature scalar is therefore given by

$$R \cong R_{KK} - R_{00} \cong \frac{3}{2}R - R_{00}$$

$$\text{Or,} \quad R \cong 2R_{00} \quad \dots\dots\dots (1.41)$$

Using (1.41) and (1.32) in (1.40) , we find

$$G_{00} \cong 2C_1 R_{00} \quad \dots\dots\dots (1.42)$$

To calculate R_{00} for weal field we may use the linear part of a $R_{\lambda\mu\upsilon\kappa}$, given by

$$R_{\lambda\mu\vartheta\kappa} = \frac{1}{2}\left[\frac{\delta^2 g_{\lambda\vartheta}}{\delta x^{\kappa}\delta x^{\mu}} - \frac{\delta^2 g_{\mu\vartheta}}{\delta x^{\kappa}\delta x^{\lambda}} - \frac{\delta^2 g_{\lambda\kappa}}{\delta x^{\vartheta}\delta x^{\mu}} + \frac{\delta^2 g_{\mu\kappa}}{\delta x^{\vartheta}\delta x^{\lambda}}\right]$$

When the field is static all time derivatives vanish, and the components we need become

$$R_{0000} \cong 0 \quad , \quad R_{i0j0} \cong \frac{1}{2}\frac{\delta^2 g_{oo}}{\delta x^i \delta x^j}$$

Hence (1.42) gives

$$G_{00} \cong 2C_1(R_{i0i0} - R_{0000}) \cong C_1 \nabla^2 g_{00}$$

And comparing this with (1.38) , we find that (E) is satisfied if and only if

$C_1 = 1$

Setting C_1in equation (1.40) completes our calculation of $G_{\mu\upsilon}$:

$$G_{\mu\vartheta} = R_{\mu\vartheta} - \frac{1}{2} g_{\mu\vartheta} R \qquad \ldots \ldots \ldots (1.43)$$

With (1.36) this gives the Einstein Field equations

$$R_{\mu\vartheta} - \frac{1}{2} g_{\mu\vartheta} R = -8\pi G T_{\mu\vartheta} \quad \ldots \ldots \ldots (1.44)$$

An alternative form is sometimes useful. Contracting (1.44) with $g^{\mu\upsilon}$ gives,

$$R - 2R = -8\pi G T_\mu^\mu$$

$$\Rightarrow R = 8\pi G T_\mu^\mu \qquad \ldots \ldots \ldots (1.45)$$

And using this in equation (1.44), we have

$$R_{\mu\vartheta} = -8\pi G (T_{\mu\vartheta} - \frac{1}{2} g_{\mu\vartheta} T_\lambda^\lambda) \quad \ldots \ldots \ldots (1.46)$$

Of course we can also go from (1.46) back (1.45) and (1.44), so (1.44) and (1.46) should be regarded as entirely equivalent forms of the Einstein Field equations.

In a vacuum T_μ vanishes, so form (1.46) we see that the Einstein field equations in empty space are just

$$R_{\mu\vartheta} = 0 \quad \ldots\ldots\ldots (1.47)$$

In a space-time of two or three dimensions this would imply the vanishing of the full curvature tensor $R_{\lambda\mu\upsilon\kappa}$ and the coonsequent absence of a gravitational field. It is only in four or more dimensions that true gravitational fields can exist in empty space.

We might be wiling to relaz assumption (B), and allow to contain terms with fewer than derivatives of the metric. The freedom to use first derivatives does not allow any new term is possible,equal to $g_{\mu\upsilon}$ times a constant λ. The field equations would then read

$$R_{\mu\vartheta} - \frac{1}{2} g_{\mu\vartheta} R - \Lambda g_{\mu\vartheta} = -8\pi G T_{\mu\vartheta}$$

The term $\Lambda g_{\mu\vartheta}$ was originally introduced by Einstein for cosmological reasons (which have since disappeared); for this reason, Λ is called the cosmological constant. This term satisfies the requirements (A), (C) and (D), but does not satisfy (E), so Λ must be very small so as not to interfere with the successes of Newton's theory of gravitation.

1.8 The General Static Isotropic Metric:

For the moment we put aside Einstein's equations and what is the most general metric tensor that can represent a static isotropic gravitational field. By "static & isotropic" we mean that it must be possible to find a set of "quasi-Minkowskian" coordinates x^1, x^2, x^3, $x^0 \equiv t$, such that the invariant proper time $d\tau^2 \equiv g_{\mu\vartheta} dx^{\mu} dx\vartheta$ does not depend on *t*, and depends on *x* and *dx* only through the rotational invariants dx^2, $x.dx$,& x^2 . The most general proper time interval is then **[63],**

$$d\tau^2 = F(r)dt^2 - 2E(r)dtx.dx - D(r)(x.dx)^2 - C(r)dx^2 \quad \ldots\ldots\ldots (1.48)$$

Where *F,E,D* and *C* are unknown functions of $r \equiv (x.x)^{\frac{1}{2}}$

(e.g. , $x.dx = x^1dx^1 + x^2dx^2 + x^3dx^3$, etc)

Ti is convenient to replace *x* with spherical polar coordinates *r,θ,φ* defined as usual by

$$x^2 = rSin\theta Cos\varphi \quad , \quad x^2 = rSin\theta Sin\varphi \quad , \quad x^3 = rCos\theta$$

The proper time interval (1.48) then becomes,

$$d\tau^2 = F(r)dt^2 - 2rE(r)dtdr - r^2D(r)dr^2 - C(r)(dr^2 + r^2d\theta^2 + r^2Sin^2\theta d\varphi^2)$$
… … … (1.49)

We are free to reset our clocks by defining a new time coordinate,

$$t' \equiv t + \Phi(r)$$

Where Φ an arbitrary function of *r*. This allows us to eliminate the off-diagonal element g_{tr} by setting $\frac{d\Phi}{dr} = -\frac{rE(r)}{F(r)}$

The proper time (1.49) then becomes

$$d\tau^2 = F(r)dt'^2 - G(r)dr^2 - C(r)(dr^2 + r^2d\theta^2 + r^2Sin^2\theta d\varphi^2) \quad \dots \dots\dots(1.50)$$

Where $$G(r) \equiv r^2\left(D(r) + \frac{E^2(r)}{F(r)}\right)$$

We are also free to redefine the radius *r*, and thereby impose one further relation on the functions *F,G* and *C* . For instance, suppose that we define

$$r'^2 \equiv C(r)r^2$$

Then the proper time (1.50) takes what is called the standard form

$$d\tau^2 = B(r')dt'^2 - A(r')dr'^2 - r'^2(d\theta^2 + Sin^2\theta d\varphi^2) \quad \dots \dots \dots (1.51)$$

Where $B(r') \equiv F(r)$

$$A(r') \equiv \left(1+\frac{G(r)}{C(r)}\right)\left(1+\frac{r}{2C(r)}\frac{dC(r)}{dr}\right)^{-2}$$

Alternatively, we could define

$$r'' = \exp \int \left(1+\frac{G(r)}{C(r)}\right)^{\frac{1}{2}} \frac{dr}{r}$$

And (1.50) would than appear in what is called the isotropic form,

$$d\tau^2 = H(r'')dt'^2 - J(r'')(dr''^2 + r''^2 + r''^2 Sin^2\theta d\varphi^2) \quad \dots \dots \dots (1.52)$$

Where $H(r'') \equiv F(r)$ $\qquad J(r'') \equiv \frac{C(r)r^2}{r''^2}$

We shall do most of our work with a metric of the "standard" form:

$$d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + Sin^2\theta d\varphi^2) \quad \dots \dots \dots (1.53)$$

(We drop primes on *r* and *t* from now on.) The metric tensor has the non-vanishing components

$$g_{rr} = A(r) \;,\;\; g_{\theta\theta} = r^2 \;,\;\; g_{\varphi\varphi} = r^2 Sin^2\theta \;,\;\; g_{tt} = -B(r) \quad \dots \dots \dots (1.54)$$

With functions *A(r)* and *B(r)* that are to be determined by solving the field equations. Since $g_{\mu\upsilon}$ is diagonal, it is easy to write down all the non-vanishing components of its inverse:

$$g^{rr} = A^{-1}(r), \quad g^{\theta\theta} = r^{-2}, \quad g^{\varphi\varphi} = r^{-2}(Sin\theta)^2 \;,\; g^{tt} = -B^{-1}(r) \quad \dots \dots \dots (1.55)$$

Furthermore, the determinant of the metric tensor is *–g,* where

$$g = r^4 A(r)\, B(r)\, Sin^2\theta \quad \dots \dots \dots (1.56)$$

so the invariant volume element is

$$\sqrt{g}\,drd\theta d\varphi = r^2\sqrt{A(r)B(r)}Sin\theta drd\theta d\varphi \quad \ldots\ldots\ldots (1.57)$$

The affine connection can be computed from the usual formula:

$$\Gamma^{\lambda}_{\mu\vartheta} = \frac{1}{2}g^{\lambda\rho}\left(\frac{\delta g_{\rho\mu}}{\delta x^{\vartheta}} + \frac{\delta g_{\rho\vartheta}}{\delta x^{\mu}} - \frac{\delta g_{\mu\vartheta}}{\delta x^{\rho}}\right)$$

Its only non -vanishing components are-

$$\Gamma^{r}_{rr} = \frac{1}{2A(r)}\frac{dA(r)}{dr} \qquad \Gamma^{r}_{\theta\theta} = -\frac{r}{A(r)}$$

$$\Gamma^{r}_{\varphi\varphi} = \frac{-rSin^2\theta}{A(r)} \qquad \Gamma^{r}_{tt} = \frac{1}{2A(r)}\frac{dB(r)}{dr}$$

$$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r} \qquad \Gamma^{\theta}_{\varphi\varphi} = -Sin\theta Cos\theta$$

$$\Gamma^{\varphi}_{\varphi r} = \Gamma^{\varphi}_{\varphi r} = \frac{1}{r} \qquad \Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} = Cot\theta$$

$$\Gamma^{t}_{tr} = \Gamma^{t}_{rt} = \frac{1}{2B(r)}\frac{dB(r)}{dr} \quad \ldots\ldots\ldots (1.58)$$

We also need the Ricci tensor. It is given by

$$R_{\mu K} = \frac{\delta\Gamma^{\lambda}_{\mu\lambda}}{\delta x^{K}} - \frac{\delta\Gamma^{\lambda}_{\mu K}}{\delta x^{\lambda}} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{K\eta} - \Gamma^{\eta}_{\mu K}\Gamma^{\lambda}_{\lambda\eta} \quad \ldots\ldots\ldots (1.59)$$

Inserting in (1.60) the components of the affine connection given by (1.58) we find,

$$
\left.
\begin{aligned}
R_{rr} &= \frac{B''(r)}{2B(r)} - \frac{1}{4}\left(\frac{B'(r)}{B(r)}\right)\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{A'(r)}{A(r)}\right) \\
R_{\theta\theta} &= -1 + \frac{r}{2A(r)}\left(-\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) + \frac{1}{A(r)} \\
R_{\varphi\varphi} &= Sin^2\theta R_{\theta\theta} \\
R_{tt} &= -\frac{B''(r)}{2A(r)} + \frac{1}{4}\left(\frac{B'(r)}{B(r)}\right)\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{B'(r)}{A(r)}\right) \\
R_{\mu\vartheta} &= 0 \quad \text{for} \quad \mu \neq \vartheta
\end{aligned}
\right\} \quad \ldots\ldots\ldots (1.60)
$$

(A prime now means differentiation with respect to r.) The result that $R_{r\theta}$, $R_{r\varphi}$, $R_{t\theta}$, $R_{t\varphi}$ & $R_{\theta\varphi}$ vanish, and that $R_{\varphi\varphi} = Sin^2\theta R_{\theta\theta}$, are merely consequences of the rotational invariance of the metric, whereas the result that R_{rt} vanishes is because we have set our clocks so that the metric is invariant under the time-reversal transformation $t \to -t$.

Neither the standard nor the isotropic coordinates are harmonic, but we can easily use the results (1.54) and (1.58) for the metric and affine connection in standard coordinates to construct harmonic coordinates X_1, X_2 ,X_3 ,t. We set

$$X_1 = R(r)Sin\theta Cos\varphi\,;\; X_2 = R(r)Sin\theta Sin\varphi\,;\; X_3 = R(r)Cos\theta \quad \ldots\ldots\ldots (1.61)$$

A straightforward calculation gives then

$$\Box^2 X_i \equiv g^{\mu\vartheta}\left[\frac{\delta^2 X_i}{\delta x^\mu \delta x^\vartheta} - \Gamma^\lambda_{\mu\vartheta}\frac{\delta X_i}{\delta x^\lambda}\right]$$

$$= \left(\frac{X_i}{AR}\right)\left[\left(\frac{B'}{2B} + \frac{2}{r} - \frac{A'}{2A}\right)R' + R'' - \frac{2A}{r^2}R\right]$$

Where $\Box^2$ is the d'Alembertian operator.

Also, the standard time coordinate t satisfies

$$\Box^2 t = 0$$

Thus the coordinate X_1, X_2, X_3 t are harmonic if R(r) satisfies the differential equation

$$\frac{d}{dr}\left(r^2 B^{\frac{1}{2}} A^{\frac{-1}{2}} \frac{dR}{dr}\right) - 2B^{\frac{1}{2}} A^{\frac{1}{2}} R = 0 \quad \dots \dots \dots (1.62)$$

In these harmonic coordinates the proper time (1.53) becomes

$$d\tau^2 = Bdt^2 - \frac{r^2}{R^2} dX^2 - \left(\frac{A}{R^2 R'^2} - \frac{r^2}{R^4}\right)(X.dX)^2 \quad \dots \dots \dots (1.63)$$

1.9 The Schwarzschild Solution:

We now apply the Einstein's field equation to the general static isotropic metric. We use the standard form, that is **[62]**,

$$d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2 Sin^2\theta d\varphi^2 \quad \dots \dots \dots (1.64)$$

The field equations for empty space are

$$R_{\mu\vartheta} = 0 \quad \dots \dots \dots (1.65)$$

The components of the Ricci tensor are given for this metric by equation (1.60). We see that will suffice to set R_{rr} , $R_{\theta\theta}$ and R_{tt} equal to zero. We also see that

$$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = -\frac{1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right) \quad \dots \dots \dots (1.66)$$

So (1.65) requires that $\frac{B'}{B} = -\frac{A'}{A}$, or

$$A(r)\,B(r) = Constant \quad \dots\dots\dots (1.67)$$

Furthermore we impose on A and B the boundary condition that for $r \to \infty$ the metric tensor must approach the Minkowski tensor in spherical coordinate, that is,

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1 \quad \dots\dots\dots \ (1.68)$$

From (1.67) & (1.68) we have then

$$A(r) = \frac{1}{B(r)} \quad \dots\dots\dots (1.69)$$

Since (1.66) now vanishes, it remains to make R_{rr} and $R_{\theta\theta}$ vanish. Using (1.69) in (1.60) we find

$$R_{\theta\theta} = -1 + rB'(r) + B(r)$$

$$R_{rr} = \frac{B''(r)}{2B(r)} + \frac{B'(r)}{rB(r)} = \frac{R'_{\theta\theta}(r)}{2rB(r)}$$

So it is sufficient to set $R_{\theta\theta}$ equal to zero, that is,

$$\frac{d}{dr}\left(rB(r)\right) = rB'(r) + B(r) = 1$$

The solution is

$$rB(r) = r + Constant \quad \dots\dots\dots (1.72)$$

To fix the constant of integration we recall great distances from a central mass M, the component $g_{tt} = -B$ must approach $-1-2\varphi$, where φ is the Newtonian potential $-MG/r$. Hence the constant of integration is $-2MG$, and our final solution is,

$$B(r) = \left(1 - \frac{2MG}{r}\right) \quad \ldots \ldots \ldots (1.73)$$

$$A(r) = \left(1 - \frac{2MG}{r}\right)^{-1} \quad \ldots \ldots \ldots (1.74)$$

The full metric is given by

$$d\tau^2 = \left(1 - \frac{2MG}{r}\right)dt^2 - \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 Sin^2\theta d\varphi^2 \quad \ldots \ldots (1.75)$$

This solution was found by K. Schwarzschild in 1916.

The Schwarzschild solution is expressed in equation (1.75) in its "standard" form. We can also express it in the equivalent "isotropic" form, by introducing a new radius variable

$$\rho \equiv \frac{1}{2}\left[r - MG + \left(r^2 - 2MGr\right)^{\frac{1}{2}}\right]$$

$$r = \rho\left(1 + \frac{MG}{2\rho}\right)^2 \quad \ldots \ldots \ldots (1.76)$$

Substituting this in equation (1.75) gives

$$d\tau^2 = \frac{\left(1 - \frac{MG}{2\rho}\right)^2}{\left(1 + \frac{MG}{2\rho}\right)^2} dt^2 - \left(1 + \frac{MG}{2\rho}\right)^4 (d\rho^2 - \rho^2 d\theta^2 - \rho^2 Sin^2\theta d\varphi^2)$$

CHAPTER -2

ON THE COSMOLOGICAL MODELS

2.1 Introduction:

No branch of science can claim to have a bigger area of interest than cosmology because of its profound implication of poets, philosophers, religious thinkers .Observational cosmology is concerned with the physical properties of the universe such as its chemical composition, density and rate of expansion as well as the distribution of galaxies and clusters of galaxies .Physical cosmology tries to understand these properties by applying known law of physics astrophysics. Theoretical cosmology involves making models that gives a mathematical description of the observed properties of the universe based on physical understanding. Mathematics heavily use to find cosmological model from Einstein equation or other theories of gravity cosmology also has philosophical or even theological aspect in that it seeks to understand why the universe has its observed properties . Active areas of research in cosmology include the large scale galaxy surveys to map the distribution of matter or cosmological scales according to the ‘redshift’ distance relations the investigation of fluctuation in the temperative of the ‘Cosmic Background Radiation’ and their implication**[43].**

2.2 Cosmology:

Cosmology from Greak Komos and logia means universe and study respectively. Cosmology is the study of the large scale structure and behavior of the universe. That is of the universe taken as a whole.The term as a whole applied to universe need to precise definition.

2.3 Cosmological Model:

A Cosmological Model is a model of our universe which taking into account and using all known physically laws predicts correctly the observed properties of the universe and in particular explain in details the phenomena in the early universe .Such as model must also explain inter alia why the universe was so homogenous and isotropic at the epoch of last scattering of the cosmic microwave background and how and when homogeneities (galaxies and stars) arose .It should also explain ,whether the spatial properties of our universe compared with other conceivable universe depend on particular initial condition or whether the laws of nature ensure that a stable and quasi permanent universe .Similar to our own must have occurred independently of the initial conditions.

In a more restricted sense cosmological model are exact solution of the Einsteinian field equations for a perfect fluid that reproduce the important feature of our universe.

There are various cosmological models some of these are mentioned below:

i) Einstein static model of universe **[11]** is

$$ds^2 = dt^2 - R^2\left[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + Sin^2\theta d\varphi^2)\right]$$

ii) Friedmann-Lemaitre, Robertson, Walker (FRW) model **[16,60]** or big-bang model or standard model of the universe is

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + Sin^2\theta d\varphi^2)\right]$$

iii) de-Sitter universe**[9]** or Steady State theory or non-standard or beyond the standard or alternative to the big-bang model is

$$ds^2 = dt^2 - R^2 e^{2Ht}[dr^2 + r^2(d\theta^2 + Sin^2\theta d\varphi^2)]$$

2.4 Modern Cosmology:

Einstein static model of the universe was one of the great miss opportunities of theoretical physics .If had stuck to his original version of general relativity without the cosmological constant he could have predicted that the universe ought to be either expanding or contracting which means collapsing .As it happened however , it was not realized that the universe was changing with time until astronomers like *Slipher* and *Hubble* began to observe the light from other galaxies visible light is made up of waves ,like radio waves only with a much shorter wave length . If on pass the light through a prism it is decomposed into its constituent wave length or color like rainbow .*Slipher* and *Hubble* found the same characteristic pattern of wave length or colors as for the light from stars in our galaxy but the patterns well all shifted towards the red or longer wave length end of the spectrum and not surprisingly different *Friedmann* models enjoyed period of fashion that is period when they were considered the least available model for our own universe. However there was one school of thought as a simple non-Friedmann model called the steady-state solution based on the perfect cosmological principal that is universe is unchanging in space and time .Most of these consideration are largely historical in nature for although the Friedmann models are still basic to much of cosmological thinking the more recent decades , our Friedmann's model has emerged as the best available , at least as far as the origin of the universe are concerned and that is the hot big-bang model .In this model it is assume that there occurred a cataclysmic event called the big-bang .Where the universe sprang into existence and expanded away from a singular point .In the earlier phases the universe consisted of radiation at incredibly high temperature and densities to the universe expanded ,latter on the temperature and density fall and protons ,electrons and neutrons emerged from the radiation both .Further the simple atoms such as hydrogen ,helium emerged first which followed later to the heavier elements .This phase can be treated mathematically and one of the general success of this abundances of the heavy element with observed abundances .

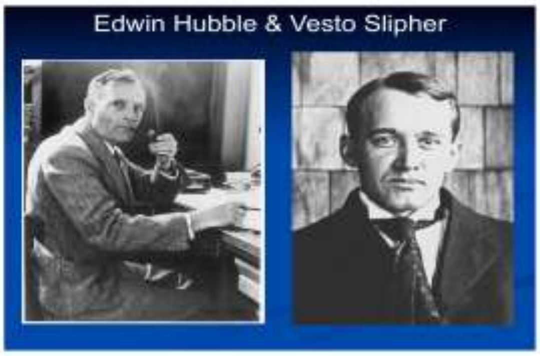


Figure: *Edwin Hubble & Vesto Slipher*

2.5 Standard model of cosmology:

Recent years have witnessed enormous advance in the quantitative understanding of cosmology and the establishment of a Standard Cosmological Model. In addition to known forms of matter, Einstein's General Relativity and an ansatz for the space-time metric (spatially homogeneous and isotropic), two mysterious elements need to be added in order to account for the observations. These give the name to the standard, ΛCDM model: Cold Dark Matter (CDM) to explain the formation and dynamics of cosmic structures and a Cosmological Constant (Λ) to account for the dimming of distant supernovae.

The standard model of cosmology is based on the general theory of relativity .Einstein's discovery of general relativity enabled us to develop a theory of the universe which is testable and can be falsified. So cosmology has become a proper science which can predict events and explain observations. The Big Bang model of the universe which is based on general relativity and is in fact the standard model of the universe at present, has successfully passed several important tests include the expansion of the universe.. The standard cosmological model also needs to account for the origins of in homogeneities such as galaxies, stars and planets. In the early 1980's the inflationary model was and subsequently shown to be able to successfully seed galaxy formation .Now this model is being put to several tests by CMB experiments such as COBE, WMAP and (soon) PLANCK.

The steadily increasing precision and wealth of data and the surprising findings call for a revision of the hypothesis made in the construction of the Standard Cosmological Model.

As a logical construction, the standard cosmological model requires six hypotheses

Standard Model = GR + FRW+ Initial Conditions + SM + CDM+ Λ

1.*General Relativity:* As gravity is always an attractive interaction, it dominates on macroscopic distances and will be crucial for cosmological phenomena.

2.***Friedman-Robertson-Walker metric*:** GR is a metric theory governed by nonlinear partial deferential equations. A simple ansatz for the metric is required, which in the Standard case is based on maximal spatial symmetry.

3.***(Inflationary) Initial Conditions*:** The paradigm of cosmic inflation is able to provide initial conditions for the perturbations around the background metric, explains the observed value of the spatial curvature and further supports the choice of the metric.

4.***Standard Matter***: The known forms of matter and their cosmological effects have to be accounted for, notoriously baryons (nucleons and electrons), photons and neutrinos.

5.***Cold Dark Matter***: Structure formation requires the presence of a form of matter does not interact with light, usually assumed to be a new, weakly interacting particle species.

6.*Cosmological Constant*: The observed acceleration of the universe can be explained in a simple way by the presence of an energy density that does not evolve in time.

2.6 Derivation of Robertson-Walker metric :

In term of Cartesian co-ordinate x_1, x_2, x_3, x_4 a 3-surface of constant negative curvature is given by an equation

$$\mathbf{x}_1^2 + \mathbf{x}_2^2 + \mathbf{x}_3^2 - \mathbf{x}_4^2 = -\mathbf{R}^2 \quad \ldots\ldots\ldots (2.1)$$

Where R is a constant .The contrary of universe is necessary in order to ensure that the properties of homogeneity and isotropy.

Let us consider the co-ordinates with their transformation

$$\mathbf{x}_1 = \mathbf{R\ Sinh\chi\ Sin\theta\ Cos\varphi}$$

$$\mathbf{x}_2 = \mathbf{R\ Sinh\chi\ Sin\theta\ Sin\varphi}$$

$$\mathbf{x}_3 = \mathbf{R\ Sinh\chi\ Cos\theta}$$

$$\mathbf{x}_4 = \mathbf{R\ Cosh\chi}$$

Therefore we get the spatial line element on the 3-surface is given by,

$$\mathbf{d\sigma}^2 = \mathbf{dx}_1^2 + \mathbf{dx}_2^2 + \mathbf{dx}_3^2 - \mathbf{dx}_4^2 \quad \ldots\ldots\ldots (2.2)$$

But,

$dx_1 = R\ (\ Cosh\chi\ Sin\theta\ Cos\varphi\ d\chi + Sinh\chi\ Cos\theta\ Cos\varphi\ d\theta - Sinh\chi\ Sin\theta\ Sin\varphi\ d\varphi\ )$

$dx_2 = R\ (\ Cosh\chi\ Sin\theta\ Sin\varphi\ d\chi + Sinh\chi\ Cos\theta\ Sin\varphi\ d\theta + Sinh\chi\ Sin\theta\ Cos\varphi\ d\varphi\ )$

$dx_3 = R\ (\ Cosh\chi\ Cos\ \theta\ d\chi - Sinh\chi\ Sin\theta\ d\theta\ )$

$dx_4 = R\ Sinh\chi\ d\chi$

Squaring we get,

$\mathbf{dx_1^2 = R^2\ (Cosh\chi\ Sin\theta\ Cos\varphi\ d\chi + Sinh\chi\ Cos\theta\ Cos\varphi\ d\theta - Sinh\chi\ Sin\theta\ Sin\varphi\ d\varphi\ )^2}$

$=R^2\ (Cosh^2\chi Sin^2\theta Cos^2\varphi d\chi^2 + Sinh^2\chi Cos^2\theta Cos^2\varphi d\theta^2 + Sinh^2\chi Sin^2\theta Sin^2\varphi d\varphi^2 + 2Cosh\chi\ Sin\theta\ Sinh\chi\ Cos\theta Cos^2\varphi d\chi\ d\theta - 2Cosh\chi\ Sin^2\theta\ Cos\varphi\ Sinh\chi Sin\varphi\ d\chi d\varphi - 2\ Sinh^2\chi\ Cos\theta\ Cos\varphi\ Sin\theta\ Sin\varphi\ d\theta d\varphi)$

$\mathbf{dx_2^2 = R^2\ (Cosh\chi\ Sin\theta\ Sin\varphi\ d\chi + Sinh\chi\ Cos\theta\ Sin\varphi\ d\theta + Sinh\chi\ Sin\theta\ Cos\varphi\ d\varphi\ )^2}$

$= R^2 (\text{Cosh}^2 \chi \text{Sin}^2 \theta \text{Sin}^2 \varphi d\chi^2 + \text{Sinh}^2 \chi \text{Cos}^2 \theta \text{Sin}^2 \varphi d\theta^2 + \text{Sinh}^2 \chi \text{Sin}^2 \theta \text{Cos}^2 \varphi d\varphi^2 + 2\text{Cosh}\chi\ \text{Sin}\theta\ \text{Sinh}\chi\ \text{Cos}\theta \text{Sin}^2 \varphi d\chi\ d\theta + 2\text{Cosh}\chi\ \text{Sin}^2 \theta\ \text{Cos}\varphi\ \text{Sinh}\chi \text{Cos}\varphi\ d\chi d\varphi + 2\ \text{Sinh}^2 \chi\ \text{Cos}\theta\ \text{Sin}\varphi\ \text{Sin}\theta\ \text{Cos}\varphi\ d\theta d\varphi)$

$\mathbf{dx_3^2 = R^2 (Cosh\chi\ Cos\ \theta\ d\chi - Sinh\chi\ Sin\theta\ d\theta)^2}$

$= R^2 (\text{Cosh}^2 \chi \text{Cos}^2 \theta d\chi^2 + \text{Sinh}^2 \chi\ \text{Sin}^2 \theta d\theta^2 - 2\text{Cosh}\chi \text{Cos}\theta \text{Sinh}\chi\ \text{Sin}\theta\ d\chi\ d\theta)$

$\mathbf{dx_4^2 = (R\ Sinh\chi\ d\chi)^2}$

$= R^2\ \text{Sinh}^2 \chi d\chi^2$

Putting these values in equation (2.1) we get,

$$d\sigma^2 = \mathbf{dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2}$$

$= R^2 [\text{Cosh}^2 \chi\ \text{Sin}^2 \theta\ \text{Cos}^2 \varphi d\chi^2 + \text{Sinh}^2 \chi\ \text{Cos}^2 \theta\ \text{Cos}^2 \varphi\ d\theta^2 + \text{Sinh}^2 \chi\ \text{Sin}^2 \theta\ \text{Sin}^2 \varphi\ d\varphi^2 + 2\text{Cosh}\chi\ \text{Sin}\theta\ \text{Sinh}\chi\ \text{Cos}\theta\ \text{Cos}^2 \varphi d\chi d\theta - 2\text{Cosh}\chi\ \text{Sin}^2 \theta\ \text{Cos}\varphi\ \text{Sinh}\chi\ \text{Sin}\varphi\ d\chi d\varphi - 2\text{Sinh}^2 \chi\ \text{Cos}\theta\ \text{Cos}\varphi\ \text{Sin}\theta\ \text{Sin}\varphi\ d\theta\ d\varphi) + (\text{Cosh}^2 \chi\ \text{Sin}^2 \theta\ \text{Sin}^2 \varphi\ d\chi^2 + \text{Sinh}^2 \chi\ \text{Cos}^2 \theta\ \text{Sin}^2 \varphi\ d\theta^2 + \text{Sinh}^2 \chi\ \text{Sin}^2 \theta\ \text{Cos}^2 \varphi\ d\varphi^2 + 2\text{Cosh}\chi\ \text{Sin}\theta\ \text{Sinh}\chi\ \text{Cos}\theta\ \text{Sin}^2 \varphi d\chi\ d\theta + 2\text{Cosh}\chi\ \text{Sin}^2 \theta\ \text{Cos}\varphi\ \text{Sinh}\chi\ \text{Cos}\varphi\ d\chi\ d\varphi + 2\text{Sinh}^2 \chi\ \text{Cos}\theta\ \text{Sin}\varphi\ \text{Sin}\theta\ \text{Cos}\varphi\ d\theta\ d\varphi) + (\text{Cosh}^2 \chi\ \text{Cos}^2 \theta\ d\chi^2 + \text{Sinh}^2 \chi\ \text{Sin}^2 \theta\ d\theta^2 - 2\text{Cosh}\chi\ \text{Cos}\theta\ \text{Sinh}\chi\ \text{Sin}\theta\ d\chi\ d\theta) - \text{Sinh}^2 \chi\ d\chi^2]$

$= R^2 [(\text{Cosh}^2 \chi\ \text{Sin}^2 \theta\ \text{Cos}^2 \varphi + \text{Cosh}^2 \chi\ \text{Sin}^2 \theta\ \text{Sin}^2 \varphi + \text{Cosh}^2 \chi \text{Cos}^2 \theta - \text{Sinh}^2 \chi) d\chi^2 + (\text{Sinh}^2 \chi\ \text{Cos}^2 \theta\ \text{Cos}^2 \varphi + \text{Sinh}^2 \chi\ \text{Cos}^2 \theta\ \text{Sin}^2 \varphi + \text{Sinh}^2 \chi\ \text{Sin}^2 \theta) d\theta^2 + (\text{Sinh}^2 \chi\ \text{Sin}^2 \theta\ \text{Sin}^2 \varphi + \text{Sinh}^2 \chi\ \text{Sin}^2 \theta\ \text{Cos}^2 \varphi) d\varphi^2]$

$$=R^2 [\, d\chi^2 + \text{Sinh}^2\chi\, d\theta^2 + \text{Sinh}^2\chi\, \text{Sin}^2\theta\, d\varphi^2]$$

$$=R^2 [\, d\chi^2 + \text{Sinh}^2\chi\, (d\theta^2 + \text{Sin}^2\theta\, d\varphi^2)]$$

Let us consider,

$$r = \text{Sinh}\chi$$

$$\text{Or, } dr = \text{Cosh}\chi\, d\chi$$

$$\text{Or, } dr = \frac{dr}{\sqrt{1+Sinh^2\chi}}$$

$$\text{Or, } d\chi = \frac{dr}{\sqrt{1+r^2}}$$

So we get ,

$$d\sigma^2 = R^2 [\frac{dr^2}{1+r^2} + r^2 (\, d\theta^2 + \text{Sin}^2\theta\, d\varphi^2)\,] \quad \dots \dots \dots (2.3)$$

We know that the Einstein-Static model of the universe as follows,

$$d\sigma^2 = R^2 [\frac{dr^2}{1-r^2} + r^2 (\, d\theta^2 + \text{Sin}^2\theta\, d\varphi^2)\,] \quad \dots \dots \dots (2.4)$$

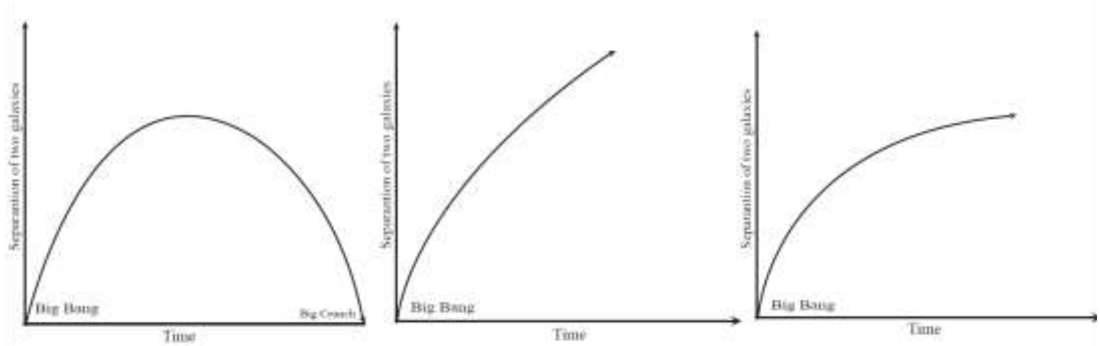
Expression (2.3) & (2.4) can be combined in a single expression by including a parameter K in general form that takes the value ± 1 ,

$$d\sigma^2 = R^2 [\frac{dr^2}{1-kr^2} + r^2 (\, d\theta^2 + \text{Sin}^2\theta\, d\varphi^2)\,] \quad \dots \dots \dots (2.5)$$

Where 3-space t = *constant* are Euclidean for k=0, closed with +ve curvature for k=1 & open with negative curvature for k=-1. Also the scale factor R (t) is often called the expansion factor, sometime for Einstein static is treated as the radius of the universe.

Geometrical Configuration :

The geometrical configuration of the universe involve R-W Model are as-



(i) **(ii)** **(iii)**

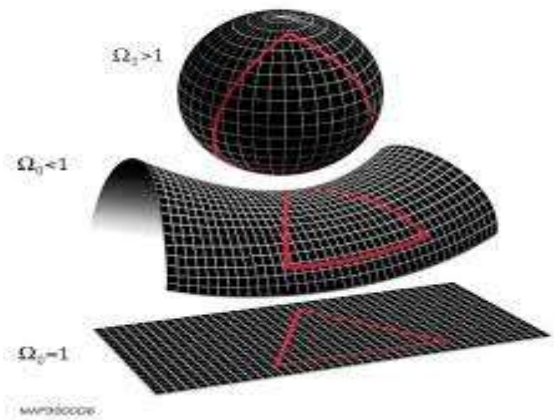


Figure: Scale-Factor R over Time t

Figure (i) *shows how the distance between two neighboring galaxies changes as time increases. It starts at zero, increases to a maximum, and then decreases to zero again. In the second kind of solution, the universe is expanding so rapidly that the gravitational attraction can never stop it, though it does slow it down a bit.*

Figure (ii): *Shows the Separation between neighboring galaxies in this model. It starts at zero and eventually the galaxies are moving apart at a steady speed. Finally, there is a third kind of solution, in which the universe is expanding only just fast enough to avoid recollapse.*

In this case the separation, shown in ***Figure (iii),*** *also starts at zero and increases forever. However, the speed at which the galaxies are moving apart gets smaller and smaller, although it never quite reaches zero.*

(i) For K=1 the space is called with positive curvature *ie*. The universe is finite or the recolpsing universe.

(ii) For K=0 the universe is flat.

(iii) For K=-1 the space is open with negative curvature the universe is infinite *ie*. The universe is eternal it will never collapse.

2.7 Non-vanishing affine connection :

The R-W metric can be written as ,

$$\mathbf{ds}^2 = \mathbf{dt}^2 - \mathbf{R}^2\,\mathbf{(t)}[\frac{dr^2}{1-kr^2} + r^2(\,\mathbf{d\theta}^2 + \mathbf{Sin}^2\,\theta\,\mathbf{d\varphi}^2)\,] \quad \ldots\ldots\ldots (2.6)$$

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$$

$$g_{00} = 1 \;,\; g_{11} = \frac{-R^2}{1-kr^2} \;,\; g_{22} = -r^2 R^2 \;,\; g_{33} = -r^2 R^2 \, Sin^2\theta$$

$$g^{00} = 1 \;,\; g^{11} = \frac{-(1-kr^2)}{R^2} \;,\; g^{22} = \frac{-1}{r^2R^2} \;,\; g^{33} = \frac{-1}{r^2R^2Sin^2\theta}$$

The Christoffel symbol can be calculated from the equation,

$$\Gamma^\mu_{\lambda\vartheta} = \frac{1}{2} g^{\mu\sigma}(\frac{\delta g_{\sigma\vartheta}}{\delta x^\lambda} + \frac{\delta g_{\sigma\lambda}}{\delta x^\vartheta} - \frac{\delta g_{\lambda\vartheta}}{\delta x^\sigma}) \quad \ldots\ldots\ldots (2.7)$$

$$= \frac{1}{2} g^{\mu\sigma}(g_{\sigma\vartheta,\lambda} + g_{\sigma\lambda,\vartheta} - g_{\vartheta\lambda,\sigma})$$

$$\Gamma^0_{11} = \frac{1}{2} g^{00}(\frac{\delta g_{10}}{\delta x^1} + \frac{\delta g_{01}}{\delta x^1} - \frac{\delta g_{11}}{\delta x^0})$$

$$= \frac{1}{2} g^{00} \frac{\delta}{\delta t}\{\frac{-R^2(t)}{1-kr^2}\}$$

$$= \frac{1}{2} . 1 . \frac{2\dot{R}R}{1-kr^2}$$

$$\therefore \Gamma^0_{11} = \frac{\dot{R}R}{1-kr^2}$$

$$\Gamma^0_{22} = \frac{1}{2} g^{00}(\frac{\delta g_{20}}{\delta x^2} + \frac{\delta g_{02}}{\delta x^2} - \frac{\delta g_{22}}{\delta x^0})$$

$$= -\frac{1}{2} g^{00} \frac{\delta}{\delta t}(-R^2 r^2)$$

$$= \frac{1}{2} . 1 . 2R\dot{R}r^2$$

$$\therefore \Gamma^0_{22} = R\dot{R}r^2$$

$$\Gamma^0_{33} = \frac{1}{2} g^{00}(\frac{\delta g_{30}}{\delta x^3} + \frac{\delta g_{03}}{\delta x^3} - \frac{\delta g_{33}}{\delta x^0})$$

$$= -\frac{1}{2} . 1 . \frac{\delta}{\delta t}(-R^2 r^2 Sin^2\theta)$$

$$= \frac{1}{2} . 2 . R\dot{R}r^2 Sin^2\theta$$

$$\therefore \Gamma^0_{33} = R\dot{R}r^2 Sin^2\theta$$

$$\Gamma^1_{01} = \frac{1}{2} g^{11}(\frac{\delta g_{11}}{\delta x^0} + \frac{\delta g_{10}}{\delta x^1} - \frac{\delta g_{01}}{\delta x^1})$$

$$= \frac{1}{2} \{\frac{-(1-kr^2)}{R^2}\} \frac{\delta}{\delta t} (\frac{-R^2}{1-kr^2})$$

$$= \frac{(1-kr^2)}{2R^2} . \frac{2R\dot{R}}{1-kr^2}$$

$$\therefore \Gamma^1_{01} = \frac{\dot{R}}{R} = \Gamma^0_{10}$$

$$\Gamma^2_{02} = \frac{1}{2} g^{22} (\frac{\delta g_{22}}{\delta x^0} + \frac{\delta g_{20}}{\delta x^2} - \frac{\delta g_{02}}{\delta x^2})$$

$$= \frac{1}{2} (\frac{-1}{R^2 r^2}) \frac{\delta}{\delta t} (-R^2 r^2)$$

$$= \frac{1}{2} . \frac{1}{R^2 r^2} . 2R\dot{R} r^2$$

$$= \frac{\dot{R}}{R}$$

$$\therefore \Gamma^2_{02} = \frac{\dot{R}}{R} = \Gamma^2_{20}$$

$$\Gamma^3_{03} = \frac{1}{2} g^{33} (\frac{\delta g_{33}}{\delta x^0} + \frac{\delta g_{30}}{\delta x^3} - \frac{\delta g_{03}}{\delta x^3})$$

$$= \frac{1}{2} (\frac{-1}{r^2 R^2 Sin^2\theta}) \frac{\delta}{\delta t} (-R^2 r^2 Sin^2\theta)$$

$$= \frac{1}{2r^2 R^2 Sin^2\theta} . 2R^2 r^2 Sin^2\theta$$

$$\therefore \Gamma^3_{03} = \frac{\dot{R}}{R} = \Gamma^3_{30}$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11}\left(\frac{\delta g_{11}}{\delta x^1} + \frac{\delta g_{11}}{\delta x^1} - \frac{\delta g_{11}}{\delta x^1}\right)$$

$$= \frac{1}{2} \cdot \left\{\frac{-(1-kr^2)}{R^2}\right\} \frac{\delta}{\delta r}\left(\frac{-R^2}{1-kr^2}\right)$$

$$= \frac{1}{2} \; \frac{(1-kr^2)}{R^2} \cdot \frac{R^2}{(1-kr^2)} .(-1) .(-2kr)$$

$$\therefore \Gamma^1_{11} = \frac{kr}{(1-kr^2)}$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}\left(\frac{\delta g_{21}}{\delta x^2} + \frac{\delta g_{12}}{\delta x^2} - \frac{\delta g_{22}}{\delta x^1}\right)$$

$$= \frac{1}{2} \cdot \left\{\frac{-(1-kr^2)}{R^2}\right\}\left\{-\frac{\delta}{\delta r}(-R^2 r^2)\right\}$$

$$= \frac{-1}{2} \cdot \frac{(1-kr^2)}{R^2} . 2 \, rR^2$$

$$= - \, r(1-kr^2)$$

$$\therefore \Gamma^1_{22} = - r(1-kr^2)$$

$$\Gamma^1_{33} = \frac{1}{2} g^{11}\left(\frac{\delta g_{31}}{\delta x^3} + \frac{\delta g_{13}}{\delta x^3} - \frac{\delta g_{33}}{\delta x^1}\right)$$

$$= \frac{1}{2} \cdot \left\{\frac{-(1-kr^2)}{R^2}\right\}\left\{-\frac{\delta}{\delta r}(-R^2 r^2 Sin^2\theta)\right\}$$

$$= - \frac{1}{2} \cdot \frac{(1-kr^2)}{R^2} . 2 \; R^2 r \; Sin^2\theta$$

$$= -r(1-kr^2) \quad Sin^2\theta$$

$$\therefore \; \Gamma^1_{33} = -r(1-kr^2) \quad Sin^2\theta$$

$$\Gamma^2_{12} = \frac{1}{2} g^{22}\left(\frac{\delta g_{22}}{\delta x^1} + \frac{\delta g_{21}}{\delta x^2} - \frac{\delta g_{12}}{\delta x^2}\right)$$

$$= \frac{1}{2} \cdot \left(\frac{-1}{r^2R^2}\right) \cdot \frac{\delta}{\delta r}(-R^2r^2)$$

$$= \frac{1}{2} \cdot \frac{1}{r^2R^2} \quad 2R^2r$$

$$= \frac{1}{r}$$

$$\therefore \; \Gamma^2_{12} = \frac{1}{r} = \Gamma^2_{21}$$

$$\Gamma^2_{33} = \frac{1}{2} g^{22}\left(\frac{\delta g_{32}}{\delta x^3} + \frac{\delta g_{23}}{\delta x^3} - \frac{\delta g_{33}}{\delta x^2}\right)$$

$$= \frac{1}{2} \cdot \left(\frac{-1}{r^2R^2}\right) \cdot \left\{-\frac{\delta}{\delta\theta} \; (-\,r^2 R^2 Sin^2 \theta)\right\}$$

$$= -\frac{1}{2} \cdot \frac{1}{r^2R^2} \quad 2R^2r^2 \sin\theta \cos\theta$$

$$= -Sin\theta Cos\theta$$

$$\therefore \; \Gamma^2_{33} = -Sin\theta Cos\theta$$

$$\Gamma^3_{13} = \frac{1}{2} g^{33}\left(\frac{\delta g_{33}}{\delta x^1} + \frac{\delta g_{31}}{\delta x^3} - \frac{\delta g_{13}}{\delta x^3}\right)$$

$$= \frac{1}{2} \cdot \left(\frac{-1}{r^2 R^2 Sin^2\theta}\right) \left\{\frac{\delta}{\delta r} (- r^2 R^2 Sin^2 \theta)\right\}$$

$$= \frac{1}{r}$$

$$\therefore \Gamma^3_{13} = \frac{1}{r} = \Gamma^3_{31}$$

$$\Gamma^3_{23} = \frac{1}{2} g^{33}\left(\frac{\delta g_{33}}{\delta x^2} + \frac{\delta g_{32}}{\delta x^3} - \frac{\delta g_{23}}{\delta x^3}\right)$$

$$= \frac{1}{2} \cdot \left(\frac{-1}{r^2 R^2 Sin^2\theta}\right) \left\{\frac{\delta}{\delta \theta} (- r^2 R^2 Sin^2 \theta)\right\}$$

$$= Cot\theta$$

$$\therefore \Gamma^3_{23} = Cot\theta = \Gamma^3_{32}$$

Therefore non-zero affine connection of the metric tensor of R-W metric using the Chrisstoffel symbol are,

$$\left.\begin{aligned}
&\Gamma^0_{11} = \frac{\dot{R}R}{1-kr^2}\ ;\ \Gamma^0_{22} = R\dot{R}r^2\ ;\ \Gamma^0_{33} = R\dot{R}r^2 Sin^2\theta \\
&\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{R}}{R} \\
&\Gamma^1_{11} = \frac{kr}{(1-kr^2)}\ ;\ \Gamma^1_{22} = -r(1-kr^2)\ ;\ \Gamma^1_{33} = -r(1-kr^2)Sin^2\theta \\
&\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}
\end{aligned}\right\} \quad \ldots(2.8)$$

$$\Gamma^2_{33} = -Sin\theta\, Cos\theta$$

$$\Gamma^3_{23} = \Gamma^3_{32} = Cot\theta$$

2.8 Calculation of finding the value of R_{00}, R_{11}, R_{22}, R_{33} & R:

The Ricci tensor is define by the contraction

$$R_{jk} = \Gamma^i_{rk} \; \Gamma^r_{ji} - \Gamma^i_{ri} \; \Gamma^r_{jk} + \frac{\delta}{\delta x^k}\Gamma^i_{ji} - \frac{\delta}{\delta x^i}\Gamma^i_{jk} \quad \ldots\ldots\ldots (2.9)$$

$$\therefore R_{00} = \Gamma^i_{r0} \; \Gamma^r_{0i} - \Gamma^i_{ri} \; \Gamma^r_{00} + \frac{\delta}{\delta x^0}\Gamma^i_{0i} - \frac{\delta}{\delta x^i}\Gamma^i_{00} \quad \ldots\ldots\ldots (2.10)$$

Now,

$$\Gamma^i_{r0} \; \Gamma^r_{0i} = \Gamma^0_{r0} \; \Gamma^r_{00} + \Gamma^1_{r0} \; \Gamma^r_{01} + \Gamma^2_{r0} \; \Gamma^r_{02} + \Gamma^3_{r0} \; \Gamma^r_{03}$$

$$= \Gamma^0_{00} \; \Gamma^0_{00} + \Gamma^1_{10} \; \Gamma^1_{01} + \Gamma^2_{20} \; \Gamma^2_{02} + \Gamma^3_{30} \; \Gamma^3_{03}$$

$$= 0 + \frac{\dot{R}}{R} \cdot \frac{\dot{R}}{R} + \frac{\dot{R}}{R} \cdot \frac{\dot{R}}{R} + \frac{\dot{R}}{R} \cdot \frac{\dot{R}}{R}$$

$$= \frac{3\dot{R}^2}{R^2}$$

$$\Gamma^i_{ri} \; \Gamma^r_{00} = 0$$

$$\frac{\delta}{\delta x^0}\Gamma^i_{0i} = \frac{\delta}{\delta t}(\Gamma^0_{00} + \Gamma^1_{01} + \Gamma^2_{02} + \Gamma^3_{03})$$

$$= \frac{\delta}{\delta t}\left[0 + \frac{\dot{R}}{R} + \frac{\dot{R}}{R} + \frac{\dot{R}}{R}\right]$$

$$= \frac{3\ddot{R}}{R} - \frac{3\dot{R}^2}{R^2}$$

And ,

$$\frac{\delta}{\delta x^i}\Gamma^{i}_{00} = 0$$

Now putting these value in equation (2.10) we get ,

$$R_{00} = \frac{3\dot{R}^2}{R^2} - 0 + \frac{3\ddot{R}}{R} - \frac{3\dot{R}^2}{R^2} - 0$$

$$\therefore R_{00} = \frac{3\ddot{R}}{R}$$

Again ,

$$\therefore R_{11} = \Gamma^{i}_{r1} \Gamma^{r}_{1i} - \Gamma^{i}_{ri} \Gamma^{r}_{11} + \frac{\delta}{\delta x^1}\Gamma^{i}_{1i} - \frac{\delta}{\delta x^i}\Gamma^{i}_{11} \quad \dots\dots\dots (2.11)$$

Now ,

$$\Gamma^{i}_{r1} \Gamma^{r}_{1i} = \Gamma^{0}_{r1} \Gamma^{r}_{10} + \Gamma^{1}_{r1} \Gamma^{r}_{11} + \Gamma^{2}_{r1} \Gamma^{r}_{12} + \Gamma^{3}_{r1} \Gamma^{r}_{13}$$

$$= \Gamma^{0}_{11} \Gamma^{1}_{10} + \Gamma^{0}_{11} \Gamma^{1}_{01} + \Gamma^{1}_{11} \Gamma^{1}_{11} + \Gamma^{2}_{12} \Gamma^{2}_{21} + \Gamma^{3}_{31} \Gamma^{3}_{13}$$

$$= \frac{\dot{R}R}{1-kr^2} \cdot \frac{\dot{R}}{R} + \frac{\dot{R}}{R} \cdot \frac{\dot{R}R}{1-kr^2} + \frac{kr}{(1-kr^2)} \cdot \frac{kr}{(1-kr^2)} + \frac{1}{r^2} + \frac{1}{r^2}$$

$$= \frac{2\dot{R}^2}{1-kr^2} + \frac{k^2r^2}{(1-kr^2)^2} + \frac{2}{r^2}$$

$$\Gamma^{i}_{ri} \Gamma^{r}_{11} = \Gamma^{r}_{11} (\Gamma^{0}_{r0} + \Gamma^{1}_{r1} + \Gamma^{2}_{r2} + \Gamma^{3}_{r3})$$

$$= \Gamma^{0}_{11} \Gamma^{0}_{00} + \Gamma^{0}_{11} \Gamma^{1}_{01} + \Gamma^{1}_{11} \Gamma^{1}_{11} + \Gamma^{0}_{11} \Gamma^{0}_{20} + \Gamma^{2}_{12} \Gamma^{1}_{11} + \Gamma^{3}_{31} \Gamma^{3}_{11} + \Gamma^{0}_{11} \Gamma^{3}_{03}$$

$$=0 + \frac{\dot{R}}{R} \cdot \frac{\dot{R}R}{1-kr^2} + \frac{k^2r^2}{(1-kr^2)^2} + \frac{\dot{R}R}{1-kr^2} \cdot \frac{\dot{R}}{R} + \frac{1}{r}\frac{kr}{(1-kr^2)} + \frac{1}{r} \cdot \frac{kr}{(1-kr^2)} + \frac{\dot{R}}{R} \cdot \frac{\dot{R}R}{1-kr^2}$$

$$= \frac{3\dot{R}^2}{1-kr^2} + \frac{2k}{(1-kr^2)} + \frac{k^2r^2}{(1-kr^2)^2}$$

$$\frac{\delta}{\delta x^i}\Gamma^{i}_{1i} = \frac{\delta}{\delta r}(\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13})$$

$$= \frac{\delta}{\delta r}\left(0 + \frac{kr}{1-kr^2} + \frac{1}{r} + \frac{1}{r}\right)$$

$$= \frac{k+k^2r^2}{(1-kr^2)^2} - \frac{2}{r^2}$$

$$\frac{\delta}{\delta x^i}\Gamma^{i}_{11} = \frac{\delta}{\delta x^0}\Gamma^0_{11} + \frac{\delta}{\delta x^1}\Gamma^1_{11} + \frac{\delta}{\delta x^2}\Gamma^2_{11} + \frac{\delta}{\delta x^3}\Gamma^3_{11}$$

$$= \frac{\delta}{\delta t}\left(\frac{\dot{R}R}{1-kr^2}\right) + \frac{\delta}{\delta r}\left(\frac{kr}{1-kr^2}\right) + 0 + 0$$

$$= \frac{\dot{R}^2 + R\ddot{R}}{1-kr^2} + \frac{k+k^2r^2}{(1-kr^2)^2}$$

Thus from equation (2.11) we get ,

$$R_{11} = \frac{2\dot{R}^2}{1-kr^2} + \frac{k^2r^2}{(1-kr^2)^2} + \frac{2}{r^2} - \frac{3\dot{R}^2}{1-kr^2} - \frac{2k}{(1-kr^2)} - \frac{k^2r^2}{(1-kr^2)^2} + \frac{k+k^2r^2}{(1-kr^2)^2} - \frac{2}{r^2} - \frac{\dot{R}^2 + R\ddot{R}}{1-kr^2} - \frac{k+k^2r^2}{(1-kr^2)^2}$$

$$\therefore R_{11} = \frac{-1}{1-kr^2}(R\ddot{R} + 2\dot{R}^2 + 2k)$$

Similarly we get,

$$R_{22} = -r^2(2\dot{R}^2 + R\ddot{R} + 2k)$$

$$R_{33} = r^2 Sin^2\theta(2\dot{R}^2 + R\ddot{R} + 2k)$$

Therefore,

$$\left.\begin{aligned} R_{00} &= \frac{3\ddot{R}}{R} \\ R_{11} &= \frac{-1}{1-kr^2}(2\dot{R}^2 + R\ddot{R} + 2k) \\ R_{22} &= -r^2(2\dot{R}^2 + R\ddot{R} + 2k) \\ R_{33} &= r^2 Sin^2\theta(2\dot{R}^2 + R\ddot{R} + 2k) \end{aligned}\right\} \quad \ldots\ \ldots\ \ldots\ (2.12)$$

Now,

$$R_0^0 = g^{00}R_{00} = 1.\frac{3\ddot{R}}{R} = \frac{3\ddot{R}}{R}$$

$$R_1^1 = g^{11}R_{11} = -\frac{(1-kr^2)}{R^2}[\frac{-1}{1-kr^2}(2\dot{R}^2 + R\ddot{R} + 2k)]$$

$$= \frac{R\ddot{R} + 2\dot{R}^2 + 2k}{R^2}$$

$$R_2^2 = g^{22}R_{22} = -\frac{1}{R^2r^2}[-r^2(2\dot{R}^2 + R\ddot{R} + 2k)] = \frac{R\ddot{R} + 2\dot{R}^2 + 2k}{R^2}$$

$$R_3^3 = g^{33}R_{33} = -\frac{1}{R^2r^2Sin^2\theta}[-r^2Sin^2\theta(2\dot{R}^2 + R\ddot{R} + 2k)]$$

$$= \frac{R\ddot{R} + 2\dot{R}^2 + 2k}{R^2}$$

$$\therefore R = R_0^0 + R_1^1 + R_2^2 + R_3^3$$

$$\Rightarrow R = \frac{3\ddot{R}}{R} + \frac{3(R\ddot{R} + 2\dot{R}^2 + 2k)}{R^2}$$

$$\therefore R = \frac{6(R\ddot{R} + \dot{R}^2 + k)}{R^2} \qquad \ldots\ldots\ldots (2.13)$$

2.9 Different components of energy momentum tensor :

We have the energy momentum tensor

$$T_{\mu\vartheta} = (p+\rho)U_{\mu}U_{\vartheta} - pg_{\mu\vartheta} \qquad \ldots\ldots\ldots (2.14)$$

We have,

$$U^{\mu} = \frac{dx^{\mu}}{dt}$$

$$U^0 = \frac{dx^0}{dt} = \frac{dt}{dt} = 1 \quad \& \quad U^1 = U^2 = U^3 = 0$$

$$U^{\mu} = (1,0,0,0) \quad \& \quad U_{\mu} = (1,0,0,0)$$

Again we know from FRW model,

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + Sin^2\theta\, d\varphi^2)\right]$$

$$g_{00} = 1 \;,\quad g_{11} = \frac{-R^2}{1-kr^2} \;,\quad g_{22} = -r^2R^2 \;,\quad g_{33} = -r^2R^2Sin^2\theta$$

$$g^{00} = 1 \;,\quad g^{11} = \frac{-(1-kr^2)}{R^2} \;,\quad g^{22} = \frac{-1}{r^2R^2} \;,\quad g^{33} = \frac{-1}{r^2R^2Sin^2\theta}$$

Now ,

$$T_{00} = (p+\rho)U_0U_0 - pg_{00}$$

$$= (p+\rho) - p$$

$$= \rho$$

$$T_{11} = (p+\rho)U_1U_1 - pg_{11}$$

$= 0 - pg_{11}$

$= -p\{\frac{-R^2(t)}{1-kr^2}\}$

$= \frac{pR^2}{1-kr^2}$

$$T_{22} = (p+\rho)U_2U_2 - pg_{22}$$

$= 0 - pg_{22}$

$= -p(-r^2R^2)$

$= pr^2R^2$

$$T_{33} = (p+\rho)U_3U_3 - pg_{33}$$

$= 0 - pg_{33}$

$= -p(-r^2R^2\,\mathrm{Sin}^2\theta)$

$= pr^2R^2\,\mathrm{Sin}^2\theta$

Again ,

$T^{00} = g^{00}g^{00}T_{00} = 1 . 1 . \rho = \rho$

$$T^{11} = g^{11}g^{11}T_{11} = \{\frac{-(1-kr^2)}{R^2}\}.\{\frac{-(1-kr^2)}{R^2}\}.\frac{pR^2}{1-kr^2} = \frac{p(1-kr^2)}{R^2}$$

$$T^{22} = g^{22}g^{22}T_{22} = \{\frac{-1}{r^2R^2}\}.\{\frac{-1}{r^2R^2}\}.\, pr^2R^2 = \frac{p}{r^2R^2}$$

$$T^{33} = g^{33}g^{33}T_{33} = \{\frac{-1}{r^2R^2Sin^2\theta}\}.\{\frac{-1}{r^2R^2Sin^2\theta}\}.\, pr^2R^2Sin^2\theta = \frac{p}{r^2R^2Sin^2\theta}$$

So we get ,

$$\left.\begin{array}{lll} T_{00} = \rho & & T^{00} = \rho \\ T_{11} = \frac{pR^2}{1-kr^2} & \& & T^{11} = \frac{p(1-kr^2)}{R^2} \\ T_{22} = pr^2R^2 & & T^{22} = \frac{p}{r^2R^2} \\ T_{33} = pr^2R^2Sin^2\theta & & T^{33} = \frac{p}{r^2R^2Sin^2\theta} \end{array}\right\} \quad \ldots\ldots\ldots (2.15)$$

Which are the components of the energy momentum tensor.

2.10 Derivation of time-time components :

Einstein's Field equation **[12]** are given by,

$$R_{\mu\vartheta} - \frac{1}{2}g_{\mu\vartheta}R = -8\pi G T_{\mu\vartheta} \quad \ldots\ldots\ldots (2.16)$$

Where $T_{\mu\vartheta}$ is the energy momentum tensor of the source producing the gravitational field and G is the Newtonian gravitational constant. For perfect fluid (means that type of fluid which has no viscosity) $T_{\mu\vartheta}$ takes the following form,

$$T_{\mu\vartheta} = (p+\rho)U_\mu U_\vartheta - p g_{\mu\vartheta} \quad \dots\dots\dots (2.17)$$

Where ρ is the mass energy density . p is the pressure and U_μ is the 4-velocity vector of matter is given by ,

$$U^\mu = \frac{dx^\mu}{dt} = (1,0,0,0)$$

So the 4-velocity are the same as the contravariant given by ,

$$U_\mu = (1,0,0,0)$$

So that zero-zero component of $T_{\mu\vartheta}$ are ,

$$T_{00} = (p+\rho)U_0 U_0 - p g_{00} = (p+\rho) - p = \rho$$

Thus equation (2.16) becomes for zero-zero component ,

$$R_{00} - \frac{1}{2} g_{00} R = -8\pi G T_{00}$$

Or, $$\frac{3\ddot{R}}{R} - \frac{1}{2}.1.\frac{6(R\ddot{R} + \dot{R}^2 + k)}{R^2} = -8\pi G\rho$$

Or , $$\frac{3R\ddot{R} - 3R\ddot{R} - 3\dot{R}^2 - 3k}{R^2} = -8\pi G\rho$$

Or, $$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{R^2}$$

$$\therefore H^2 = \frac{8\pi G\rho}{3} - \frac{k}{R^2} \quad \dots\dots\dots (2.18)$$

Which is the solution of Einstein equation for **time-time** or **zero-zero** component .Again we have Einstein equation,

$$R_{\mu\vartheta} - \frac{1}{2} g_{\mu\vartheta} R = -8\pi G T_{\mu\vartheta}$$

For one-one component,

$$R_{11} - \frac{1}{2} g_{11} R = -8\pi G T_{11}$$

Or, $R_{11} - \frac{1}{2} g_{11} R = -8\pi G p g_{11}$ $\quad [\because T_{11} = -p g_{11}]$

Or, $\frac{R_{11}}{g_{11}} - \frac{1}{2} R = -8\pi G p$

Or, $\frac{-1}{1-kr^2} (R\ddot{R} + 2\dot{R}^2 + 2k).\{\frac{-(1-kr^2)}{R^2}\}- \frac{1}{2}.1.\frac{6(R\ddot{R} + 2\dot{R}^2 + 2k)}{R^2} = 8\pi G p$

Or, $\frac{R\ddot{R} + 2\dot{R}^2 + 2k - 3R\ddot{R} - 3\dot{R}^2 - 3k}{R^2} = 8\pi G p$

Or, $\frac{2R\ddot{R}}{R^2} = -8\pi G p - (\frac{\dot{R}^2}{R^2} + \frac{k^2}{R^2})$

Or, $\frac{2R\ddot{R}}{R^2} = -8\pi G p - \frac{8\pi G \rho}{3}$

$$\therefore \ddot{R} = -\frac{4\pi G}{3}(\rho + 3p)R \qquad \dots \dots \dots (2.19)$$

Which is the Space-space component.

Again, we have *Einstein* field equation can be written as,

$$R_{\mu\vartheta}-\frac{1}{2}g_{\mu\vartheta}R=-8\pi GT_{\mu\vartheta}$$

This can be written as,

$$R^{\mu\vartheta}-\frac{1}{2}g^{\mu\vartheta}R=-8\pi GT^{\mu\vartheta}$$

Taking the co-varient derivative on both sides we get,

$$(R^{\mu\vartheta}-\frac{1}{2}g^{\mu\vartheta}R)_{;\vartheta}=-8\pi GT^{\mu\vartheta}_{;\vartheta}$$

Or, $-8\pi GT^{\mu\vartheta}_{;\vartheta}=0$

Or, $T^{\mu\vartheta}_{;\vartheta}=0$

Again we have,

$$T^{\mu\vartheta}_{;\vartheta}=T^{\mu\vartheta}_{\vartheta}+\Gamma^{\mu}_{\sigma\vartheta}T^{\sigma\vartheta}+\Gamma^{\vartheta}_{\sigma\vartheta}T^{\sigma\mu}$$

$$\therefore T^{\mu\vartheta}_{\vartheta}+\Gamma^{\mu}_{\sigma\vartheta}T^{\sigma\vartheta}+\Gamma^{\vartheta}_{\sigma\vartheta}T^{\sigma\mu}=0$$

Or, $T^{00}_{0}+\Gamma^{0}_{00}T^{00}+\Gamma^{0}_{11}T^{11}+\Gamma^{0}_{22}T^{22}+\Gamma^{0}_{33}T^{33}+T^{00}(\Gamma^{0}_{00}+\Gamma^{1}_{10}+\Gamma^{2}_{20}+\Gamma^{3}_{30})=0$

Or, $\dot{\rho}+0+\frac{\dot{R}R}{1-kr^2}\ \frac{p(1-kr^2)}{R^2}+R\dot{R}r^2\,.\frac{p}{r^2R^2}+R\dot{R}r^2Sin^2\theta\,.\frac{p}{r^2R^2Sin^2\theta}+$

$\rho[\frac{\dot{R}}{R}+\frac{\dot{R}}{R}+\frac{\dot{R}}{R}]=0$

Or, $\dot{\rho}+\frac{p\dot{R}}{R}+\frac{p\dot{R}}{R}+\frac{p\dot{R}}{R}+\frac{3\rho\dot{R}}{R}=0$

Or, $\dot{\rho}+\frac{3p\dot{R}}{R}+\frac{3\rho\dot{R}}{R}=0$

$$\therefore \dot{\rho}+\frac{3\dot{R}}{R}(p+\rho)=0 \qquad \ldots\ldots\ldots (2.20)$$

We have equation of state,

$$p=(\gamma-1)\rho \quad ; 1\le\gamma\le 2 \qquad \ldots\ldots\ldots (2.21)$$

Putting, $\gamma=1, p=o$

$$\therefore \dot{\rho}+\frac{3\dot{R}}{R}(p+\rho)=0$$

Or, $\dot{\rho}+\frac{3\rho\dot{R}}{R}=0$

Or, $\frac{\dot{\rho}}{\rho}+\frac{3\dot{R}}{R}=0$

Integrating,

$$\log\rho+3\log R=\log R_0$$

Or, $\log\rho=\log R_0-3\log R$

Or, $\log\rho=\log R_0+\log R^{-3}$

Or, $\rho=R_0R^{-3}$

$$\therefore \rho\propto R^{-3} \qquad \ldots\ldots\ldots (2.22)$$

Which is known as matter-dominant universe.

Putting, $\gamma = \frac{4}{3}$ in equation (2.6) we get $p = \frac{\rho}{3}$

Then,

$$\therefore\ \dot{\rho} + \frac{3\dot{R}}{R}(p+\rho) = 0$$

$$\text{Or,}\quad \dot{\rho} + \frac{3\dot{R}}{R}.\frac{4\rho}{3} = 0$$

$$\text{Or,}\quad \frac{\dot{\rho}}{\rho} + \frac{4R}{R} = 0$$

Integrating,

$$\log\rho + 4\log R = \log R_0$$

$$\text{Or,}\quad \log\rho = \log R_0 - 4\log R$$

$$\text{Or,}\quad \rho = R_0 R^{-4}$$

$$\therefore\ \rho \propto R^{-4} \qquad \ldots\ \ldots\ \ldots\ (2.23)$$

Which is the radiation dominant universe.

Combining equation (2.22) & (2.23) generalizes the following relation ,

$$\rho \propto R^{-3\gamma}$$

If we put $\gamma = 2$ in equation (2.6) we get $p = \rho$

$$\therefore\ \dot{\rho} + \frac{3\dot{R}}{R}(p+\rho) = 0$$

$$\text{Or,}\quad \dot{\rho} + \frac{3\dot{R}}{R}.2\rho = 0 \qquad [\because p = \rho]$$

$$\text{Or, } \dot{\rho} + \frac{6\rho\dot{R}}{R} = 0$$

$$\text{Or, } \frac{\dot{\rho}}{\rho} + \frac{6\dot{R}}{R} = 0$$

Integrating,

$$\log \rho + 6 \log R = \log R_0$$

$$\text{Or, } \log \rho = \log R_0 - 6 \log R$$

$$\text{Or, } \log \rho = \log R_0 + \log R^{-6}$$

$$\text{Or, } \rho = R_0 R^{-6}$$

$$\therefore \rho \propto R^{-6} \qquad \ldots \ldots \ldots (2.24)$$

Which is condition for that stiff fluid.

2.11 The Cosmic Microwave Background:

The Big Bang theory predicts that the early universe was a very hot place and that as it expands, the gas within it cools. Thus the universe should be filled with radiation that is literally the remnant heat left over from the Big Bang, called the "Cosmic microwave background radiation", or CMB.

2.12 Discovery of the Cosmic Microwave Background:

The existence of the CMB radiation was first predicted by *Ralph Alpher* , *Robert Herman* and *George Gamow* in 1948, as part of their work on Big Bang Nucleosynthesis **[1]**. It was first observed inadvertently in 1965 by

Arno Penzias and *Robert Wilson* at the Bell Telephone Laboratories in Murray Hill, New Jersey. The radiation was acting as a source of excess noise in a radio receiver they were building. Coincidentally, researchers at nearby Princeton University, led by *Robert Dicke* and including *Dave Wilkinson* of the WMAP science team, were devising an experiment to find the CMB. When they heard about the Bell Labs result they immediately realized that the CMB had been found. The result was a pair of papers in the Astrophysical Journal (vol. 142 of 1965): one by Penzias and Wilson detailing the observations, and one by *Dicke* ,*Peebles, Roll*, and *Wilkinson* giving the cosmological interpretation. Penzias and Wilson shared the 1978 Nobel Prize in physics for their discovery **[47]**.

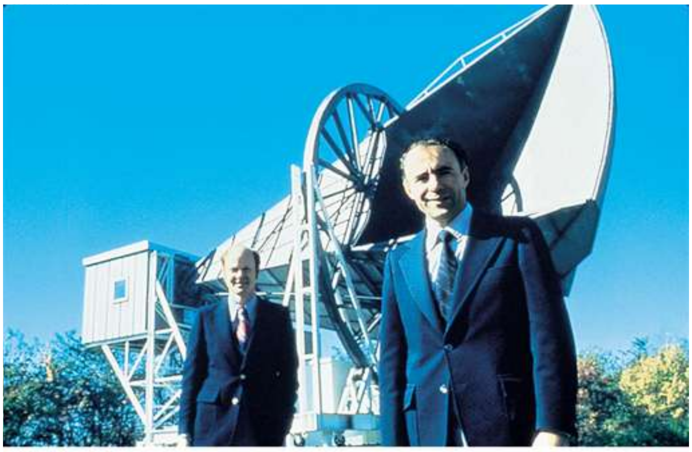


Figure: ***Arno Penzias*** and ***Robert Wilson*** (***Discovery of the Cosmic Microwave Background)***

Today, the CMB radiation is very cold, only 2.725° above absolute zero, thus this radiation shines primarily in the microwave portion of the electromagnetic spectrum, and is invisible to the naked eye. However, it fills the universe and can be detected we look. In fact, if we could see microwaves, the entire sky would glow with a brightness that was astonishingly uniform in every direction. The picture at left shows a false color depiction of the temperature (brightness) of the CMB over the full sky (projected onto an oval, similar to a map of the Earth). The temperature is uniform to better than one part in a thousand! This uniformity is one compelling reason to interpret the radiation as remnant heat from the Big Bang; it would be very difficult to imagine a local source of radiation that was this uniform. In fact, many scientists have tried to devise alternative explanations for the source of this radiation but none have succeeded**[65].**

2.13 Why study the Microwave Background :

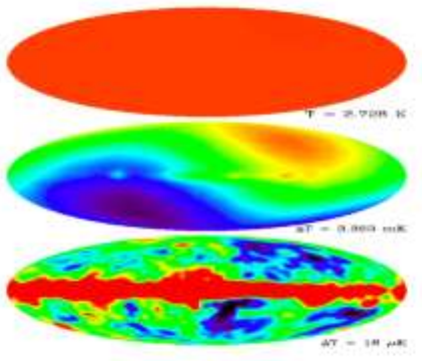
Since light travels at a finite speed, astronomers observing distant objects are looking into the past. Most of the stars that are visible to the naked eye in the night sky are 10 to 100 light years away. Thus, we see them as they were 10 to 100 years ago .We observe Andromeda, the nearest big galaxy, as it was about 2.5 million years ago. Astronomers observing distant galaxies with the Hubble Space Telescope can see them as they were only a few billion years after the Big Bang. (Most cosmologists believe that the universe is between 12 and 14 billion years old.)

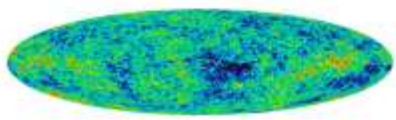
The CMB radiation was emitted only a few hundred thousand years after the Big Bang, long before stars or galaxies ever existed. Thus, by studying the detailed physical properties of the radiation, we can learn about conditions in the universe on very large scales, since the radiation we see today has traveled over such a large distance, and at very early times.

2.14 The origin of the Cosmic Microwave Background:

One of the profound observations of the 20th century is that the universe is expanding. This expansion implies the universe was smaller, denser and hotter in the distant past. When the visible universe was half its present size, the density of matter was eight times higher and the cosmic microwave background was twice as hot. When the visible universe was one hundredth of its present size, the cosmic microwave background was a hundred times hotter (273 degrees above absolute zero or 32degrees Fahrenheit, the temperature at which water freezes to form ice on the Earth's surface). In addition to this cosmic microwave background radiation, the early universe was filled with hot hydrogen gas with a density of about 1000 atoms per cubic centimeter. When the visible universe was only one hundred millionth its present size, its temperature was 273 million degrees above absolute zero and the density of matter was comparable to the density of air at the Earth's surface. At these high temperatures, the hydrogen was completely ionized into free protons and electrons.

Since the universe was so very hot through most of its early history, there were no atoms in the early universe, only free electrons and nuclei. (Nuclei are made of neutrons and protons). The cosmic microwave background photons easily scatter off of electrons. Thus, photons wandered through the early universe, just as optical light wanders through a dense fog. This process of multiple scattering produces what is called a “thermal” or “blackbody” spectrum of photons. According to the Big-Bang theory, the frequency spectrum of the CMB should have this blackbody form. This was indeed measured with tremendous accuracy by the FIRAS experiment on NASA's COBE satellite**[65].**





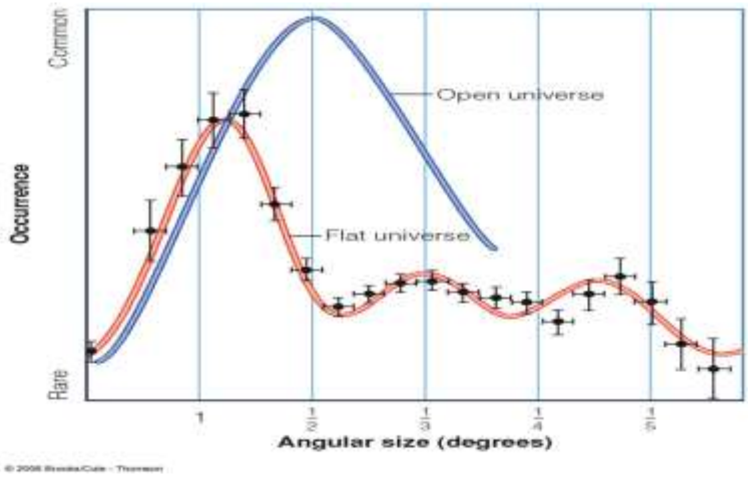


Figure: *This figure shows the prediction of the Big Bang theory for the energy spectrum of the cosmic microwave background radiation compared to the observed energy spectrum.*

The FIRAS experiment measured the spectrum at 34 equally spaced points along the blackbody curve. The error bars on the data points are so small that they cannot be seen under the predicted curve in the figure! There is no alternative theory yet proposed that predicts this energy spectrum. The accurate measurement of its shape was another important test of the Big Bang theory.

2.15 Cosmological parameters :

There are some parameters which define the evolution and dynamics of the system qualitatively. In fact a model suggested to explain any phenomena is a basic frame which only by defining its parameters can be confronted with observations and tested. A good model is a model which can explain the evolution of a system can predict correctly the events and has also a few numbers of free parameters. Our standard model of cosmology (together with its companion theories such as in inflation), has also several free parameters which need to be determined from observations. In the standard model, the universe is a perturbed FRW space-time with dynamics governed by the Einstein equations.

All cosmological components with different densities and different equations of state ($w_i = p_i/\rho_i$) , p_i is the pressure and ρ_i is the energy density of the *ith* component) are responsible for the overall dynamics of our cosmological system. The Friedmann equation, expressed in terms of the cosmological redshift z(*t*) = $\frac{R_0}{R_t} - 1$ becomes

$$\frac{H^2}{H_0^2} = \sum_i \Omega_{0i}(1+z)^{3(1+w_i)} + (1 - \sum_i \Omega_{0i})(1+z)^2 \quad \ldots \ldots \ldots (2.25)$$

Below we briefly comments on some important cosmological parameters-

(i) Hubble parameter:

The value of the Hubble parameter as a function of redshift was given in eq^n (2.25). Its present value $H_0 = (\frac{\dot{R}_0}{R_t})_{t=t_0}$, denotes the expansion rate at the current epoch. A recent measurement of the Hubble constant from the Hubble Space Telescope Key Project estimated H_0 = 72 ± 3(statistical) ± 7(systematic) $km\ sec^{-1}\ Mpc^{-1}$ by using the empirical period-luminosity relation for Cepheid variable stars to obtain distances to 31 galaxies. This result is in very good agreement with estimates of H_0 derived from observation of the cosmic microwave background made by the WMAP satellite **[64].**

(ii)Curvature parameter:

The observed value of the curvature parameter $\Omega_{0k} = -k/R_0^2H_0^2 = 1-\sum_i \Omega_{0i}$ provides strong support for a spatially flat universe as originally predicted by the inflationary scenario .This has important implications for the total matter density of the universe.

2.16 Tensor to scalar ratio :

Inflation generates perturbations by amplifying quantum fluctuations and stretching them to astrophysical scales through rapid expansion. The simplest inflationary scenarios based on a single scalar field generate both scalar field perturbation as well as tensor metric fluctuations (gravity waves).

Fluctuations in the scalar field subsequently result in structure formation in the universe via gravitational instability, while the tensor metric fluctuations give rise to a relic gravity background. Both tensor and scalar fluctuations perturb the CMB. The ratio between the tensor power spectrum and scalar power spectrum in the CMB is characterized by the tensor to scalar ratio, *r*. This ratio can be derived by analyzing the cosmic microwave background data and comparing with theoretical predictions from different inflationary scenarios. In the following we briefly discuss the tensor to scalar ratio for a single scalar field (slow roll) inflation. Inflation leads to a period of early acceleration $\ddot{R} > 0$ during which

$$\frac{d}{dt}\left(\frac{1}{RH}\right) < 0 \quad \ldots \ldots \ldots (2.26)$$

in other words, during the inflationary epoch the comoving Hubble length, $1/RH$ decreases with time**[57].**

But from equation (2.26) we have

$$\frac{d}{dt}\left(\frac{1}{RH}\right)=\frac{d}{dt}\left(\frac{1}{R.\dot{R}/R}\right)=\frac{d}{dt}\left(\frac{R}{R\dot{R}}\right)=\frac{d}{dt}\left(\frac{1}{\dot{R}}\right)=-\dot{R}^{-2}.\ddot{R}=-\frac{\ddot{R}}{\dot{R}^{2}}$$

$$ie.-\frac{\ddot{R}}{\dot{R}^{2}}<0$$

$\therefore$ We can write $\ddot{R}<0$ …. … … (2.27)

Where $\ddot{R}$ is the acceleration

We also know that acceleration can not be less than zero *ie.* it is positive . So from equation (2.27) we can say that $\ddot{R}$ is deceleration.

Again we have the equation

$$\frac{\ddot{R}}{\dot{R}}=\frac{-4\pi G}{3}\sum(\rho+3p)$$

From this equation we find a necessary condition for inflation is ,

$$\rho+3p<0$$

But it is not possible .Because the pressure (p) & density (ρ) is always positive and so their addition will also be positive.

So we see that, in order to obtain inflation, we need a material with the very unusual property of negative pressure (similar to Λ or any other candidate of dark energy, but here it must be a dominant agent by a huge factor and also it needs to decay subsequently to baryons, dark matter and radiation). Scalar fields are good candidates for our purpose.

From energy momentum tensor we get ,

$$T_{\mu\vartheta} = (p+\rho)U_{\mu}U_{\vartheta} - pg_{\mu\vartheta} \quad \ldots\ldots\ldots (2.28)$$

And we also have ,

$$T_{\mu\vartheta} = \delta_{\mu}\phi\delta_{\vartheta}\phi - g_{\mu\vartheta}[\frac{1}{2}\delta_{\sigma}\phi\delta^{\sigma}\phi - V(\phi)] \quad \ldots\ldots\ldots (2.29)$$

From these equation we get ,

$$(p+\rho)U_{\mu}U_{\vartheta} - pg_{\mu\vartheta} = \delta_{\mu}\phi\delta_{\vartheta}\phi - g_{\mu\vartheta}[\frac{1}{2}\delta_{\sigma}\phi\delta^{\sigma}\phi - V(\phi)] \ldots (2.30)$$

$$\Rightarrow (p+\rho)U_{\mu}U_{\vartheta} - pg_{\mu\vartheta} = \frac{\delta\phi}{\delta x^{\mu}}\frac{\delta\phi}{\delta x^{\vartheta}} - g_{\mu\vartheta}[\frac{1}{2}\delta_{\sigma}\phi\delta^{\sigma}\phi - V(\phi)]$$

$$\ldots\ldots\ldots(2.31)$$

Again we know,

$$U^{\mu} = \frac{dx^{\mu}}{dt}$$

$$U^{0} = \frac{dx^{0}}{dt} = \frac{dt}{dt} = 1 \quad \& \quad U^{1} = U^{2} = U^{3} = 0$$

$$U^{\mu} = (1,0,0,0) \quad \& \quad U_{\mu} = (1,0,0,0)$$

$\therefore$ for $\mu = \vartheta = 0$ we get from equation (2.31),

$$(p+\rho)U_0U_0 - pg_{00} = \frac{\delta\phi}{\delta x^0}\frac{\delta\phi}{\delta x^0} - g_{00}[\frac{1}{2}\delta_\sigma\phi\delta^\sigma\phi - V(\phi)]$$

$$\Rightarrow (p+\rho) - p = \frac{\delta\phi}{\delta t}\frac{\delta\phi}{\delta t} - [\frac{1}{2}\delta_\sigma\phi\delta^\sigma\phi - V(\phi)]$$

$$\Rightarrow \rho = \dot{\phi}.\dot{\phi} - [\frac{1}{2}\delta_\sigma\phi\delta^\sigma\phi - V(\phi)]$$

For , $\sigma = 0$ we get,

$$\rho = \dot{\phi}^2 - [\frac{1}{2}\delta_0\phi\delta^0\phi - V(\phi)]$$

$$\Rightarrow \rho = \dot{\phi}^2 - [\frac{1}{2}\frac{\delta\phi}{\delta x^0}\frac{\delta\phi}{\delta x^0} - V(\phi)]$$

$$\Rightarrow \rho = \dot{\phi}^2 - [\frac{1}{2}\dot{\phi}\dot{\phi} - V(\phi)]$$

$$\therefore\ \rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad \ldots\ldots\ldots (2.32)$$

Which is the density of the dynamic system.

Again putting $\mu = \vartheta = 1$ we get from equation (2.6) ,

$$(p+\rho)U_1U_1 - pg_{11} = \frac{\delta\phi}{\delta x^1}\frac{\delta\phi}{\delta x^1} - g_{11}[\frac{1}{2}\delta_\sigma\phi\delta^\sigma\phi - V(\phi)]$$

$$\Rightarrow 0 - pg_{11} = \frac{\delta\phi}{\delta r}\frac{\delta\phi}{\delta r} - g_{11}[\frac{1}{2}\delta_\sigma\phi\delta^\sigma\phi - V(\phi)]$$

$$\Rightarrow -pg_{11} = 0 - g_{11}[\frac{1}{2}\delta_\sigma\phi\delta^\sigma\phi - V(\phi)]$$

$$\Rightarrow p = [\frac{1}{2}\delta_{\sigma}\phi\delta^{\sigma}\phi - V(\phi)] \quad \ldots\ldots\ldots (2.33)$$

For , $\sigma = 0$ we get,

$$p = [\frac{1}{2}\delta_{o}\phi\delta^{o}\phi - V(\phi)]$$

$$\Rightarrow p = [\frac{1}{2}\frac{\delta\phi}{\delta t}\frac{\delta\phi}{\delta t} - V(\phi)]$$

$$\Rightarrow p = [\frac{1}{2}\dot{\phi}\dot{\phi} - V(\phi)]$$

$$\therefore p = \frac{1}{2}\dot{\phi}^{2} - V(\phi) \quad \ldots\ldots\ldots (2.34)$$

If we put $\sigma = 1,2,3$ then the first part of R.H.S of equation (2.33) will be zero because ϕ is a function of t only.

Similarly putting $\mu = \vartheta = 2,3$ we will get the same result .

So finally we can write ,

$$\left.\begin{aligned} \therefore \rho &= \frac{1}{2}\dot{\phi}^{2} + V(\phi) \\ \therefore p &= \frac{1}{2}\dot{\phi}^{2} - V(\phi) \end{aligned}\right\} \quad \ldots\ldots\ldots (2.35)$$

where $V(\phi)$ is the potential of the scalar field and $\frac{1}{2}\dot{\phi}^{2}$ is its kinetic energy. Next, by assuming the scalar field to be the dominant component in a spatially flat universe, We can derive the following equations describing the inflationary era:

$$(\frac{\dot{R}}{R})^2 = \frac{8\pi G\rho}{3} - \frac{k}{R^2}$$

For flat universe, k=0

So, $$(\frac{\dot{R}}{R})^2 = \frac{8\pi G\rho}{3} - 0$$

$$\Rightarrow H^2 = \frac{8\pi G\rho}{3} \cdot \frac{1}{3}\rho_\phi$$

$$\Rightarrow H^2 = \frac{1}{M_{pl}^2} \cdot \frac{1}{3}[\frac{1}{2}\dot{\phi}^2 + V(\phi)]$$

$$\Rightarrow H^2 = \frac{1}{3M_{pl}^2}[\frac{1}{2}\dot{\phi}^2 + V(\phi)] \quad \ldots\ldots\ldots (2.36)$$

Where $\frac{1}{M_{pl}^2} = \sqrt{8\pi G} = 4.342 \times 10^{-6} g$

Which is called the reduced Planck mass .From energy conjurvative we have ,

$$T^{\mu\vartheta}_{;\vartheta} = 0$$

$$\dot{\rho} + \frac{3\dot{R}}{R}(p+\rho) = 0$$

… … … (2.37)

From equation (2.35) we get ,

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad \ldots\ldots\ldots (2.38)$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

Adding, $$p + \rho = \dot{\phi}^2 \qquad \dots\dots\dots (2.39)$$

Differentiating (2.38) with respect to t we get,

$$\frac{d}{dt}(\rho) = \frac{d}{dt}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]$$

$$\Rightarrow \dot{\rho} = \frac{1}{2}(2\dot{\phi}\ddot{\phi}) + \frac{d}{dt}\{V(\phi)\}$$

$$\Rightarrow \dot{\rho} = \dot{\phi}\ddot{\phi} + \frac{d}{d\phi}\{V(\phi)\}\frac{d\phi}{dt}$$

$$\Rightarrow \dot{\rho} = \dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi} \qquad \dots\dots\dots (2.40)$$

Now putting the values of (2.39) & (2.40) in equation (2.37) we get ,

$$\dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi} + \frac{3\dot{R}\dot{\phi}^2}{R} = 0$$

$$\Rightarrow \ddot{\phi} + V'(\phi) + 3H\dot{\phi} = 0$$

$$\Rightarrow \ddot{\phi} + 3H\dot{\phi} = -V'(\phi)$$

$$\therefore \ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi} \qquad \dots\dots\dots (2.41)$$

To satisfy the inflationary condition, $\rho + 3p < 0$ we must have $\dot{\phi}^2 < V(\phi)$ which arises for sufficiently flat potentials. It is interesting that even if we start with a non-flat spatially geometry, a suitably flat potential makes the curvature term in the Friedmann equation less important as inflation gets underway. In fact it is one of the predictions of inflation that the geometry of the universe must flatten towards the end of inflation which is indeed in agreement with recent CMB observations. The most common approach to studying single field inflation is by means of the slow-roll approximation. In this approximation we neglect the $\dot{\phi}^2$ term in comparison with $V(\phi)$ and we also assume that $\ddot{\phi}$ must be negligible in equation (2.41)

$$\therefore H^2 \cong \frac{V(\phi)}{3M_{pl}^2}$$

$$\& \qquad 3H\dot{\phi} \cong -V'(\phi)$$

Where $V'(\phi) = \frac{dV}{d\phi}$ .The slow-roll approximation translates into the following rewuirements for the slow-roll parameters ε & η

$$\varepsilon(\phi) = \frac{M_{pl}^2}{2}(\frac{V'}{V})^2 << 1$$

$$\& \qquad |\eta(\phi)| = M_{Pl}^2\left|\frac{V''}{V}\right| << 1$$

In the slow-roll approximation the scalar and tensor spectra can be written as ,

$$P_R(k) \cong \frac{1}{24\pi^2 M_{pl}^4}\frac{V}{\varepsilon}\Big|_{k=RH} \quad \dots \dots \dots (2.42)$$

$$P_{grav}(k) \cong \frac{2}{3\pi^2 M_{pl}^4}V\Big|_{k=RH} \quad \dots \dots \dots (2.43)$$

Where in each case, the expressions in the right hand side are calculated when the scale k is equal to the Hubble radius during inflation. The symbol $\cong$ indicates the slow roll approximation has been used, which is expected to be accurate to a few percent. As a result one can compute the spectral index in

$$P_R(k) = A_s[\frac{k}{k_*}]^{n_s-1}$$

Becomes , $n_s \cong 1-6\varepsilon+2\eta$

One can also compute n_{grav} for the gravity waves by defining $P_{grav}(k) = A_{grav}(k/k_*)^{n}{}_{grav}$

$$n_{grav} \cong -2\varepsilon$$

and the tensor to scalar ratio

$$r \equiv \frac{P_{grav}(k_*)}{P_R(k_*)} \cong 16\varepsilon \cong -8n_{grav}$$

which is also known as the consistency equation. The five-year WMAP data has given the upper limit on the tensor to scalar ratio r < 0:43 (with 95% CL), for the standard ΛCDM model assuming a power-law primordial spectrum.

2.17 Beyond the Standard model :

In this section we briefly discuss extensions to the standard cosmological model .There are possibilities that our universe may be much more complicated than the standard model introduced in the previous section

.In addition the standard of cosmology assumes as adiabatic & Gaussian initial perturbations .In general primodial perturbations can be both adiabatic & isocurvature & the possibility of non-Gaussian fluctuations has also been widely discussed in the literature.

The ionizations history of the universe too can be much more complicated than in the standard model which assumes rapid ionization .But observational evidence is mixed and may be we have to improve our analysis with a better & more complicated approximation which may result in a change to our basic set of cosmological parameters.

Properties of dark matter can differ from the assumptions of the standard model where we assume the dark matter has no significant interaction with other matter and that its particles have a low velocities .Any change in this assumption will directly affect gravitational clustering & the properties of large scale structure .Variation of the fundamental constants on cosmological scales can be another extension to the standard model of cosmology .We can also ask whether the general theory of relativity is valid at all epochs or not Braneworld models and f(R) theories address this important issue .Topology of the universe is another open question which could add some more parameters to the above points there are two very important extensions to the standard model which are within our observational reach .The first one is model of dark energy with different properties in comparison with cosmological constant &the second one is non-power law form of the primordial perturbation spectrum .

2.18 The redshift :

At first we try to understand how the nebular redshift found by *Hubble & Humason* is accounted for by the *Robertson-Walker* model .We begin by recalling that the basic units of *Weyl's* postulates are galaxies with constant co-ordinates x^{μ} . We can easily identify the x^{μ} with the (r,θ,φ) of the *Robertson-Walker* spacetime .Thus each galaxy has a constant set of co-ordinates (r,θ,φ). This co-ordinate frame is often referred to as the cosmological rest frame .As observers we are located in our galaxy which also has constant (r,θ,φ) co-ordinate without loss of generality we can take $r = 0$ for our galaxy .Although this assumption suggests that we are placing ourselves at the centre of the universe , it does not confer any special status on us .Because of the assumption of homogeneity any galaxy could be chosen to have $r = 0$ .Our particular choice is simply dictated by convenience .

Consider a galaxy G ,at (r,θ,φ) emitting light waves towards us .Let us denote by t_0 the present epoch of observation .At what time should a light wave leave G_1 in order to arrive at r=0 at the present time $t = t_0$.

To find the answer to this question we need to know the path of the wave from G_1 to us .Since light travels along null geodesics .We need to calculate the null geodesic from G_1 to us.

From the symmetry of a speccetime we can guess that a null geodesic from $r = 0$ to $r = r_1$ will maintain a constant spatial direction .That is we expect to have $\theta = \theta_1$ & $\varphi = \varphi_1$ all along the null geodesic .This guess proves to be correct when we substitute these values into the geodesic equations.

Accordingly we will assume that only r & t change along the null geodesic .Next we recall that a first integral of the null geodesic equation is simply ds=0 .For the *Robertson-Walker* line element this gives us,

$$Cdt = \pm \frac{Sdr}{\sqrt{1-kr^2}} \quad \ldots \ldots \ldots (2.44)$$

Since r decreases as t increases along this null geodesic .We should take the minus sign in the above relation .Suppose that the null geodesic left G_1 at time t_1 .then we get from the above relation ,

$$\int_{t_1}^{t_0} \frac{Cdt}{S(t)} = \int_{0}^{r_1} \frac{dr}{\sqrt{1-kr^2}} \quad \ldots \ldots \ldots (2.45)$$

Thus if we know s(t) & k ,we know the answer to our question .

However consider what happens to successive wave creates emitted by G_1 .Suppose that wave creates were emitted at t_1 & $t_1+\Delta t_1$ & received by us at t_0 & $t_0+\Delta t_0$ respectively .Then similarly to equation (2.45) we have ,

$$\int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{Cdt}{S(t)} = \int_{0}^{r_1} \frac{dr}{\sqrt{1-kr^2}} \quad \ldots \ldots \ldots (2.46)$$

If S(t) is a slowly varying function so that it effectively remains unchanged over the small internals Δt_0 & Δt_1 .We get by subtraction of (2.45) & (2.46)

$$\frac{C\Delta t_0}{S(t_0)} - \frac{C\Delta t_1}{S(t_1)} = 0$$

$$ie.\ \frac{C\Delta t_0}{C\Delta t_1} = \frac{S(t_0)}{S(t_1)} = 1 + z \quad \ldots \ldots \ldots (2.47)$$

2.19 The Luminosity distance:

Since the *Friedmann* model are frequently used to interpret cosmological observations .We will now derive some of the observable quantities in these models ,starting with the Luminosity distance .Our aim is to express the final answer in terms of the two parameters the characterize a Friedmann model : H_0 & q_0

We use equation $\int_{t_1}^{t_0} \frac{Cdt}{S(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}}$ to relate r_1 the radial coordinates of the galaxy G_1 to the time t_1 & to its redshift z :

$$r_1 = \int_{t_1}^{t_0} \frac{Cdt}{S(t)}$$

$$\Rightarrow r_1 = \frac{C}{S_0} \int_{t_1}^{t_0} t_0^{2/3} . t^{-2/3} dt$$

$$\Rightarrow r_1 = \frac{C}{S_0} . t_0^{2/3} \int_{t_1}^{t_0} t^{-2/3} dt$$

$$\Rightarrow r_1 = \frac{3C}{S_0} . t_0 [1 - (\frac{t_1}{t_0})^{\frac{1}{3}}]$$

We now use equation $\frac{C\Delta t_0}{C\Delta t_1} = \frac{S(t_0)}{S(t_1)} = 1 + z$ to note that ,

$$1 + z = \frac{S(t_0)}{S(t_1)} = (\frac{t_0}{t_1})^{\frac{2}{3}}$$

Then we have ,

$$r_1 = \frac{3Ct_0}{S_0}[1-(1+z)^{-\frac{1}{2}}]$$

$$\Rightarrow r_1 = \frac{2C}{S_0 H_0}[1-(1+z)^{-\frac{1}{2}}] \qquad [\because t_0 = \frac{2}{3H_0}]$$

The Luminosity distance is therefore given by,

$$D_1 = r_1 S_0 (1+z)$$

$$= \frac{2C}{H_0}[(1+z)-(1+z)^{\frac{1}{2}}]$$

2.20 The Hoyle-Narlikar cosmologies :

We consider a theory of gravitation that may claim to have given the most direct quantative expression to Mach's principal .This theory was first proposed in 1964 *Fred Hoyle & J. V. Narlikar* .We will refer to it here as the HN theory and to the cosmological model based on it as HN cosmologies .Throughout this discussion we will set c =1 .

Figure: Fred Hoyle & J. V. Narlikar

Like general relativity & *Brans-Dicke* theory the HN theory is formulated in the Riemannian spacetime .There is one important difference however between this theory & all the cosmological theories we have discussed so far .The difference lies in the fact that general relativity , the *Brans-Dicke* theory & so on are pure field theories .Whereas the HN theory arose from the concept of direct interparticle action .The difference between the two types of theories is best seen from a description of electromagnetism to which we will frequently refer in this section & the next comparison until the advent of Maxwell's field theory , it was customary to describe electric & magnetic interactions as instance of direct action at a distance between particles .The success of *Maxwell's* theory established the concept of action at a distance .

Since the Mach's principal (implying as it does a connection between the local and distant) suggest action at a distance even an early convert to it like Einstein later became skeptical regarding its validity .Einstein's objections were based on the belief that action at a distance was supposed to be instantaneous and hence inconsistent with relativity .By the early 1960s however it had become clear that action at a distance can be made consistent with relativity and also successfully describe electrodynamics , besides having interesting cosmological implications .Since *Hoyle & Narlikar* had played an active role in these development , they naturally adopted an action at a distance approach to *Mach's* principal .

Accordingly we use here the some what unfamiliar notation of action at a distance .Let us denote by a, b, … … … the particles in the universe m_a & e_a being the mass and charge of the a^{th} particle .As implied by Mach the mass m_a is not entirely an intrinsic properties of particle a , it also owes its origin to the background provided by the rest of the universe**[43].**

To express this idea quantitatively write,

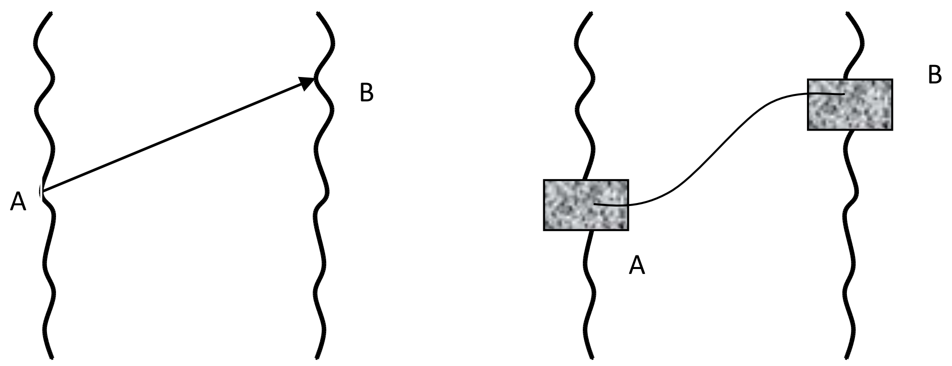
$$m_a(A) = \lambda_a \sum_{b \neq a} m^{(b)}(A)$$

The above expression means the following.

At atypical world point A on the world line of particle a , the mass acquired by a is the nett sum of contributions from all other particle b($\neq$ a) in the universe .The contribution from b at A is given by the scalar function $m^{(b)}$ (A) **.**

The coupling constant λ_a is intrinsic to the particle *a* .Notice however that if *a* were the only particle in the universe m_a **= *0*** & we have conclusion arrived.

Action of distance ***Field theory***



(a) **(b)**

Figure : **(a)** In the action at a distance picture the influence from the point A or the world line of particle a is transmitted directly across spacetime (along the arrow line) to the point B on the world line of particle b .

Figure : **(b)** In field theory the field in the neighborhood of A (shown by shaded region) is disturbed the disturbance prorogates across spacetime as a wave in the ambient field & reaches the neighborhood of B (also shaded in the shaded region) .The disturbance then exerts a force on particle b at B**[43].**

2.21 Derivation of de-Sitter model of the Universe :

For the *Robertson-Walker* metric in which the matter is in the from of a perfect fluid of mass energy density ρ & pressure p .So that energy momentum tensor is given by ,

$$T_{\mu\vartheta} = (p+\rho)U_{\mu}U_{\vartheta} - pg_{\mu\vartheta} \quad \dots \dots \dots (2.48)$$

With $U_{\mu} = (1,0,0,0)$ as we are in co-moving co-ordinates.

The cosmological principal which leads to the *Robertson-Walker* line element **[15,60]** namely,

$$ds^2 = c^2 \ dt^2 - R^2(t)[\frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + Sin^2\theta \ d\varphi^2)] \quad \dots \dots \dots (2.49)$$

The Einstein modified field equation are,

$$G_{\mu\vartheta} + \Lambda g_{\mu\vartheta} = -8\pi G T_{\mu\vartheta} \quad \dots \dots \dots (2.50)$$

Where Λ is the cosmological constant & $T_{\mu\vartheta}$ is the energy momentum tensor of the source producing radiation , gravitational matter dust , cloud , clusters , super clusters etc. and G is the Newtonian gravitational constant which is equal to one & $G_{\mu\vartheta}$ is called the Einstein tensor .

Now putting equation (2.48) in equation (2.50) we get ,

$$R_{\mu\vartheta} - \frac{1}{2} g_{\mu\vartheta} R + \Lambda g_{\mu\vartheta} = -8\pi \ [(p+\rho)U_{\mu}U_{\vartheta} - pg_{\mu\vartheta}]$$

When $\mu = \vartheta = 0$, we get

$$R_{00} - \frac{1}{2} g_{00} R + \Lambda g_{00} = -8\pi[(p+\rho)U_0 U_0 - p g_{00}]$$

Or, $\frac{3\ddot{R}}{R}$ - $\frac{1}{2}.1.\frac{6(R\ddot{R} + \dot{R}^2 + k)}{R^2} + \Lambda.1 = -8\pi[(p+\rho) - p]$

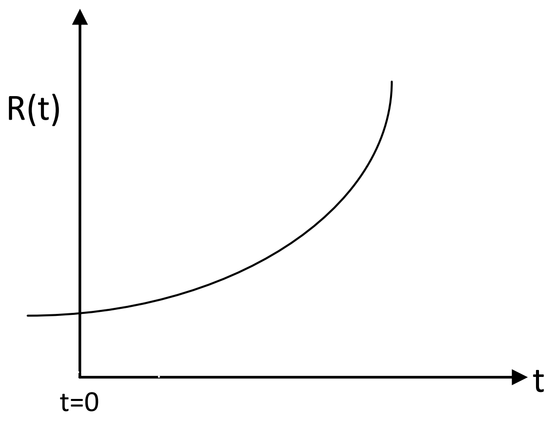
Or, $\left(\frac{\dot{R}}{R}\right) + \frac{k}{R^2}{}^{2} - \frac{\Lambda}{3} = \frac{8\pi\rho}{3}$

Or, $\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi\rho}{3} - \frac{k}{R^2} + \frac{\Lambda}{3}$

For early universe, ρ = 0 , P = 0 & k = 0

So the above equation becomes,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{\Lambda}{3}$$



Or, $H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{\Lambda}{3}$

Or, $H^2 = \frac{\Lambda}{3} = constant$

Now , H = $\frac{\dot{R}}{R}$

$$\Rightarrow \frac{1}{R}\frac{dR}{dt} = H$$

$$\Rightarrow \int \frac{dR}{R} = \int H dt$$

$$\Rightarrow \; logR = Ht + logR_0$$

$\therefore$ $\boldsymbol{R = R_0 \; exp(Ht)}$

Which is called the de-Sitter model or steady-state model of the universe.

Thus the de-Sitter line element is given by ,

$$ds^2 = dt^2 - R_0^2 e^{2Ht}[dr^2 + r^2(d\theta^2 + Sin^2\theta d\theta^2)] \quad \ldots \ldots \ldots (2.51)$$

Where, $R = R_0 \; exp(Ht)$ is not an an expantion but the expantion is an exponential .

2.22 The steady-state theory of the universe :

In 1948 there appeared a strong alternative to the evolutionary cosmologies based on general relativity in the form of the steady-state theory .The theory was proposed in two different versions , one by Fred Hoyle and the other by *Hermann Bondi* and *Thomas Gold* .Both versions adopted as a starting point the "Perfect Cosmological Model" ,meaning the assumption that as far as the large-scale features of the universe are concerned there is neither a privileged position nor a privileged time : the universe is spatially and temporally homogenous .Since Hoyle , Bondi **[4]** and Gold accepted the expansion of the universe , they were forced to introduce the radical hypothesis that matter's continually created throughout the universe assumedly in the form of hydrogen atoms or protons & electrons (without the hypothesis the matter density would decrease over time , contrary to the perfect cosmological principle) .As shown by Hoyle ,it followed from the theory that the average density of matter was given by $\rho = \frac{3H^2}{8\pi G}$

Where G is Newton's gravitational constant & the Hubble parameter H was a true constant contrary to the situation in the Big-Bang models (where H is a measure of the age of the universe) .To cancel the thinning out of matter as a result the expansion ,it was necessary to postulate creation of new matter at a rate of 3ρHor about 10^{-43}g/cm^3s .This exceedingly small value corresponding to the formation of three new hydrogen atoms per cubic meter per million years was far too small to have direct observational effects .*Bondi* and *Gold* further concluded that metric of the steady-state universe must be of the same type as in de-Sitter model of 1917 *ie.* a flat space (curvature parameter k=0) expanding exponentially .It followed that the deceleration parameter ,which is a dimensionless measure of the slowing down of the expansion and given by $q_0 = -[\frac{\ddot{R}}{RH^2}]_0$ had the value q_0=-1 .

(i)

(ii)

Figure : ***Thomas Gold*** , ***Hermann Bondi*** & ***Fred Hoyle*** (*from left*) in both picture (i) & (ii)

Shortly after the publication of the papers of *Hoyle* ,*Bondi* and *Gold* ,the steady-state theory was met with strong opposition the result being a protracted controversy the lasted until the mid -1960s.The critics of the theory accused it of building on impulsible hypothesis (such as the continual creation of matter) and being unable to account for observations .In order to counter these and other objections to the steady-state theory, Hoyle proposed to modify it in various ways, whereas the more rigid version of *Bondi* and Gold allowed virtually no changes. *William McCrea,* an early convert to the theory, argued in 1951-53 that continual creation of matter could be understood within the framework of standard general relativity. According to McCrea, matter creation did not really conflict with the law of energy conservation, and the theory also promised a unification of quantum mechanics and cosmology.

As far as observational tests were concerned, the situation remained unsettled for several years. The critics of the steady-state theory believed it could be shot down quickly, but this turned out not to be the case; on the contrary, the theory fared remarkably well. For example, attempts by *Allan Sandage* and his collaborators to test the theory by comparing its predicted redshift-magnitude relationship with observations failed to yield an unambiguous result. Different world models have different deceleration parameters and the steady-state value of $q_0 = -1$ distinguished the theory from most relativistic models. By measuring the redshift and magnitude of distant galaxies it would in principle be possible to determine q_0, and hence to decide if the steady-state theory was allowed. However, although *Sandage* concluded that

$q_0 > -1$, his data were not good enough to clearly rule out the theory.

The most serious challenge came from the new science of radio astronomy, and in the late 1950s data of radio sources obtained by Martin Ryle's group in Cambridge indicated an incurable disagreement with the prediction of the steady-state theory. A few years later, when the data had stabilized, nearly all radio astronomers agreed that they provided conclusive evidence against the steady-state theory. The supporters of the theory responded by producing alternative explanations of the radio source counts or by suggesting modified steady-state versions designed to cope with the problems. For a short while these responses kept the theory alive, but not much more.

In 1965 *Robert Wilson* and *Arno Penzias* unexpectedly detected an isotropic cosmic background of microwaves at wavelength 7.3 cm, which immediately was interpreted as a relic of the hot big bang. In fact, the background radiation had been predicted by *Ralph Alpher* and *Robert Herman* as early as 1948, but without attracting any attention. The sensational discovery of 1965 had no natural explanation within the framework of classical steady-state theory. In effect, the cosmic microwave background killed an already dying theory. However, the refutation of the classical steady-state theory, whether in the Hoyle version or the *Bondi-Gold* version, did not imply that the general idea of an eternally expanding universe with continual creation of matter had to be abandoned.

2.23 Steady-state model:

Astronomer Fred Hoyle frid a noble way of deploing field equation of general relativity starting with original research by Herman-Bondi and Thomas Gold in 1948 .He devised a non-static model of the universe whose general appearance remain unaltered forever .This is the steady-state model of continuous creation .Hoyle extended the cosmological principle to arrive at following –

"Not only does the universe appear the same to all observers but it looks the same in perpetuity."

The steady-state model has no singularity no beginning & no end .Space expands exponentially with time forward infinity .The Hubble constant (H) does not vary with time as in the evolving models where it decreases with time .Galaxies form evolve & disappear while the average density of matter in space remain constant .

To keep the population of galaxies or average density of matter, constant .We have to assume that new matter hydrogen is continuously being created.

As we know that Hubble constant $\frac{\dot{R}(t_0)}{R(t)}$ is an observable parameter .So that it must be independent of the present time in a steady-state model.

Letting H denotes the permanent value of the Hubble constant .We have thus, $H = \frac{\dot{R}(t_0)}{R(t)}$ for all t

Integrating with respect to t we get,

$$Ht + c = \ln R(t)$$

$$\Rightarrow \ln R(t) = Ht + c \quad \dots \dots \dots (2.52)$$

But if for t_0

$$R(t_0) = R_0$$

$$\therefore\ \ln R(t_0) = Ht_0 + c$$

$$\Rightarrow\ \ln R_0 = Ht_0 + c \quad \ldots\ \ldots\ \ldots\ (2.53)$$

Subtracting (2.53) from (2.52) we have,

$$\ln R(t) - \ln R(t_0) = H(t- t_0)$$

$$\Rightarrow \frac{R(t)}{R(t_0)} = e^{H(t-t_0)}$$

$$\Rightarrow R(t) = R_0 e^{H(t-t_0)}$$

$$\Rightarrow R(t) = R_0 e^{Ht}$$

$$\therefore R(t) \propto e^{Ht} \quad \ldots\ \ldots\ \ldots\ (2.54)$$

Which is the steady-state model of the universe where $H = \frac{1}{T}$ the red-shift parameter.

CHAPTER -3

CLASSICAL & QUANTIZATION PROBLEM

3.1 Introduction:

It is important to emphasize that our discussion in this chapter is based entirely on Classical and Quantization concept.

The critical fact that we shall need from quantum field theory is that quantum fields can produce an energy density that mimics a cosmological constant. The discussion will be restricted to the case of a scalar field φ (complex in general, but real field). As it is presently understood, and stated in the most general terms, inflation involves the dynamical evolution of a weakly coupled scalar field that was at one time displaced from the minimum of its potential. As such, the key to understanding the mechanics of inflation is scalar field dynamics in the expanding universe. The restriction of scalar fields is not simply for reasons of simplicity, but because the scalar sector of particle physics is relatively unexplored. While vector fields such as electromagnetism are well understood, it is expected in many theories of unification that additional scalar fields such as the Higgs field will exit. We now look at what these can do for cosmology. We have also use the Klein-Gordon field to describe it.

In order to make the analysis of the evolution of a scalar field manageable, it is necessary to make some simplifying assumptions. For simplicity we consider a Higgs field, which take to be a scalar field φ. The goal of this chapter is to explain a new potential by applying the scalar field and equation of motion and some new interior solutions are provided.

3.2 Classical Klein-Gordan Field :

The *Klein-Gordon* field defined with the *Lagrangian* density **[44]** ,

$$L_{KG} = \frac{1}{2}(\delta_{\mu}\varphi\delta^{\mu}\varphi - m^2\varphi^2) \quad \dots\dots\dots (3.1)$$

$$= \frac{1}{2}[\dot{\varphi}^2 - (\nabla\varphi)^2 - m^2\varphi^2]$$

The co-variant momentum density is more easily evaluated by re-writing,

$$L_{KG} = \frac{1}{2} g^{\mu\vartheta}(\delta_\mu \varphi \delta_\vartheta \varphi - m^2 \varphi^2)$$

Thus ,

$$\Pi^\mu = \frac{\delta L}{\delta(\delta_\mu \varphi)} = \frac{1}{2} g^{\mu\vartheta}(\delta_\mu^\alpha \delta_\vartheta \varphi + \delta_\mu \varphi \delta_\vartheta^\alpha)$$

$$= \frac{1}{2}(\delta_\mu^\alpha \delta^\mu \varphi + \delta^\vartheta \varphi \delta_\vartheta^\alpha)$$

$$= \frac{1}{2}(\delta^\alpha \varphi + \delta^\alpha \varphi)$$

$$= \delta^\alpha \varphi$$

Thus for the *Klein-Gordon* field we have,

$$\Pi^\alpha = \delta^\alpha \varphi \quad \ldots \ldots \ldots (3.2)$$

Giving the canonical momentum

$$\Pi = \Pi^0 = \delta^0 \varphi = \delta_0 \varphi = \dot{\varphi}$$

$$\therefore \Pi = \dot{\varphi} \quad \ldots \ldots \ldots (3.3)$$

$$L_{KG} = \frac{1}{2} g^{\mu\vartheta}(\delta_\mu \varphi \delta_\vartheta \varphi - m^2 \varphi^2)$$

$$\therefore L_{KG} = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\nabla \varphi)^2 - \frac{m^2 \varphi^2}{2}$$

$$\Rightarrow \frac{\delta L}{\delta \varphi} = 0 - 0 - \frac{2m^2 \varphi}{2}$$

$$\therefore \quad \frac{\delta L}{\delta \varphi} = -m^2 \varphi \quad \ldots \ldots \ldots (3.4)$$

The *Euler-Lagrange* equations give the field equation as $\delta_\mu \delta^\mu \varphi + m^2 \varphi$ or,

$$(\Box^2 + m^2)\varphi = 0 \quad \dots \dots \dots (3.5)$$

Which is the Klein-*Gordon* equation for a free massive scalar field .In momentum space

$$P^2 = - \Box^2$$

Then we get ,

$$(P^2 - m^2)\varphi = 0$$

The energy momentum tensor is,

$$T_{\mu\vartheta} = \Pi_\mu \delta_\vartheta \varphi - g_{\mu\vartheta} L$$

$$= \delta_\mu \varphi \delta_\vartheta \varphi - g_{\mu\vartheta} L$$

$$= \delta_\mu \varphi \delta_\vartheta \varphi - \frac{1}{2} g_{\mu\vartheta} (\delta_\alpha \varphi \delta^\alpha \varphi - m^2 \varphi^2) \quad \dots \dots \dots (3.6)$$

Therefore the *Hamiltonian* density,

$$H \equiv T_{00} = \delta_0 \varphi \delta_0 \varphi - \frac{1}{2} g_{00} (\delta_\alpha \varphi \delta^\alpha \varphi - m^2 \varphi^2)$$

$$\Rightarrow T_{00} = \frac{\delta\varphi}{\delta x^0} \frac{\delta\varphi}{\delta x^0} - \frac{1}{2} (\delta_\alpha \varphi \delta^\alpha \varphi - m^2 \varphi^2)$$

$$\Rightarrow T_{00} = \frac{\delta\varphi}{\delta t} \frac{\delta\varphi}{\delta t} - \frac{1}{2} (\delta_\alpha \varphi \delta^\alpha \varphi - m^2 \varphi^2)$$

$$\Rightarrow T_{00} = \dot{\varphi}^2 - \frac{1}{2} [\delta_0 \varphi \delta^0 \varphi - (\delta_1 \varphi \delta^1 \varphi + \delta_2 \varphi \delta^2 \varphi + \delta_3 \varphi \delta^3 \varphi) - m^2 \varphi^2]$$

[Considering in Minkowski-space]

$$\Rightarrow T_{00} = \dot{\varphi}^2 - \frac{1}{2}\left[\frac{\delta\varphi}{\delta x^0}\frac{\delta\varphi}{\delta x^0} - (\frac{\delta\varphi}{\delta x1}\frac{\delta\varphi}{\delta x^1} + \frac{\delta\varphi}{\delta x^2}\frac{\delta\varphi}{\delta x^2} + \frac{\delta\varphi}{\delta x^3}\frac{\delta\varphi}{\delta x^3}) - m^2\varphi^2\right]$$

$$\Rightarrow T_{00} = \dot{\varphi}^2 - \frac{1}{2}\left[\frac{\delta\varphi}{\delta t}\frac{\delta\varphi}{\delta t} - \{(\frac{\delta\varphi}{\delta x^1})^2 + (\frac{\delta\varphi}{\delta x^2})^2 + (\frac{\delta\varphi}{\delta x^3})^2\} - m^2\varphi^2\right]$$

$$\Rightarrow T_{00} = \dot{\varphi}^2 - \frac{1}{2}[\dot{\varphi}^2 - (\nabla\varphi)^2 - m^2\varphi^2]$$

$$\Rightarrow H \equiv T_{00} = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 \quad \ldots\ldots\ldots (3.7)$$

$$\therefore H = \frac{1}{2}[\Pi^2 + (\nabla\varphi)^2 + m^2\varphi^2] \quad \text{[From (3.3)]}$$

For free particle case namely $\Pi^\alpha = \delta^\alpha\varphi$ & $\Pi = \dot{\varphi}$

Solving *Euler-Lagrangian* equation now gives ,

$$(\Box^2 + m^2)\varphi + V' = 0$$

$$\Rightarrow \ddot{\varphi} - \nabla^2\varphi + m^2\varphi + V' = 0 \qquad [\because \ddot{\varphi} - \nabla^2\varphi = \Box^2]$$

The energy momentum tensor is the same as for the free particle case equation except for the addition of $g_{\mu\upsilon}V(\varphi)$ as in ,

$$T_{\mu\vartheta} = \delta_\mu\phi\delta_\vartheta\phi - g_{\mu\vartheta}[\frac{1}{2}(\delta_\alpha\phi\delta^\alpha\phi - m^2\varphi^2) - V(\varphi)] \ldots\ldots (3.8)$$

Yielding the *Hamiltonian* density the same as for the free particle case,

$$\therefore H \equiv T_{00} = \frac{1}{2}[\Pi^2 + (\nabla\varphi)^2 + m^2\varphi^2] + V(\varphi)$$

$$\therefore H = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 + V(\varphi) \quad \ldots\ldots\ldots (3.9)$$

The purely spatial components are ,

$$T_{ii} = \delta_i\phi\delta_i\phi - g_{ii}[\frac{1}{2}(\delta_\alpha\phi\delta^\alpha\phi - m^2\phi^2) - V(\varphi)] \quad \ldots \ldots (3.10)$$

With g_{ii} = - 1 we obtain ,

$$T_{ii} = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2}m^2\varphi^2 - V(\varphi) \quad \ldots \ldots \ldots (3.11)$$

Note that even though T_{ii} has repeated indices.

Let us not assume $\sum_i$ is implied in this case .That is T_{ii} refers to $T_{ii} = T_{11}=T_{22} = T_{33}$ and not $T_{ii} = T_{11} + T_{22} + T_{33}$.

Let us assume that the effects of the scalar field are averaged so as to behave like a perfect fluid.

Then we make the identification,

$$\varepsilon \equiv p \equiv \langle T_{oo} \rangle$$

$$\text{Or, } p \equiv \langle T_{oo} \rangle$$

Where $\varepsilon \equiv \rho$ is the energy density & p is the pressure.

So, from equation (3.9) we can write,

$$p \equiv \langle T_{oo} \rangle = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 + V(\varphi) \quad \ldots \ldots \ldots (3.12)$$

From equation (3.10) we get,

$$T_{ii} = \delta_i\varphi\delta_i\varphi - g_{ii}[\frac{1}{2}(\delta_\alpha\varphi\delta^\alpha\varphi - m^2\varphi^2) - V(\varphi)]$$

$$\Rightarrow (p+\rho)U_iU_i - pg_{ii} = \delta_i\varphi\delta_i\varphi - g_{ii}[\frac{1}{2}(\delta_\alpha\varphi\delta^\alpha\varphi - m^2\varphi^2) - V(\varphi)]$$

Putting i = 1,

$$(p+\rho)U_1U_1 - pg_{11} = \delta_1\varphi\delta_1\varphi - g_{11}\left[\frac{1}{2}(\delta_\alpha\varphi\delta^\alpha\varphi - m^2\varphi^2) - V(\varphi)\right]$$

$$\Rightarrow (p+\rho).0 - pg_{11} = 0 - g_{11}\left[\frac{1}{2}(\delta_\alpha \varphi \delta^\alpha \varphi - m^2\varphi^2) - V(\varphi)\right]$$

$$\Rightarrow p = \frac{1}{2}(\delta_\alpha \phi \delta^\alpha \phi - m^2\phi^2) - V(\varphi)$$

Putting α = 0, 1, 2 we get,

$$\Rightarrow p = \frac{1}{2}(\delta_0 \phi \delta^0 \phi + \delta_1 \phi \delta^1 \phi + \delta_2 \phi \delta^2 \phi + \delta_3 \phi \delta^3 \phi) - \frac{1}{2}m^2\phi^2 - V(\varphi)$$

$$\Rightarrow p = \frac{1}{2}[\dot{\varphi}^2 + (\frac{\delta\varphi}{\delta x^1})^2 + (\frac{\delta\varphi}{\delta x^2})^2 + (\frac{\delta\varphi}{\delta x^3})^2] - \frac{1}{2}m^2\phi^2 - V(\varphi)$$

$$\therefore p = \frac{1}{2}[\dot{\varphi}^2 + (\nabla\varphi)^2] - \frac{1}{2}m^2\phi^2 - V(\varphi) \quad \ldots \ldots \ldots (3.13)$$

So we finally get,

$$\therefore \rho = \frac{1}{2}[\dot{\varphi}^2 + (\nabla\varphi)^2] + \frac{1}{2}m^2\varphi^2 + V(\varphi) \quad \ldots \ldots \ldots (3.14)$$

$$\therefore p = \frac{1}{2}[\dot{\varphi}^2 + (\nabla\varphi)^2] - \frac{1}{2}m^2\varphi^2 - V(\varphi) \quad \ldots \ldots \ldots (3.15)$$

Let us also assume that the scalar field massless and that φ = φ (t) only *ie.*

$$\varphi \neq \varphi(x)$$

So the spatial derivative disappear

Therefore, we finally obtain,

$$\therefore \rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \quad \ldots \ldots \ldots (3.16)$$

$$\therefore p = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \quad \ldots \ldots \ldots (3.17)$$

3.3 Equation of states:

An expression for the rate of change of density $\dot{\rho}$ will be needed in terms of density ρ & pressure p **[44]**.

The first law of thermodynamics is ,

$$dU + dW = dQ \qquad \ldots\ldots\ldots (3.18)$$

Where U is the internal energy, W is the work & Q is the heat transfer.

Ignoring any heat transfer & *written*

$$dW = Fdr = pdV$$

Where F is the force, r is the distance, p is the pressure & V is the volume. Then,

$$dU = -pdV \quad \ldots\ldots\ldots \ (3.19)$$

Assuming that ρ is a relativistic energy density means that energy is expressed as,

$$U = \rho V$$

Differentiating,

$$\dot{U} = \dot{\rho}V + \rho\dot{V}$$

$$\Rightarrow -p\dot{V} = \dot{\rho}V + \rho\dot{V} \quad \text{[using (3.19)]} \quad \ldots\ldots\ldots (3.20)$$

Now writing , $V \propto r^3$ implies that $\dfrac{\dot{V}}{V} = \dfrac{3\dot{r}}{r}$

Thus we get , from (3.20),

$$\dot{\rho} + \frac{\rho\dot{v}}{V} = -\frac{\rho\dot{v}}{V}$$

$$\Rightarrow \dot{\rho} + 3\rho\frac{\dot{r}}{r} = -\rho.3.\frac{\dot{r}}{r}$$

$$\therefore \dot{\rho} = -3(\rho + \rho)\frac{\dot{r}}{r} \quad \ldots\ldots\ldots (3.21)$$

<u>*3.4 Velocity & Acceleration equation :*</u>

The *Friedmann* equation which specifies the speed of recession is obtained by writing the total energy E as the sum of kinetic plus potential energy terms (and using M = $\frac{4}{3}\pi r^3 \rho$)

$$\therefore E = T + V$$

$$= \frac{1}{2}m\dot{r}^2 - \frac{GMm}{r}$$

$$= \frac{1}{2}mr^2(\frac{\dot{r}^2}{r} - \frac{2GM}{r^3})$$

$$= \frac{1}{2}mr^2(H^2 - \frac{2G}{r^3} \times \frac{4}{3}\pi r^3 \rho)$$

$$= \frac{1}{2}mr^2(H^2 - \frac{8\pi G\rho}{3})$$

$$\Rightarrow \frac{2E}{mr^2} = H^2 - \frac{8\pi G\rho}{3}$$

$$\therefore H^2 = \frac{8\pi G\rho}{3} + \frac{2E}{mr^2} \quad \ldots\ldots\ldots (3.22)$$

Where the *Hubble* constant $H = \frac{\dot{r}}{r}$, m is the mass of a test particle in the potential energy field enclosed by a gas of dust of mass M, r the distance from the centre of dust to the test particle & G is Newtonian constant .

Defining K = $\frac{-2E}{m}$ & Writing the distance in terms of the scale factor R & a constant length S as r(t)=R(t) . It follows that, $\frac{\dot{r}}{r}=\frac{\dot{R}}{R}$ & $\frac{\ddot{r}}{r}=\frac{\ddot{R}}{R}$

Giving the *Friedmann* equation [from equation (3.22)]

$$\therefore H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G\rho}{3} - \frac{K}{R^2} \quad \ldots\ldots\ldots (3.23)$$

This specifies the speed of recession .The scale factor is introduced because in general relativity it is space itself .Even though this equation is derived for matter ,it is also true for radiation .in fact it is also true for vacuum ,with $\Lambda \equiv 8\pi G\rho_{vac}$ where Λ is the cosmological constant & ρ_{vac} is the vacuum energy density which just replaces the ordinary density .Exactly the same equation is obtained from the general relativistic Einstein field equation .According to *Guth* .K can be rescaled so that instead of being negative ,zero or positive it takes on the values -1 ,0 or +1.

From a Newtonian point of view this corresponds to unbound, critical or bound trajectories as mentioned above. From a geometric, general relativistic point of view this corresponds to an open, flat or closed universe. In elementary mechanics the speed *v* of a ball dropped from a height *r* is evaluated from the conservation of energy equation as *v* = $\sqrt{2gr}$, where *g* is the acceleration due to gravity. The derivation shown above is exactly analogous to such a calculation. Similarly the acceleration *a* of the ball is calculated as *a* = *g* from Newton's equation *F* = $m\ddot{r}$, where *F* is the force and the acceleration is $\ddot{r}=\frac{d^2r}{dt^2}$. The acceleration for the universe is obtained from Newton's equation

$$-G\frac{Mm}{r^2} = m\ddot{r} = F$$

Again using M $=\frac{4}{3}\pi r^3\rho$ & $\frac{\ddot{r}}{r}=\frac{\ddot{R}}{R}$ gives the acceleration equation,

$$-G\frac{m}{r^2}\times\frac{4}{3}\pi r^3\rho = m.\frac{\ddot{R}}{R}$$

$$\Rightarrow \frac{\ddot{R}}{R}\equiv\frac{\ddot{r}}{r}=-\frac{4\pi G\rho}{3} \quad \dots\dots\dots (3.24)$$

However because M $=\frac{4}{3}\pi r^3\rho$ was used, it is clear that this acceleration equation holds only for matter. In our example of the falling ball instead of the acceleration being obtained from Newton's Law, it can also be obtained by taking the time derivative of the energy equation.

Now taking the time derivative of equation (3.23) ,

$$\dot{R}^2 = \frac{8\pi G\rho R^2}{3} \qquad \text{[for k=o]}$$

$$\Rightarrow \frac{d}{dt}(\dot{R}^2) = \frac{8\pi G}{3}\frac{d}{dt}(\rho R^2)$$

$$\Rightarrow 2\dot{R}\ddot{R} = \frac{8\pi G}{3}(\dot{\rho}R^2 + 2R\dot{R}\rho)$$

$$\Rightarrow 2\dot{R}\ddot{R} = \frac{8\pi G}{3}[-3(\rho + p)\frac{\dot{R}}{R}.R^2 + 2R\dot{R}\rho]$$

$$\Rightarrow 2\dot{R}\ddot{R} = -\frac{8\pi G}{3}(\rho + 3p)\dot{R}R$$

$$\Rightarrow \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p)$$

$$\Rightarrow \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + \gamma\rho) \qquad [\because p = \frac{\gamma\rho}{3} \Rightarrow 3p = \gamma\rho]$$

$$\dots\dots\dots. (3.25)$$

$$\therefore \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(1+\gamma)\rho \quad \dots\dots\dots (3.26)$$

Which reduces to equation (3.24) for the matter equation of state (γ = 0).Exactly the same equation is obtained from the Einstein field equations.

3.5 Cosmological constant :

In both Newtonian and relativistic cosmology the universe is unstable to gravitational collapse. Both Newton and Einstein believed that the Universe is static. In order to obtain this Einstein introduced a *repulsive* gravitational force, called the cosmological constant, and Newton could have done exactly the same thing, had he believed the universe to be finite **[44].**

In order to obtain a possibly zero acceleration, a positive term (conventionally taken as $\frac{\Lambda}{3}$) is added to the acceleration equation (3.26) as ,

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \quad \ldots\ldots\ldots (3.27)$$

Which, with the proper choice of Λ can give the required zero acceleration for a static universe. Again exactly the same equation is obtained from the Einstein field equations. What has been done here is entirely equivalent to just adding a repulsive gravitational force in Newton's Law. The question now is how this repulsive force enters the energy equation (3.23).

Identifying the force from,

$$\frac{\ddot{r}}{r} = \frac{\ddot{R}}{R} \equiv \frac{F_{repulsive}}{mr} \equiv \frac{\Lambda}{3} \quad \ldots\ldots\ldots (3.28)$$

$$\Rightarrow F_{repulsive} = \frac{\Lambda}{3} mr \equiv -\frac{dv}{dr} \quad \ldots\ldots\ldots (3.29)$$

$$\Rightarrow \frac{dv}{dr} = -\frac{\Lambda}{3} mr$$

Integrating,

$$V = -\frac{\Lambda}{3} \cdot \frac{1}{2} mr^2$$

$$\therefore V = -\frac{\Lambda}{3} \cdot \frac{1}{2} mr^2 \quad \ldots \ldots \ldots (3.30)$$

This is just a repulsive simple harmonic oscillator. Substituting this into the conservation of energy equation,

E = T + V

$$= \frac{1}{2} m\dot{r}^2 - \frac{GMm}{r} - \frac{\Lambda}{3} \cdot \frac{1}{2} mr^2$$

$$= \frac{1}{2} mr^2 \left(\frac{\dot{r}^2}{r^2} - \frac{2G}{r^3} \times \frac{4}{3} \pi r^3 \rho - \frac{\Lambda}{3}\right)$$

$$= \frac{1}{2} mr^2 \left(H^2 - \frac{8\pi\rho G}{3} - \frac{\Lambda}{3}\right)$$

$$\Rightarrow \frac{2E}{mr^2} = H^2 - \frac{8\pi\rho G}{3} - \frac{\Lambda}{3}$$

$$\therefore H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi\rho G}{3} + \frac{\Lambda}{3} - \frac{k}{R^2} \qquad [\because \frac{-2E}{mr^2} = \frac{k}{R^2}]$$

$$\ldots \ldots \ldots (3.31)$$

Equations (3.31) and (3.27) constitute the fundamental equations of motion that are used in all discussions of Friedmann models of the Universe.

Let us comment on the repulsive harmonic oscillator obtained above. Recall one of the standard problems often assigned in mechanics courses. The problem is to imagine that a hole has been drilled from one side of the Earth, through the center and to the other side. One is to show that if a ball is dropped into the hole, it will execute harmonic motion. The solution is obtained by noting that whereas gravity is an inverse square law for point masses M and m separated by a distance r as given by $F = \frac{GMm}{r^2}$, yet if one of the masses is a continuous mass distribution represented by a density then $F = G\frac{4\pi\rho mr}{3}$. The force rises linearly as the distance is increased because the amount of matter enclosed keeps increasing. Thus the gravitational force for a continuous mass distribution rises like Hooke's law and thus oscillatory solutions are encountered. This sheds light on our repulsive oscillator found above. In this case we want the gravity to be repulsive, but the cosmological constant acts just like the uniform matter distribution.

Finally authors often write the cosmological constant in terms of a vacuum energy density as $\Lambda \equiv 8\pi G\rho_{vac}$ so that the velocity and acceleration equations become,

$$\therefore H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi\rho G}{3} + \frac{\Lambda}{3} - \frac{k}{R^2} = \frac{8\pi G}{3}(\rho_{vac} + \rho) - \frac{k}{R^2} \quad \ldots\ldots\ldots (3.32)$$

And

$$\frac{\ddot{R}}{R} = \frac{-4\pi G}{3}(1+\gamma)\rho + \frac{\Lambda}{3} = \frac{-4\pi G}{3}(1+\gamma)\rho + \frac{8\pi G}{3}\rho_{vac} \quad \ldots\ldots\ldots (3.33)$$

From equation (3.31) & (3.33) we get,

$$\therefore H^2 = 8\pi G[\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 + V(\varphi)] - \frac{k}{R^2} + \frac{\Lambda}{3} \quad \ldots\ldots\ldots (3.34)$$

And,

$$\frac{\ddot{R}}{R} = \frac{-4\pi G}{3}[\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2}m^2\varphi^2 - V(\varphi)] + \frac{\Lambda}{3} \quad \ldots\ldots\ldots (3.35)$$

Again we have,

$$\rho = \frac{1}{2}[\dot{\varphi}^2 + (\nabla\varphi)^2] + \frac{1}{2}m^2\phi^2 + V(\varphi) \quad \ldots\ldots\ldots (3.36)$$

$$p = \frac{1}{2}[\dot{\varphi}^2 + (\nabla\varphi)^2] - \frac{1}{2}m^2\varphi^2 - V(\varphi) \quad \ldots\ldots\ldots (3.37)$$

And we know,

$$\dot{\rho} + \frac{3\dot{R}}{R}(p+\rho) = 0 \quad \ldots\ldots\ldots (3.38)$$

Adding (3.36) & (3.37) we get ,

$$p + \rho = \dot{\varphi}^2 + (\nabla\varphi)^2$$

Differentiating equation (3.36) we get,

$$\dot{\rho} = \dot{\varphi}\ddot{\varphi} + V'(\varphi)\dot{\varphi} + m^2\varphi\dot{\varphi} + 0$$

$$\Rightarrow \dot{\rho} = [\ddot{\varphi} + V'(\varphi) + m^2\varphi]\dot{\varphi}$$

Now putting the values in equation (3.38) we get ,

$$[\ddot{\varphi} + V'(\varphi) + m^2\varphi]\dot{\varphi} + \frac{3\dot{R}}{R}[\dot{\varphi}^2 + (\nabla\varphi)^2] = 0$$

$$\Rightarrow \ddot{\varphi} + V'(\varphi) + m^2\varphi + 3H\left(\dot{\varphi} + \frac{(\nabla\varphi)^2}{\dot{\varphi}}\right) = 0$$

The difference occurs because we have now incorporated gravity via the Friedmann and conservation equation. We shall derive this equation again.

Again assuming the field is massless and ignoring spatial derivatives we have,

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0 \quad \ldots\ldots\ldots (3.39)$$

Notice that this is the equation for a damped harmonic oscillator

$(V = \frac{1}{2}kx^2 \;\&\; \frac{dv}{dx} \equiv V' = kx$ with $F = V')$ as,

$$m\ddot{x} + d\dot{x} + kx = 0$$

Kolb & Turmer also include a particle creation term due to decay of the scalar field,

$$\ddot{\varphi} + 3H\dot{\varphi} + \Gamma\dot{\varphi} + V' = 0 \quad \ldots\ldots\ldots (3.40)$$

Alternative derivation :

Consider a Lagrangian for φ which already has the scale factor built into it as

$$L = R^3[\frac{1}{2}(\delta_\mu \varphi \delta^\mu \varphi - m^2\varphi^2) - V(\varphi)] \quad \ldots\ldots\ldots (3.41)$$

Where the R^3 factor comes from $R^3 = \sqrt{-g}$ for a Robertson-Walker metric. The equation of motion is,

$$\ddot{\varphi} + V'(\varphi) + m^2\varphi + 3H\dot{\varphi} - (\nabla\varphi)^2 = 0 \quad \ldots\ldots\ldots (3.42)$$

However if $m = 0$ and $\nabla\varphi = 0$ it is the same as $\ddot{\varphi} + 3H\dot{\varphi} + V' = 0$

Let's only consider,

$$L = R^3[\frac{1}{2}\dot{\varphi}^2 - V(\varphi)]$$

which results from setting $m = 0$ and $\nabla\varphi = 0$ in (3.41). The equation of motion is,

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0 \quad \ldots\ldots\ldots (3.43)$$

Identifying the Lagrangian as,

$$L = R^3(T - V)$$

we immediately write down the total energy density,

$$\rho = T + V = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)$$

Taking the time derivative,

$$\dot{\rho} = \frac{1}{2}.2\dot{\varphi}\ddot{\varphi} + \frac{d}{dt}V(\varphi)$$

$$\Rightarrow \dot{\rho} = \dot{\varphi}\ddot{\varphi} + \frac{dV}{d\varphi}.\frac{d\varphi}{dt}$$

$$\Rightarrow \dot{\rho} = \dot{\varphi}\ddot{\varphi} + V'\dot{\varphi}$$

$$\Rightarrow \dot{\rho} = (\ddot{\varphi} + V')\dot{\varphi}$$

$$\therefore \dot{\rho} = -3H\dot{\varphi}^2 \qquad \text{[From (3.43)]}$$

& substituting into the conservation equation,

$$\dot{\rho} + 3H(p + \rho) = 0$$

$$\Rightarrow \dot{\rho} = -3H(p + \rho)$$

We obtain the pressure as,

$$p = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)$$

Thus our energy density and pressure derived here agree with our results above,

$$\rho = \frac{1}{2}[\dot{\varphi}^2 + (\nabla\varphi)^2] + \frac{1}{2}m^2\phi^2 + V(\varphi)$$

$$p = \frac{1}{2}[\dot{\varphi}^2 + (\nabla\varphi)^2] - \frac{1}{2}m^2\phi^2 - V(\varphi)$$

Notice that the pressure is nothing more than,

$$p = \frac{L}{R^3}$$

3.6 Limiting solution :

We have the Friedmann equation,

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi\rho G}{3} + \frac{\Lambda}{3} - \frac{k}{R^2}$$

Assuming that $k = \Lambda = 0$ the Friedmann equation becomes,

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi\rho G}{3} + 0 - 0$$

$$\Rightarrow H^2 = \frac{8\pi G}{3}[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)]$$

This equation together with equation,

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0 \quad \ldots\ldots\ldots (3.44)$$

form a set of coupled equations where solutions give $\varphi(t)$ and $R(t)$. We solve the coupled equations in the standard way by first eliminating one variable, then solving one equation, then substituting the solution back into the other equation to solve for the other variable.

Let's write equation (3.44) purely in terms of φ by eliminating R which appears in the form $H = \frac{\dot{R}}{R}$.

We eliminate R by substituting H in equation (3.44),

$$\ddot{\varphi} + V' = -3H\dot{\varphi}$$

$$\Rightarrow (\ddot{\varphi} + V')^2 = 9H^2\dot{\varphi}^2 = 9\dot{\varphi}^2 \frac{8\pi G}{3}[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)]$$

$$\Rightarrow \ddot{\varphi}^2 + 2\ddot{\varphi}V' + V'^2 = 3\dot{\varphi}^2 8\pi G \left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right]$$

$$\Rightarrow \ddot{\varphi}^2 + 2\ddot{\varphi}V' + V'^2 - 12\pi G(\dot{\varphi}^2 + 2V)\dot{\varphi}^2 = 0 \; \ldots \ldots \ldots \; (3.45)$$

$$\Rightarrow (\ddot{\varphi} + V')^2 = 12\pi G(\dot{\varphi}^2 + 2V)\dot{\varphi}^2$$

$$\Rightarrow \ddot{\varphi} + V' = \pm\sqrt{12\pi G(\dot{\varphi}^2 + 2V)}\dot{\varphi} \quad \ldots \ldots \ldots \; (3.46)$$

$$\therefore \ddot{\varphi} - \sqrt{12\pi G(\dot{\varphi}^2 + 2V)}\dot{\varphi} + V' = 0 \quad \text{(Taking only positive)} \ldots \ldots \; (3.47)$$

Notice that this is a *non-linear* differential equation for *φ*, which is difficult to solve in general. In this section we shall study the solutions for certain limiting cases.

Potential Energy = 0

Setting $V = 0$ we then have,

$$\rho = \frac{1}{2}\dot{\varphi}^2 + 0 = \frac{1}{2}\dot{\varphi}^2 \qquad \& \qquad p = \frac{1}{2}\dot{\varphi}^2 - 0 = \frac{1}{2}\dot{\varphi}^2$$

$$\therefore \rho = p$$

Or, γ= 3.

With V = V' = 0 then equation (3.45) becomes,

$$\ddot{\varphi}^2 + 0 + 0 - 12\pi G(\dot{\varphi}^2 + 0)\dot{\varphi}^2 = 0$$

$$\Rightarrow \ddot{\varphi}^2 = 12\pi G\dot{\varphi}^4$$

$$\Rightarrow \ddot{\varphi} = \sqrt{12\pi G}\dot{\varphi}^2$$

$$\therefore \ddot{\varphi} - \sqrt{12\pi G}\dot{\varphi}^2 = 0 \quad \ldots \ldots \ldots \; (3.48)$$

Which has the solution,

$$\varphi(t) = \varphi_0 + \frac{1}{\sqrt{12\pi G}}\ln[1 + \sqrt{12\pi G}\dot{\varphi}(t - t_0)] = 0 \; \ldots \ldots \ldots \; (3.49)$$

Upon substituting this solution back into the Friedmann equation and solving the differential equation we have the process,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\varphi}_0^{\ 2} + V\right]$$

$$\Rightarrow \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\,[\frac{1}{2}\dot{\varphi}_0^{\ 2} + 0] = \frac{4\pi G}{3}\ \dot{\varphi}_0^{\ 2}$$

$$\Rightarrow \frac{\dot{R}}{R} = \sqrt{\frac{4\pi G}{3}}\dot{\varphi}_0$$

$$\Rightarrow \frac{1}{R}(\frac{dR}{dt}) = \sqrt{\frac{4\pi G}{3}}\dot{\varphi}_0$$

$$\Rightarrow \frac{dR}{R} = \sqrt{\frac{4\pi G}{3}}\dot{\varphi}_0 dt$$

Integrating,

$$\log R = \sqrt{\frac{4\pi G}{3}}\dot{\varphi}_0 t + \log R_0$$

$$\Rightarrow R = R_0 e^{\sqrt{\frac{4\pi G}{3}}\dot{\varphi}_0 t}$$

$$\Rightarrow \frac{R}{R_0} = e^{\sqrt{\frac{4\pi G}{3}}\dot{\varphi}_0 t}$$

$$\Rightarrow \left(\frac{R}{R_0}\right)^3 = (e^{\sqrt{\frac{4\pi G}{3}}\dot{\varphi}_0 t})^3$$

$$\Rightarrow \left(\frac{R}{R_0}\right)^3 = e^{\sqrt{\frac{9\times 4\pi G}{3}}\dot{\varphi}_0 t}$$

$$\Rightarrow \left(\frac{R}{R_0}\right)^3 = e^{\sqrt{12\pi G}\dot{\varphi}_0 t}$$

$$\Rightarrow \left(\frac{R}{R_0}\right)^3 = (1+\sqrt{12\pi G}\dot{\varphi}_0 t + ...)$$ [by expansion of e^x]

$$\therefore R(t) = R_0[1+\sqrt{12\pi G}\dot{\varphi}(t-t_0)]^{1/3} \qquad \ldots\ \ldots\ \ldots\ (3.50)$$

This result may be understood from another point of view. Writing the Friedmann equations as,

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi\rho G}{3} \qquad \& \qquad \rho = \frac{\alpha}{R^m}$$

Then the solution is always $R \propto t^{\frac{2}{m}}$ which always gives $\rho \propto \frac{1}{t^2}$

If ρ = constant then the solution is $R \propto e^t$ for m < 2, one obtains power law inflation. For ordinary matter (m = 3) we have $R \propto t^{\frac{2}{3}}$, or radiation (m = 4) we have and $R \propto t^{\frac{1}{2}}$ respectively.

Returning to the scalar field solution the density is $\rho = \frac{1}{2}\dot{\varphi}^2$ for V=0 thus differentiating equation (3.49) we get ,

$$\dot{\varphi}(t) = 0 + \frac{1}{\sqrt{12\pi G}} \cdot \frac{\sqrt{12\pi G}\dot{\varphi}_0}{1+\sqrt{12\pi G}\dot{\varphi}(t-t_0)}$$

$$\Rightarrow \dot{\varphi}(t) = \frac{\dot{\varphi}_0}{1+\sqrt{12\pi G}\dot{\varphi}(t-t_0)}$$

But,

$$(\frac{R}{R_0})^3 = 1+\sqrt{12\pi G}\dot{\varphi}_0(t-t_0)$$

this yields,

$$\dot{\varphi}(t) = \dot{\varphi}_0 \frac{R_0^3}{R^3}$$

To give the density,

$$\rho = \frac{1}{2}\dot{\varphi}^2 + 0$$

$$\therefore \rho = \frac{1}{2}\dot{\varphi}^2 \frac{R_0^6}{R^6}$$

Corresponding to m = 6 & thus $R = t^{\frac{1}{3}}$ in agreement with,

$$R(t) = R_0[1 + \sqrt{12\pi G}\dot{\varphi}(t - t_0)]^{\frac{1}{3}}$$

Note also that this density $\rho \propto \frac{1}{R^6}$ also gives, $\rho \propto \frac{1}{t^2}$.

Thus for a scalar field with v=0 we have $\rho = p\ (\gamma = 3)$ & $\rho \propto \frac{1}{R^6}$ contrast this with matter for which p=0 $(\gamma = 0)$ & $\rho \propto \frac{1}{R^3}$ or radiation for which $p = \frac{1}{3}\rho(\gamma = \frac{1}{3})$ & $\rho \propto \frac{1}{R^4}$.

However equation

$$\therefore \rho = \frac{1}{2}\dot{\varphi}^2 \frac{R_0^6}{R^6}$$

may not be interpreted as a decaying cosmological constant because $p \neq \rho$.

Kinetic energy:

Here we take $\dot{\varphi} = \ddot{\varphi} = 0$

$$\rho = \frac{1}{2} \times 0 + V \qquad\qquad p = \frac{1}{2} \times 0 - V$$

$$\therefore \rho = V \qquad \& \qquad \therefore p = -V$$

So we can write, p = - ρ

Or γ =-3 which is a negative pressure equation of state .Our equation of motion for scalar field

$$\ddot{\varphi}^2 + 2\ddot{\varphi}V' + V'^2 - 12\pi G(\dot{\varphi}^2 + 2V)\dot{\varphi}^2 = 0$$

becomes $V' = 0$.

Which meaning that V=V_0 which is constant .Now putting this value in Freidman equation we get ,

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3} V_0$$

which acts as a cosmological constant & which has the solution,

$$R(t) = R_0 e^{\sqrt{\frac{8\pi G}{3} V_0}\,(t-t_0)} \quad \ldots\ldots\ldots (3.50^*)$$

Which is an inflationary solution valid for any V .

3.7 Exactly solvable model of Inflation:

We shall examine the model of *Barrow* which can be solved exactly leads to power law inflation. The advantage of an exactly solvable model is that one can develop ones physical inflation better **[44].**

Any scalar field model is specified by writing down the potential V(φ). *Barrow's* potential is,

$$V(\varphi) \equiv \beta e^{-\lambda\varphi} \quad \ldots\ldots\ldots (3.51)$$

Where β & λ are constants to be determined *Barrow* claims that a particular solution to equation (3.47) is,

$$\varphi(t) = \sqrt{2A}\ln t \quad \ldots\ldots\ldots (3.52)$$

Where $\sqrt{2A}$ is just some constant .we check this by putting (3.51) & (3.52) in (3.47) we get,

$$\ddot{\varphi}(t) = \frac{\sqrt{2A}}{t}$$

$$\Rightarrow \dot{\varphi}(t) = -\frac{\sqrt{2A}}{t^2} \qquad \text{[Differentiating]}$$

We get ,

$$\ddot{\varphi}^2 + 2\ddot{\varphi}V' + V'^2 - 12\pi G(\dot{\varphi}^2 + 2V)\dot{\varphi}^2 = 0$$

$$\Rightarrow \frac{2A}{t^4} - \frac{2\sqrt{2A}}{t^2}V' - 12\pi G(\frac{2A}{t^2} + 2V)\frac{2A}{t^2} + V'^2 = 0$$

$$\Rightarrow \frac{2A}{t^4} - \frac{2\sqrt{2A}}{t^2}V' - 12\pi G.\frac{2A}{t^2} - 24\pi G.\frac{2A}{t^2}.\beta e^{-\lambda\varphi} + V'^2 = 0$$

From the above result we get,

$$\lambda = \sqrt{\frac{2}{A}}$$

& $\qquad \beta = -A$

Or, $\beta = A(24\pi GA - 1)$ … … … (3.53)

Note that *Barrow* is wrong when he writes $\lambda A=\sqrt{2}$ .Also he uses units with $8\pi G=1$.

So equation (3.53) can be written as,

$$\beta = A(3A - 1)$$

having solved for φ(t) we now substitute in , $H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi\rho G}{3}$ to solve for R(t).

Substituting, V = $\frac{\Lambda}{t^2}$ & $\dot{\varphi} = \frac{\sqrt{2A}}{t}$ we get,

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}(\frac{1}{2}\dot{\varphi}^2 + V)$$

$$= \frac{8\pi G}{3}(\frac{1}{2}.\frac{2A}{t^2} + \frac{\beta}{t^2})$$

$$= \frac{8\pi G}{3}(A + \beta)\frac{1}{t^2} \quad \text{……… (3.54)}$$

So we can write,

$$\rho = (A + \beta)\frac{1}{t^2}$$

Clearly we reject β = -A .It would give zero density .using β = A(24πGA -1) yields,

$$\rho = \frac{24\pi G A^2}{t^2} \quad \text{……… (3.55)}$$

Now from equation (3.54) we get,

$$R \propto t^{8\pi G A}$$

Where D is some constant .Setting 8πG =1 we have,

$$R \propto t^A$$

In agreement with *Barrow*'s solution .Power law inflation results for A > 1 .

Inverting solution $R \propto t^A$ we have,

$$t^2 = c' R^{\frac{2}{A}}$$

Where c' is some constant .Substituting this in equation (3.55) we get,

$$\rho = \frac{24\pi G A^2}{C' R^{2/A}}$$

$$\text{Or,} \quad \rho \propto \frac{1}{R^{2/A}}$$

Which corresponds to weak decaying cosmological constant .For the inflationary result A >1 .We have $\frac{2}{A} \equiv m < 2$ which corresponds to the quantum tunneling solution.

Again, $\rho \propto \frac{1}{R^{2/A}}$ can also be obtained via

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V$$

We have,

$$V = \frac{\beta}{t^2} \propto \frac{1}{R^{2/A}}$$

$$\& \qquad \dot{\varphi}(t) = \frac{\sqrt{2A}}{t}$$

Giving,

$$\dot{\varphi}^2 \propto \frac{1}{R^{2/A}} \qquad \ldots\ldots\ldots (3.56)$$

3.8 Cosmological constant & scalar field :

We have,

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)$$

$$p = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)$$

For the case where potential energy V= 0 then we have, P = ρ

The pressure is positive & therefore we get,

$$\rho \propto \frac{1}{R^6}$$

Which can not be interpreted as a cosmological constant.

For the case where kinetic energy $= \frac{1}{2}\dot{\varphi}^2 = 0$

We have P = - ρ meaning that ρ can be interpreted as a strong cosmological constant.

These results are true in general (assuming m =∇φ=0) as we have not yet specified V(φ).

Let us now consider the *Barrow* model,

$$V(\varphi) = \beta e^{-\lambda\varphi}$$

We found that,

$$\varphi(t) = \frac{2}{\lambda}\ln t \quad \& \quad \beta = \frac{2}{\lambda^2}(\frac{6}{\lambda^2} - 1) \text{ for } 8\pi G = 1$$

Introducing $A \equiv \frac{2}{\lambda}$ we can re-write as

$$\beta = A(3A - 1) \quad \& \quad \varphi(t) = \sqrt{2A}\ln t \quad \& \quad V(\varphi) = A(3A-1).e^{\varphi.\frac{2}{A}}$$

Substituting we get,

$$\rho_{Barrow} = \frac{3A^2}{t^2} \quad \& \quad \rho_{Barrow} = \frac{3A^2}{t^2}(\frac{2A}{3} - 1) = (\frac{2A}{3} - 1)\rho$$

The general equation of state is $P = \frac{\gamma}{3}\rho$ giving the *Barrow* equation of state,

$$\gamma_{Barrow} = \frac{2}{A} - 3 \quad \dots\dots\dots (3.57)$$

Now in A > 1 we conclude that power law inflation results for A > 1 . Substituting this in equation (3.57) we get ,

$$\gamma_{Barrow} < -3$$

Which expect because power law inflation implies $\ddot{R} > 0$ .Thus for A > 1 the *Barrow* pressure is negative with γ < -1 & $\rho_{Barrow} = \frac{3A^2}{t^2}$ corresponds to a weak cosmological constant .Furthermore this cosmological constant is variable & decays with time.

Now we can write,

$$\rho_{Barrow} \propto \frac{1}{R^{2/A}} \quad \dots\dots\dots (3.58)$$

<u>3.9 Density fluctuation:</u>

An important result that we shall use without proof is that fluctuations of the scalar field are given approximately by,

$$\delta_{\varphi} \approx \frac{H}{2\pi}$$

Using $\rho = \frac{1}{2}\dot{\varphi}^2 + V$ We have, $\frac{d\rho}{d\varphi} = 0 + V'(\varphi)$

$$\Rightarrow \delta\rho = V'(\varphi)\delta\varphi \approx V'\frac{H}{2\pi}$$

Now assuming $\dot{\varphi} = 0$ gives , We know, $$H^2 = \frac{8\pi G}{3}(\frac{1}{2}\dot{\varphi}^2 + V)$$

$$\Rightarrow H^2 = \frac{8\pi G}{3}(0 + V)$$

$$\therefore H^2 = \frac{8\pi GV}{3}$$

$$\therefore H = \sqrt{\frac{8\pi GV}{3}}$$

$$\therefore \frac{\delta\rho}{\rho} = \frac{\frac{V'\sqrt{\frac{8\pi GV}{3}}}{2\pi}}{0 + V} \qquad [\because \rho = \frac{1}{2}\dot{\varphi}^2 + V = 0 + V(\varphi)]$$

$$\Rightarrow \frac{\delta\rho}{\rho} = \frac{\frac{V'\sqrt{\frac{8\pi GV}{3}}}{2\pi}}{V} = V'\sqrt{\frac{2G}{3\pi V}} = \frac{V'}{M_p}\sqrt{\frac{2}{3\pi V}}$$

where we have used $G \equiv \frac{1}{M_p}$

For the chaotic inflation model $V(\varphi) = \frac{1}{2}m^2\phi^2$ this yields,

$$\Rightarrow \frac{\delta\rho}{\rho} = \frac{V'}{M_p}\sqrt{\frac{2}{3\pi}}\frac{1}{\sqrt{\frac{1}{2}m^2\varphi^2}} = \frac{V'}{M_p}\sqrt{\frac{2}{3\pi}}\frac{\sqrt{2}}{m\varphi}$$

$$\Rightarrow \frac{\delta\rho}{\rho} = \frac{m^2\varphi}{M_p}\sqrt{\frac{2}{3\pi}}\frac{\sqrt{2}}{m\varphi} \qquad [\because V' = m^2\varphi]$$

$$\Rightarrow \frac{\delta\rho}{\rho} = \frac{m}{M_p}\sqrt{\frac{4}{3\pi}}$$

This is an intensely important formula often written as,

$$\frac{\delta\rho}{\rho} \approx m\sqrt{G} = \frac{m}{M_p}$$

The density fluctuations observed by COBE are $\frac{\delta\rho}{\rho} \approx 10^{-5}$ yielding m = 10^{-5} M_p.

The above formula is not very useful for Barrow model where a well defined inflation means m = $m_{inflation}$ is not present .In that case the formula is written more usefully as,

$$\frac{\delta\rho}{\rho} = \frac{\rho_{\text{inflation}}}{\rho_P} \quad \dots\dots\dots (3.59)$$

3.10 Equation of state for variable cosmological constant:

In this section we wish to demonstrate that variable cosmological constant models have negative pressure.

Firstly if one assumes, $P \equiv \frac{\gamma\rho}{3}$ ……… (3.60)

Then the conservation law follows as ,

$$\frac{1}{R^{3+\gamma}}\frac{d}{dR}(\rho R^{3+\gamma}) = -\rho_v' \quad \dots\dots\dots (3.61)$$

Let's assume that

$$\rho_v \equiv \frac{\alpha}{R^m} = \rho_{v_0}(\frac{R_0}{R})^m \quad \dots\dots\dots (3.62)$$

Integrating equation (3.61) we have,

$$\rho = \frac{1}{R^{3+\gamma}}[\rho_0 - \frac{m}{3+\gamma-m}\rho_{v_0}]R_0^{3+\gamma} + \frac{m}{3+\gamma-m}\rho_v$$

$$\Rightarrow \rho = \frac{A}{R^{3+\gamma}} + \chi\rho_v \quad \ldots\ldots\ldots (3.63)$$

Where , $\chi = \dfrac{m}{3+\gamma-m}$ and A is a constant given by ,

$$A = [\rho_0 - \chi\rho_{v_0}]R_0^{3+\gamma}$$

Now from equation (3.60) we get the pressure is ,

$$P \equiv \frac{\gamma\rho}{3} = \frac{\gamma}{3}.\frac{A}{R^{3+\gamma}} + \frac{\gamma}{3}.\frac{m}{3+\gamma-m}\rho_v \quad \ldots\ldots\ldots (3.64)$$

This looks like bad news .Assuming that ρ_v dominate over the first term at some stage of evolution it looks like the pressure only get negative for m > 3+γ . However there are two things to keep in mind .Firstly the pressure P is not the pressure of radiation of matter or vacuum because $\rho = \dfrac{A}{R^{3+\gamma}} + \chi\rho$ & $P = \dfrac{\gamma\rho}{3}$.

The pressure that we would want to be negative would be the vacuum pressure P_γ which shall work out below .Secondly the key point is not as much having P negative but rather having $\ddot{R}$ positive.

The equation,

$$-qH^2 \equiv \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(p+3\rho) + \frac{8\pi G}{3}\rho_v \quad \ldots\ldots\ldots (3.65)$$

Can still give positive $\ddot{R}$ even if P is not negative, because the ρ_v term has to be considered .The *Friedman* equation is ,

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = -\frac{8\pi G}{3}(\rho_v + \rho) - \frac{k}{R^2} \quad \ldots\ldots\ldots (3.66)$$

Now putting the value of ρ from the equation (3.74) in this equation,

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}[\frac{A}{R^{3+\gamma}} + (\chi + 1)\rho_v] - \frac{k}{R^2} \quad \ldots \ldots \ldots (3.67)$$

Where,

$$\chi + 1 = \frac{m}{3+\gamma-m} + 1 = \frac{3+\gamma}{3+\gamma-m}$$

Also assuming $P = \frac{\gamma\rho}{3} \Rightarrow 3P = \gamma\rho$ in equation (3.65) we get ,

$$-qH^2 \equiv \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(1+\gamma)\rho + \frac{8\pi G}{3}\rho_v$$

$$= -\frac{4\pi G}{3}(1+\gamma)(\frac{A}{R^{3+\gamma}} + \chi\rho_v) + \frac{8\pi G}{3}\rho_v$$

$$= -\frac{4\pi G}{3}[(1+\gamma)(\frac{A}{R^{3+\gamma}} + \chi\rho_v) - 2\rho_v]$$

$$= -\frac{4\pi G}{3}[(1+\gamma)\rho - 2\rho_v] \quad \ldots \ldots \ldots (3.68)$$

$$= -\frac{4\pi G}{3}\left[(1+\gamma)\frac{A}{R^{3+\gamma}} + (1+\gamma)\frac{m}{3+\gamma-m}\rho_v - 2\rho_v\right]$$

$$= -\frac{4\pi G}{3}\left[(1+\gamma)\frac{A}{R^{3+\gamma}} + (\frac{m+\gamma m-6-2\gamma+2m}{3+\gamma-m})\rho_v\right]$$

$$= -\frac{4\pi G}{3}\left[(1+\gamma)\frac{A}{R^{3+\gamma}} + (\frac{3m+\gamma m-6-2\gamma}{3+\gamma-m})\rho_v\right]$$

$$= -\frac{4\pi G}{3}\left[(1+\gamma)\frac{A}{R^{3+\gamma}} + \frac{(3+\gamma)(m-2)}{3+\gamma-m}\rho_v\right] \quad \ldots \ldots \ldots (3.69)$$

Having established that a decaying cosmological constant can lead to negative pressure .Let us now work out the vacuum equation of state for a decaying cosmological constant.

From equation (3.68) let us define,

$$\tilde{\rho} \equiv \frac{A}{R^{3+\gamma}} \quad \& \quad \tilde{\rho}_v \equiv (1+\chi)\rho_v = \frac{3+\gamma}{3+\gamma-m}\rho_v \quad \ldots \ldots \ldots (3.70)$$

So , $\tilde{\rho} + \tilde{\rho}_v = \rho + \rho_v$

Giving, $$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}(\rho_v + \rho) - \frac{k}{R^2}$$

$$\Rightarrow H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}(\tilde{\rho}_v + \tilde{\rho}) - \frac{k}{R^2} \quad \ldots \ldots \ldots (3.71)$$

From equation (3.66) we get ,

$$-qH^2 \equiv \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}[(1+\gamma)\rho - 2\rho_v]$$

$$= -\frac{4\pi G}{3}[(1+\gamma)\tilde{\rho} + (m-2)\tilde{\rho}_v] \quad \text{[Using equation (3.70)]}$$

… … … (3.72)

Which we should like to write as ,

$$-qH^2 \equiv \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\tilde{\rho} + 3\tilde{\wp}) - \frac{4\pi G}{3}(\tilde{\rho}_v + 3\tilde{\wp}_v) \quad \ldots \ldots \ldots (3.73)$$

This is achieved if we make the following definitions,

$$\widetilde{P} = \frac{\gamma\widetilde{\rho}}{3} \quad \ldots\ldots\ldots (3.74) \qquad \& \qquad \widetilde{\rho}_v \equiv \frac{m-3}{3}\widetilde{\rho}_v \equiv \frac{\gamma_v}{3}\widetilde{\rho}_v \ \ldots\ldots (3.75)$$

Which is our vacuum equation of state for a decaying cosmological constant.

We see that for m < 3 we have $\widetilde{\rho}_v$ & γ_v negative .For a m< 2 we have γ_v < -1 which we saw previously is the condition for inflation assuming vacuum domination of the density & pressure .

It is also satisfying to note that the equation of state for the non-vacuum component is identical to the equation of sate for a perfect fluid that we encountered for models without a cosmological constant.

3.11 Wheeler-DeWitt equation :

The discussion of the *Wheeler-DeWitt* equation in the minisuperspace approximation is usually restricted to closed (k=+1) and empty (ρ=0) universes. We consider closed, open & flat & non-empty universe .It is important to consider the possible presence of matter & radiation as they might otherwise changes the conclusions. Thus present below is a derivation of the *Wheeler-DeWitt* equation in the minisuperspace approximation which also includes matter and radiation & arbitrary values of K **[44].**

The Lagrangian is,

$$L = -\kappa R^3[(\frac{\dot{R}}{R})^2 - \frac{k}{R^2} + \frac{8\pi G}{3}(\rho + \rho_{vac})] \quad \ldots\ldots\ldots (3.76)$$

With $\kappa = \frac{3\pi}{4G}$. The momentum conjugate to R is $P = \frac{\delta L}{\delta \dot{R}}$

$$P = -\kappa R^3[\frac{2\dot{R}}{R^2} - 0 + 0] = -\kappa 2\dot{R}R \quad \ldots\ldots\ldots (3.77)$$

Substituting L & P into the *Euler-Lagrange* equation ,

$$\dot{\rho} - \frac{\delta L}{\delta R} = 0$$

Equation $H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} + \frac{\Lambda}{3} = \frac{8\pi G}{3}(\rho_{vac} + \rho) - \frac{k}{R^2}$

is recovered .[Note the calculation of $\frac{\delta L}{\delta R}$ is simplified by using the conservation equation $P = \frac{\gamma\rho}{3}$ with equation $\rho' + 3(\rho + p)\frac{1}{R} = 0$ namely $\rho' + \rho'_{vac} = -(3+\gamma)\frac{\rho}{R}$]

The *Hamiltonian* $H \equiv P\dot{R} - L$ is ,

$$H(\dot{R}, R) = -2\kappa R\dot{R}.\dot{R} - \{-\kappa R^3[(\frac{\dot{R}}{R})^2 + \frac{8\pi G}{3}(\rho_{vac} + \rho) - \frac{k}{R^2}]\}$$

$$\Rightarrow H(\dot{R}, R) = -2\kappa R\dot{R}^2 + \kappa R\dot{R}^2 + \kappa R^3\{\frac{8\pi G}{3}(\rho_{vac} + \rho) - \frac{k}{R^2}\}$$

$$\Rightarrow H(\dot{R}, R) = -\kappa R\dot{R}^2 + \kappa R^3\{\frac{8\pi G}{3}(\rho_{vac} + \rho) - \frac{k}{R^2}\}$$

$$\Rightarrow H(\dot{R}, R) = -\kappa R^3[(\frac{\dot{R}}{R})^2 - \frac{8\pi G}{3}(\rho_{vac} + \rho) + \frac{k}{R^2}] \equiv 0 \quad \dots \quad \dots (3.78)$$

Which has been written in terms of $\dot{R}$ to show explicitly that the *Hamiltonian* is identically zero & is not equal to the total energy as before.

In terms of the conjugate momentum,

$$H(P,R) = -\kappa R^3 [\frac{(-\frac{P}{2\kappa R})^2}{R^2} - \frac{8\pi G}{3}(\rho_{vac} + \rho) + \frac{k}{R^2}] \equiv 0$$

$$\Rightarrow H(P,R) = -\kappa R^3 [\frac{P^2}{4\kappa^2 R^4} - \frac{8\pi G}{3}(\rho_{vac} + \rho) + \frac{k}{R^2}] = 0 \quad \dots \dots \dots (3.79)$$

Which is of course is also equal to zero .Making the replacement $P \to -i\frac{\delta}{\delta R}$ & imposing $H\psi = 0$ results in the *Wheeler-DeWitt* equation in the minisuperspace approximation for arbitrary K & with matter or radiation (ρ term) included gives .

From equation (3.79),

$$-\kappa R^3 [\frac{P^2}{4\kappa^2 R^4} - \frac{8\pi G}{3}(\rho_{vac} + \rho) + \frac{k}{R^2}] = 0$$

$$\Rightarrow \frac{P^2}{4\kappa^2 R^4} - \frac{8\pi G}{3}(\rho_{vac} + \rho) + \frac{k}{R^2} = 0$$

$$\Rightarrow P^2 + 4\kappa^2 R^4 [\frac{k}{R^2} - \frac{8\pi G}{3}(\rho_{vac} + \rho)] = 0$$

$$\Rightarrow \{(-i\frac{\delta}{\delta R})^2 + 4(\frac{3\pi}{4G})^2 R^4 [\frac{k}{R^2} - \frac{8\pi G}{3}(\rho_{vac} + \rho)]\}\psi = 0$$

$$\Rightarrow \{-\frac{\delta^2}{\delta R^2} + \frac{9\pi^2}{4G^2}[kR^2 - \frac{8\pi G}{3}(\rho_{vac} + \rho)R^4]\}\psi = 0 \quad \dots \dots (3.80)$$

Using $\rho = \frac{C}{R^{3+\gamma}}$ the *Wheeler-DeWitt* equation becomes ,

$$\{-\frac{\delta^2}{\delta R^2} + \frac{9\pi^2}{4G^2}[kR^2 - \frac{\Lambda}{3}R^4 - \frac{8\pi G}{3}CR^{1-\gamma}]\}\psi = 0 \quad \dots \dots (3.81)$$

This just looks like zero energy *Schrodinger* equation with a potential given by ,

$$V(R) = kR^2 - \frac{\Lambda}{3}R^4 - \frac{8\pi G}{3}CR^{1-\gamma}$$

For the empty Universe case of no matter or radiation ($c = 0$) the potential $V(R)$. For the cases $k = +1, 0, -1$ respectively corresponding to closed, open and flat universes. It can be seen that only the closed universe case provides a potential barrier through which tunneling can occur. This provides a clear illustration of the idea that only closed universes can arise through quantum tunneling. If radiation ($\gamma = 1$ and $C \neq 0$) is included then only a negative constant will be added to the potential (because the term $R^{1-\gamma}$ will be constant for $\gamma = 1$) and these conclusions about tunneling will not change. For matter ($\gamma = 0$ and $C \neq 0$) a term growing like R will be included in the potential which will only be important for very small R and so the conclusions again will not be changed.) To summarize, only closed universes can arise from quantum tunneling even if matter or radiation are present.

<u>3.12 Quantization:</u>

All of our proceeding work with the scalar field was at the classical level. In this section we wish to consider quantum effects. We derived the *Wheeler-DeWitt* equation in minisuperspace approximation. We began with the Lagrangian in equation (3.80)

$$L = -\kappa R^3[(\frac{\dot{R}}{R})^2 - \frac{k}{R^2} - \frac{8\pi G}{3}(\rho + \rho_v)] \quad \dots\dots\dots (3.82)$$

and identified the conjugate momentum $\rho \equiv \frac{\delta L}{\delta \dot{R}} = -2\kappa R\dot{R}$ and derived the *Wheeler-DeWitt* equation, after quantizing with $P \to -i\frac{\delta}{\delta R}$ as ,

$$\{-\frac{\delta^2}{\delta R^2} + 4\kappa^2[kR^2 - \frac{8\pi G}{3}(\rho_v + \rho)R^4]\}\psi = 0 \quad \ldots\ldots\ldots (3.83)$$

Notice that our quantization 'didn't do anything to the density."

In the work that we have done in the present chapter we have made an effort to write the scalar field as a function of R, *ie*. *φ=φ(R)* and using $\rho = \frac{1}{2}\dot{\varphi}^2 + V$ we have written as an effective density *ρ*(*R*) for the scalar field. Our intention has been to simply insert this *ρ*(R) into the *Wheeler-DeWitt* equation. In our work on inflation we found that for $\rho \propto \frac{1}{R^m}$ dominating the *Friedman* equation then inflation occurs for $m < 3$. If this density also dominates $\rho + \rho_v$ in the Wheeler-DeWitt equation, then a tunneling potential will only be present for $m < 3$. Thus Inflation and quantum tunneling require the same condition. This leads us to the hypothesis that inflation and quantum tunneling are identical! Or in other words, inflation is simply a classical description of quantum tunneling. We call this hypothesis Quantum inflation.

Quantum inflation is easy to validate for ordinary densities, either ρ or ρ_v , that behave like $\rho \propto \frac{1}{R^m}$. With our discussion of the scalar field we have written $\rho_\varphi \propto \frac{1}{R^m}$ .So it would seem that the idea of quantum inflation also works for scalar fields **[44].**

But we know that $\rho = \frac{1}{2}\dot{\varphi}^2 + V$ so from equation (3.82) we get ,

$$L = -\kappa R^3[(\frac{\dot{R}}{R})^2 - \frac{k}{R^2}] + \kappa R^3.\frac{8\pi G}{3}(\rho + 0)]$$

$$\Rightarrow L = -\kappa R^3[(\frac{\dot{R}}{R})^2 - \frac{k}{R^2}] + \frac{3\pi}{4G}R^3.\frac{8\pi G}{3}[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)]$$

$$\Rightarrow L = -\kappa R^3[(\frac{\dot{R}}{R})^2 - \frac{k}{R^2}] + 2\pi^2 R^3[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)] \quad \ldots\ldots (3.84)$$

From this we can deduce that,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right] - \frac{k}{R^2} \quad \ldots\ldots\ldots (3.85)$$

& also $$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0 \quad \ldots\ldots\ldots (3.86)$$

Provided one uses $\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3P)$ with $\rho = \frac{1}{2}\dot{\varphi}^2 + V$ & $P = \frac{1}{2}\dot{\varphi}^2 - V$.

The canonical momenta are ,

$$\Pi_R \equiv \frac{\delta L}{\delta \dot{R}} = -2\kappa R\dot{R} \quad \ldots\ldots\ldots (3.87)$$

$$\Pi_\varphi \equiv \frac{\delta L}{\delta \dot{\varphi}} = 2\pi^2 R^3 \dot{\varphi} \quad \ldots\ldots\ldots (3.88)$$

The *Hamiltonian* $(H = p_i \dot{q}_i - L)$ becomes & also using $\kappa = \frac{3\pi}{4G}$

$$H = \Pi_R \dot{R} + \Pi_\varphi \dot{\varphi} - L$$

$$= -\kappa R^3\left[\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} - \frac{8\pi G}{3}\left\{\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right\}\right] \equiv 0 \quad \ldots\ldots (3.89)$$

Where the result H = 0 is obtained by comparing the expression for the *Hamiltonian* to the *Friedman* equation (3.85) .This *Hamiltonian* is exactly analogues to the *Hamiltonian* we had ρ instead of $\frac{1}{2}\dot{\varphi}^2 + V(\varphi)$

writing H in terms of the conjugate momentum we have ,

$$H = -\kappa R^3 [\frac{\Pi_R^2}{4\kappa^2 R^4} + \frac{k}{R^2} - \frac{8\pi G}{3}\{\frac{\Pi_\varphi^2}{8\pi^4 R^6} + V(\varphi)\}] = 0 \quad \ldots (3.90)$$

Which of course is also equal to zero .This equation is re-arranged as,

$$\Pi_R^2 - \frac{3}{4\pi G} . \frac{2}{R^2} \Pi_\varphi^2 + \frac{9\pi^2}{4G^2}\left(kR^2 - \frac{8\pi G}{3} VR^4 \right) = 0 \quad \ldots \ldots (3.91)$$

In order to compare to our signal *Wheeler-DeWitt* equation .Let's replace Π_φ with $\Pi_\varphi = 2\pi^2 R^3 \dot{\varphi}$ which results in,

$$\Pi_R^2 + \frac{9\pi^2}{4G^2}\left[kR^2 - \frac{8\pi G}{3} (\frac{1}{2}\dot{\varphi}^2 + V) R^4 \right] = 0 \quad \ldots \ldots \ldots (3.92)$$

Which is exactly analogues to our original *Wheeler-DeWitt* equation .Where we had ρ instead of $\frac{1}{2}\dot{\varphi}^2 + V(\varphi)$.

Equation (3.91) is quantized by making the replacements.

$$\Pi_R \to -i\frac{\delta}{\delta R} \qquad \& \qquad \Pi_\varphi \to -i\frac{\delta}{\delta \varphi}$$

And setting $H\psi = 0$ to give ,

$$\left\{ -\frac{\delta^2}{\delta R^2} + \frac{3}{4\pi G}\frac{1}{R^2}\frac{\delta^2}{\delta\varphi^2} + \frac{9\pi^2}{4G^2}\left(kR^2 - \frac{8\pi G}{3} VR^4 \right) \right\} \psi = 0 \quad \ldots \ldots (3.93)$$

Which is *Wheeler-DeWitt* equation in minisuperspace approximation for a quantized scalar field φ .

We identify the potential as ,

$$U(R,\varphi) = \frac{9\pi^2}{4G^2}(kR^2 - \frac{8\pi G}{3} VR^4) \quad \ldots \ldots \ldots (3.94)$$

We can see that the above method of quantizing the scalar field φ *directly* is still consistent with our idea of Quantum Inflation. Recall that ρ(*R*) and *V* (*R*) in terms of φ(*R*) obviously ρ, *V* and $\dot{\varphi}^2$ must have the same R dependence. Thus if $\rho \propto \frac{1}{R^m}$ then also $V \propto \frac{1}{R^m}$ in the same way. Thus our potential *U*(*R*, φ) will always exhibit a tunneling shape for $m < 3$. Thus Quantum Inflation still works for *U*(*R*, φ) when φ is quantized separately.

CHAPTER -4

AN EXACT SCALAR FIELD INFLATIONARY COSMOLOGICAL MODEL WHICH SOLVES COSMOLOGICAL CONSTANT PROBLEM

4.1 Introduction:

Inflation was proposed by Alan Guth although the idea of an exponential type expansion was due to Starobinsky and others. The modern form of inflationary cosmology is due to A.Linde, A. Albrecht and P. Steinhardt. In Guth's original model the inflation field Φ was assumed to be trapped in a false vacuum and assumed a local value which is minimum. The inflation field comes out from the local minimum value by quantum tunneling and as universe inflates, tunneling takes place. However, these ideas when pursued gave empty universe and therefore rejected. Guth **[19]** further tried to improve the idea but they led to others difficulties.

Linde and Steinhardt proposed new inflationary model where the inflation field varies slowly and undergoes a phase transition of second order. New inflationary models do not require the idea of tunneling. Most of the modern models depend on the idea of chaotic inflation due to Linde. In these models the initial value of the inflation field Φ is set chaotically when the universe exits from Planck era. The field then rolls downhill and if the potential is enough flat then inflation can take place.

There is another class of models known as hybrid inflationary models in which two fields are considered. These models introduce extra difficulties but they can speculate some features of single field models.

Inflationary cosmology is important because it offers solution to some great puzzles of cosmology. The puzzles are Flatness problem, Horizon problem and Monopole problem.

Flatness problem is basically why the density parameter $\Omega(t) = \dfrac{\rho}{\rho_C}$ is extremely close to unity i.e. why $\Omega \approx 1$? Horizon problem is why the universe is extremely smooth and isotropic on large scales? Monopole and the unwanted relics are the problems associated with standard hot Big

Bang Theory. They are trivially solved when Flatness and Horizon problems are solved.

The above problems namely Flatness problem and Horizon problem are problems of Standard Big Bang theory are solved by assuming an accelerated expansion in early universe for a very short duration. This accelerated expansion is named as inflation. The starting time of inflation is model dependent. However, it occurred when the universe was extremely young. Inflation ended around the time when universe was 10^{-33} sec old. From this time (10^{-33} sec) radiation domination started. The phenomenon of ending inflation and then entering into radiation dominated era is known as graceful exit. And its mechanism requires explanations. An entirely different mechanism of graceful exit will be given in this work.

Lot of scalar field inflationary cosmological models has been proposed so far to explain the above scenarios. Expansion of universe is assumed to be driven by a scalar field Φ and an associated potential V (Φ). Many forms of potentials have been used to solve the associated field equations. In some models a kind of approximation is used to solve the difficult equations. This approximation is known as slow roll approximation which assumes that the field rolls very slowly. Mathematically this is equivalent to assuming $\dot{\Phi}^2 << V(\Phi)$ where the overhead dot represents derivative with respect to time. A few models find exact solutions to the field equations. All the above models explain the mechanism of inflation and solve Horizon and Flatness problems. Further it is found that solution of these problems is equivalent to produce an e-folding [defined as $\ln \frac{R_f}{R_i}$ during inflation] N ≥ 65-70. Here R_i and R_f are values of scale factor when inflation starts and ends respectively.

However these above models fail to explain cosmological constant problem and dark matter problem. The cosmological constant problem is why the measured vacuum energy density is small by a factor 10^{120} of about from its theoretical value. This is in language of Weinberg; "Worst failure of an order of magnitude estimate in the history of physics".

The dark matter problem is another unsolved puzzle in modern cosmology. Our present knowledge asserts that the energy density of matter/energy content of our universe is: dark energy ~ 74%, dark matter ~ 22% and ordinary matter ~ 4%. No cosmological model predicts or accounts for this observation.

Finally, there is the problem of present acceleration of the universe found from the observation of distant Supernovae Ia.

The present work addresses all the above problems listed from the beginning and provides solutions in a single framework. Further, the solution of cosmological evolution equations are exact and no sort of approximations like slow roll approximation etc. is used to derive the solutions.

It may be mentioned here that slow roll is not the necessary and sufficient condition of inflation. However, if slow roll is valid, inflation takes place. It will be shown in this work that without slow roll one can have plenty of exact inflationary models.

***4.2 The Scalar field equation and its exact solutions*:**

We suppose that after tunneling there exists a scalar field Φ and an associated potential V (Φ), which is responsible for the evolution of the universe. It is further assumed that initially there existed some other type of fields ψ_i with potentials $\boldsymbol{\chi}(\psi_i)$ **.** But these fields were hanged up initially which means $\dot{\psi}_i(\frac{d\psi_i}{dt})$ and $\chi'(\psi_i)[=\frac{d\chi}{d\psi_i}]$ are negligible and they did not contribute to field equations initially. The number and nature of the ψ_i fields are not important for the purpose of cosmological predictions. The interactions of the scalar field Φ with other fields are assumed to be ignorable and consequently the ψ_i fields are assumed to interact among themselves only.

Now if the inflation field Φ has no spatial variation and depends only on time then we can write the equations of motion of the scalar field and the Friedmann equation ignoring the curvature term as:
From energy conservative we have,

$$T^{\mu\vartheta}_{;\vartheta}=0$$

$$\Rightarrow \quad \dot{\rho}+\frac{3\dot{R}}{R}(p+\rho)=0 \qquad \ldots\ldots\ldots (4.i)$$

From equation (4.10) we get,

$$\rho = \frac{1}{2}\dot{\varphi}^2+V(\varphi) \quad \ldots\ldots\ldots (4.ii)$$

$$p=\frac{1}{2}\dot{\varphi}^2-V(\varphi)$$

Adding, $\quad p+\rho=\dot{\varphi}^2$

Differentiating we get,

$$\frac{d}{dt}(\rho)=\frac{d}{dt}[\frac{1}{2}\dot{\varphi}^2+V(\varphi)]$$

$$\Rightarrow \dot{\rho}=\frac{1}{2}\times 2\dot{\varphi}\ddot{\varphi}+\frac{d}{dt}\{V(\varphi)\}$$

$$\Rightarrow \dot{\rho}=\dot{\varphi}\ddot{\varphi}+\frac{d}{d\phi}\{V(\varphi)\}\frac{d\varphi}{dt}$$

$$\Rightarrow \dot{\rho} = \dot{\varphi}\ddot{\varphi} + V'(\varphi)\dot{\varphi}$$

Now putting the values in equation (4.i) we get ,

$$\dot{\varphi}\ddot{\varphi} + V'(\varphi)\dot{\varphi} + \frac{3\dot{R}\dot{\varphi}^2}{R} = 0$$

$$\Rightarrow \ddot{\varphi} + V'(\varphi) + 3H\dot{\varphi} = 0$$

$$\Rightarrow \ddot{\varphi} + 3H\dot{\varphi} = -V'(\varphi) \quad \dots\dots\dots (4.iii)$$

And from the time-time component we have,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho_{\varphi}}{3} \qquad [\text{ for K = o }]$$

$$\Rightarrow \left(\frac{\dot{R}}{R}\right)^2 = \frac{1}{3}\rho_{\varphi} \qquad [\text{Considering } 8\pi G = 1]$$

$$\therefore \left(\frac{\dot{R}}{R}\right)^2 = \frac{1}{3}[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)] \quad \dots\dots\dots (4.iv)$$

Where R is the scale factor, φ is the inflation field and V (Φ) is the potential. Overhead dot represents derivative with respect to time and overhead prime represents derivative w.r.to φ.

Equation (4.iii) follows from Lagrangian ,

$$L_{KG} = \frac{1}{2}g^{\mu\vartheta}(\delta_{\mu}\varphi)(\delta_{\vartheta}\varphi) - V(\varphi) \quad \dots\dots\dots (4.v)$$

Solution of equation (4.iii) and (4.iv) are in some ways similar to the solution of Diophantine equations in Classical Algebra, where the numbers of unknowns are more than the number of equations given.

Here a method will be shown by which one can find exact solution of equation (4.iii) and (4.iv). In principle we will choose an arbitrary function from which we can construct some form of potentials for which equations (4.iii) and (4.iv) are exactly solvable.

The Friedmann & Scalar field equations are,

$$\frac{\dot{R}^2}{R^2} = \frac{1}{3}[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)] \quad \dots\dots\dots (4.1)$$

$$\ddot{\varphi} + 3\frac{\dot{R}}{R}\dot{\varphi} + V'(\varphi) = 0 \quad \dots\dots\dots (4.2)$$

Where the dots represent derivative with respect to time t and prime represents derivative with respect to Φ.

From equation (4.1) we obtain,

$$\frac{\dot{R}^2}{R^2} = \frac{1}{3}[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)]$$

$$\Rightarrow \frac{\dot{R}}{R} = \frac{1}{\sqrt{3}}[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)]^{\frac{1}{2}} \quad \dots\dots\dots (4.3)$$

From equation (4.2) we have,

$$\ddot{\varphi} + V'(\varphi) = -3\frac{\dot{R}}{R}\dot{\varphi}$$

$$\Rightarrow \ddot{\varphi} + V'(\varphi) = -3\dot{\varphi}\,\frac{1}{\sqrt{3}}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right]^{\frac{1}{2}} \quad \text{[Using (4.3)]}$$

$$\Rightarrow \ddot{\varphi} + V'(\varphi) = -\sqrt{3}\dot{\varphi}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right]^{\frac{1}{2}}$$

$$\Rightarrow [\ddot{\varphi} + V'(\varphi)]^2 = 3\dot{\varphi}^2\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right]^{\frac{1}{2}} \quad \text{[By Squaring]}$$

$$\Rightarrow \ddot{\varphi}^2 + 2\ddot{\varphi}V'(\varphi) + V'^2(\varphi) = \frac{3}{2}\dot{\varphi}^4 + 3V(\varphi)\dot{\varphi}^2$$

$$\Rightarrow \ddot{\varphi}^2 + 2\ddot{\varphi}V'(\varphi) + V'^2(\varphi) - \frac{3}{2}\dot{\varphi}^4 - 3V(\varphi)\dot{\varphi}^2 = 0 \quad \dots\dots\dots (4.5)$$

To solve this equation let us put

$$\dot{\varphi} = \sqrt{u(\varphi)} \quad \dots\dots\dots (4.6)$$

$$\Rightarrow \ddot{\varphi} = \frac{1}{2}u^{-1/2}(\varphi)u'(\varphi)\dot{\varphi}$$

$$\Rightarrow \ddot{\varphi} = \frac{1}{2}u^{-1/2}(\varphi)u'(\varphi)u^{1/2}(\varphi) \quad \text{[By equation (4.6)]}$$

$$\therefore \ddot{\varphi} = \frac{1}{2}u'(\varphi) \quad \dots\dots\dots (4.7)$$

Now putting the values of (4.6) & (4.7) in equation (4.5) we get,

$$\left\{\frac{1}{2}u'(\varphi)\right\}^2 + 2.\frac{1}{2}u'(\varphi)V'(\varphi) + V'^2(\varphi) - \frac{3}{2}(\sqrt{u(\varphi)})^4 - 3V(\varphi)(\sqrt{u(\varphi)})^2 = 0$$

$$\Rightarrow \frac{1}{4}u'^2 + V'^2 + u'V' - \frac{3}{2}u^2 - 3uV = 0$$

$$\Rightarrow u'^2 + 4V'^2 + 4u'V' - 6u^2 - 12uV = 0 \quad \dots\dots\dots (4.8)$$

Here, $u' = \frac{du}{d\varphi}$ & $V' = \frac{dv}{d\varphi}$

A solution of equation (4.8) is $u = -2V$ … … … (4.9)

$$\Rightarrow u' = -2V' \quad \dots\dots\dots (4.10)$$

When equation (4.9) and (4.10) is substituted in equation (4.8) the result is verified.

Therefore the conclusion is that *u = - 2V* is a solution of equation (4.8) However the solution *u* = - 2V is rejected because when this solution is substituted in equation (4.3), we obtain,

$$\frac{\dot{R}}{R} = \frac{1}{\sqrt{3}}$$

$$\left[\frac{1}{2}(u^{1/2})^2 + v\right]^{\frac{1}{2}} = \frac{1}{\sqrt{3}}\left[\frac{1}{2}u + v\right]^{\frac{1}{2}} = \frac{1}{\sqrt{3}}\left[-2.\frac{1}{2}v + v\right] = \frac{1}{\sqrt{3}}(-v+v) = 0$$

$$\Rightarrow \dot{R} = 0$$

$$\therefore R = \text{Constant}$$

Which is the condition of static universe.

So to obtain a sensible solution of equation (4.8) let us assume,

$u = -2V + \theta(\varphi)$ where $\theta(\varphi)$ is an arbitrary function of φ

$$\Rightarrow u' = -2V' + \theta'$$

Putting this in equation (4.8) we get,

$$u'^2 + 4V'^2 + 4u'V' - 6u^2 - 12uV = 0$$

$$\Rightarrow (-2V' + \theta')^2 + 4V'^2 + 4(-2V' + \theta')V' - 6(-2V + \theta)^2 - 12(-2V + \theta)V = 0$$

$$\Rightarrow 4V'^2 - 4V'\theta' + \theta'^2 + 4V'^2 + 4V'\theta' - 8V'^2 - 24V^2 + 24V\theta - 6\theta^2 + 24V^2 - 12V\theta = 0$$

$$\Rightarrow \theta'^2 + 12V\theta - 6\theta^2 = 0 \qquad \dots\dots\dots (4.12)$$

From this equation we can also find,

$$12V\theta = 6\theta^2 - \theta'^2$$

$$\Rightarrow V = V(\varphi) = \frac{6\theta^2 - \theta'^2}{12\theta} = \left(\frac{\theta}{2} - \frac{\theta'^2}{12\theta}\right) \quad \dots\dots\dots (4.13)$$

Hence we conclude that equation (4.11) is the solution of equation (4.8) i.e. solutions of equation (4.1) and (4.2) if *V (φ)* is given by equation (4.13).It is to be noted that equation (4.8) is the consequence of equation (4.1) and (4.2).

The function θ (φ) is of course arbitrary.

Now from equation (4.3) we have,

$$\frac{\dot{R}}{R} = \frac{1}{\sqrt{3}}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right]^{\frac{1}{2}} = \frac{1}{\sqrt{6}}[\dot{\varphi}^2 + 2V]^{\frac{1}{2}} = \frac{1}{\sqrt{6}}[u + 2V]^{\frac{1}{2}} \; [\because \dot{\varphi} = \sqrt{u(\varphi)}\,]$$

$$\therefore \frac{\dot{R}}{R} = \frac{1}{\sqrt{6}}\theta^{\frac{1}{2}}(\varphi) \quad \dots\dots\dots (4.14) \qquad [\text{using eq}^n\ (4.11)]$$

Now we like to calculate the scalar field potential V (φ) in terms of time t.

To do this we write, *θ(φ)=f(t)* ... … … (4.15)

V(φ)=w(t) … … … (4.16)

Since φ depends on *t* only.

So equation (4.14) can be rewritten as,

$$\frac{\dot{R}}{R} = \frac{1}{\sqrt{6}} f^{\frac{1}{2}}(t) \quad \dots\dots\dots (4.17) \qquad [\text{by using } (4.15)]$$

Again we can write,

$$\theta'(\varphi) = \frac{d\theta}{d\varphi} = \frac{d\theta}{dt}.\frac{dt}{d\varphi} = \frac{df}{dt} \times \frac{1}{\dot{\varphi}} = \frac{\dot{f}}{\dot{\varphi}}$$

$$\therefore \theta'(\varphi) = \frac{\dot{f}(t)}{\dot{\varphi}} \quad \dots\dots\dots (4.18)$$

So from equation (4.6) & (4.7) we obtain,

$$u = \dot{\varphi}^2 = -2V + \theta(\varphi) = -\theta + \frac{\theta'^2}{6\theta} + \theta \qquad \text{[By using (4.13)]}$$

$$\therefore u = \frac{\theta'^2}{6\theta} \quad \dots\dots\dots (4.19)$$

ie. $\dot{\varphi}^2 = u(\varphi) = \dfrac{\theta'^2}{6\theta} = \dfrac{\dot{f}^2(t)}{\dot{\varphi}^2} \times \dfrac{1}{6f(t)}$ [By (4.6), (4.18) & (4.15)]

$$\Rightarrow \dot{\varphi}^4 = \frac{\dot{f}^2}{6f}$$

$$\therefore \dot{\varphi}^2 = \frac{-\dot{f}}{\sqrt{6f}} \quad \dots\dots\dots (4.20)$$

Here negative sign is considered for convenience.

Now from equation (4.13) & (4.16),

$$V(\varphi) = w(t) = \frac{\theta}{2} - \frac{\theta'^2}{12\theta} = \frac{f}{2} - \frac{\dot{f}^2}{\dot{\varphi}^2} \times \frac{1}{12f} \qquad \text{[using (4.15) \& (4.18)]}$$

$$\Rightarrow V(\varphi) = \frac{f}{2} - \frac{\dot{f}^2}{12f} \times \frac{1}{\dot{\varphi}^2} = \frac{f}{2} - \frac{\dot{f}^2}{12f} \times \left(-\frac{\sqrt{6f}}{\dot{f}}\right) \qquad \text{[by (4.20)]}$$

$$\therefore V(\varphi) = w(t) = \frac{f(t)}{2} + \frac{\dot{f}(t)}{2\sqrt{6f}} \quad \dots\dots\dots (4.21)$$

The above calculations assure that the exact solution of (4.1) and (4.2) can be found from the following prescription:

Choose an arbitrary function $f = f(t)$. For this arbitrary function $f(t)$ the exact solutions of (4.1) and (4.2) are:

$$\left.\begin{aligned} \frac{\dot{R}}{R} &= \frac{1}{\sqrt{6}}\, f^{\frac{1}{2}}(t) \\ V(\varphi) = w(t) &= \frac{f(t)}{2} + \frac{\dot{f}(t)}{2\sqrt{6f}} \\ \dot{\varphi}^2 &= \frac{-\dot{f}(t)}{\sqrt{6f}} \end{aligned}\right\} \qquad \dots\ \dots\ \dots\ (4.22)$$

We can choose the arbitrary function in an infinite number of ways. Hence we can get an infinite number of exact models from (4.22).

The functions $f(t)$ is arbitrary so that one can have an infinite number of choices of $f(t)$ and can have an infinite number of exact solutions.

4.3 The exact scalar field model & solution of flatness & Horizon problems:

We have,

$$\dot{\varphi}^2 = \frac{-\dot{f}}{\sqrt{6f}} \qquad \text{[from equation (4.20)]}$$

$$\Rightarrow 2\dot{\varphi}\ddot{\varphi} = \frac{-1}{\sqrt{6}}\left(\frac{\ddot{f}}{f^{1/2}} - \frac{f^{-3/2}\dot{f}^2}{2}\right)$$

$$\Rightarrow \dot{\varphi}\ddot{\varphi} = \frac{-1}{2\sqrt{6}}\left(\frac{\ddot{f}}{f^{1/2}} - \frac{f^{-3/2}\dot{f}^2}{2}\right) \qquad \dots\ \dots\ \dots\ (4.23)$$

Therefore we find from (4.2),

$$\ddot{\varphi}+3\frac{\dot{R}}{R}\dot{\varphi}+V'(\varphi)=\ddot{\varphi}+3\frac{\dot{R}}{R}\dot{\varphi}+\frac{dV(\varphi)}{d\varphi}$$

$$\Rightarrow \ddot{\varphi}+3\frac{\dot{R}}{R}\dot{\varphi}+V'(\varphi)=\ddot{\varphi}+3\frac{\dot{R}}{R}\dot{\varphi}+\frac{dw(t)}{d\varphi} \quad \text{[by (4.16)]}$$

$$\Rightarrow \ddot{\varphi}+3\frac{\dot{R}}{R}\dot{\varphi}+V'(\varphi)=\ddot{\varphi}+3\frac{\dot{R}}{R}\dot{\varphi}+\frac{dw(t)}{dt}\times\frac{dt}{d\varphi}$$

$$\Rightarrow \ddot{\varphi}+3\frac{\dot{R}}{R}\dot{\varphi}+V'(\varphi)=\ddot{\varphi}+3\frac{\dot{R}}{R}\dot{\varphi}+\dot{w}(t)\times\frac{1}{\dot{\varphi}}$$

$$\Rightarrow \ddot{\varphi}+3\frac{\dot{R}}{R}\dot{\varphi}+V'(\varphi)=\frac{1}{\dot{\varphi}}[\ddot{\varphi}\dot{\varphi}+3\frac{\dot{R}}{R}\dot{\varphi}^2+\dot{w}(t)] \quad \dots\dots\dots (4.24)$$

Now from equation (4.21) we get,

$$w(t)=\frac{f(t)}{2}+\frac{\dot{f}(t)}{2\sqrt{6f}}$$

$$\Rightarrow \dot{w}(t)=\frac{\dot{f}(t)}{2}+\frac{\ddot{f}(t)}{2\sqrt{6}f^{1/2}}-\frac{1}{4\sqrt{6}}f^{-3/2}\dot{f}^2 \quad \dots\dots\dots (4.25)$$

It is now easy to verify from (4.24) that,

$$\ddot{\varphi}+3\frac{\dot{R}}{R}\dot{\varphi}+V'(\varphi)=\frac{1}{\dot{\varphi}}[\ddot{\varphi}\dot{\varphi}+3\frac{\dot{R}}{R}\dot{\varphi}^2+\dot{w}(t)]$$

$$=\frac{1}{\dot{\varphi}}\left[\frac{-1}{2\sqrt{6}}\frac{\ddot{f}}{f^{1/2}}+\frac{1}{4\sqrt{6}}\frac{\dot{f}^2}{f^{3/2}}+\frac{3f^{1/2}}{\sqrt{6}}\times(-\frac{1}{\sqrt{6}}\frac{\dot{f}}{f^{1/2}})+\frac{\dot{f}(t)}{2}+\frac{\ddot{f}(t)}{2\sqrt{6}f^{1/2}}-\frac{1}{4\sqrt{6}}f^{-3/2}\dot{f}^2\right]$$

[using (4.22),(4.23) & (4.25)]

$$= \frac{1}{\dot{\varphi}}\left[-\frac{1}{2\sqrt{6}}\frac{\ddot{f}}{f^{1/2}} + \frac{1}{4\sqrt{6}}\frac{\dot{f}^{2}}{f^{3/2}} - \frac{3\dot{f}}{6} + \frac{\dot{f}(t)}{2} + \frac{\ddot{f}(t)}{2\sqrt{6}f^{1/2}} - \frac{1}{4\sqrt{6}} f^{-3/2}\dot{f}^{2} \right]$$

$$\Rightarrow \ddot{\varphi} + 3\frac{\dot{R}}{R}\dot{\varphi} + V'(\varphi) = \frac{1}{\dot{\varphi}} \times 0$$

$$\therefore \ddot{\varphi} + 3\frac{\dot{R}}{R}\dot{\varphi} + V'(\varphi) = 0$$

Finally one can check in a straight forward way from (4.1) that,

$$\frac{\dot{R}^2}{R^2} = \frac{1}{3}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right] = \frac{1}{3}\left[\frac{1}{2}\dot{\varphi}^2 + w(t)\right]$$

$$\Rightarrow \frac{\dot{R}^2}{R^2} = \frac{1}{3}\left[\frac{1}{2}\left(-\frac{1}{\sqrt{6}}\frac{\dot{f}}{f^{1/2}}\right) + \frac{f}{2} + \frac{\dot{f}}{2\sqrt{6}f^{1/2}}\right] \quad \text{[by using (4.20) \& (4.21)]}$$

$$\Rightarrow \frac{\dot{R}^2}{R^2} = \frac{1}{3}\left[-\frac{1}{2\sqrt{6}}\frac{\dot{f}}{f^{1/2}} + \frac{f}{2} + \frac{\dot{f}}{2\sqrt{6}f^{1/2}}\right]$$

$$\Rightarrow \frac{\dot{R}^2}{R^2} = \frac{1}{3} \times \frac{f}{2} = \frac{f(t)}{6}$$

$$\Rightarrow \frac{\dot{R}}{R} = \frac{f^{1/2}(t)}{\sqrt{6}}$$

Now we will construct an exact inflationary model from the exact solutions obtained before.

Let us choose the arbitrary function *f (t)* as,

$$f(t) = \left(\frac{A}{t} + B\right)^2 = A^2 + \frac{2AB}{t} + B^2$$

$$\Rightarrow f^{1/2}(t) = \frac{A}{t} + B \quad \ldots\ldots\ldots (4.26)$$

It has to be remembered that *f(t)* is arbitrary.

Then from equation (4.14) & (4.15) we get,

$$\frac{\dot{R}}{R} = \frac{1}{\sqrt{6}}\theta^{\frac{1}{2}}(\varphi) = \frac{1}{\sqrt{6}} f^{\frac{1}{2}}(t) = \frac{1}{\sqrt{6}}\left(\frac{A}{t} + B\right)$$

$$\Rightarrow \frac{dR}{R} = \frac{1}{\sqrt{6}}\left(\frac{A}{t} + B\right)dt$$

Integrating, $\ln R = \frac{A}{\sqrt{6}}\ln t + \frac{Bt}{\sqrt{6}} + \ln R_0$

$$\Rightarrow \ln R = \ln t^{\frac{A}{\sqrt{6}}} + \ln e^{\frac{Bt}{\sqrt{6}}} + \ln R_0$$

$$\therefore R = R_0 t^{\frac{A}{\sqrt{6}}} e^{\frac{Bt}{\sqrt{6}}} \quad \ldots\ldots\ldots (4.27)$$

For $A > \sqrt{6}$ (of course $A_0 > 0$) & B > 0 we can observe that $\ddot{R} > 0$ always. Therefore the above scale factor gives inflation. We will choose later on A such that $A > \sqrt{6}$ & B > 0 .

Now from equation (4.26) we get,

$$f^{\frac{1}{2}}(t) = At^{-1} + B$$

$$\Rightarrow \frac{1}{2} f^{-\frac{1}{2}}.\dot{f} = -At^{-2} \quad \ldots\ldots\ldots (4.28)$$

Using equation (4.25) & (4.28) we find from,

$$w(t) = V(\varphi) = \frac{A^2}{2t^2} + \frac{AB}{t} + \frac{B^2}{2} - \frac{A}{\sqrt{6}t^2}$$

$$\Rightarrow w(t) = \left(\frac{A^2}{2} - \frac{A}{\sqrt{6}}\right)\frac{1}{t^2} + \left(\frac{AB}{t} + \frac{B^2}{2}\right) \quad \ldots\ldots\ldots (4.29)$$

This Equation (4.29) gives the time dependent form of the potential which gives the scale factor R.

Next we will find the scalar field φ dependence of the potential in the following way:

From the equation (4.20), we have

$$\dot{\varphi}^2 = \frac{-\dot{f}}{\sqrt{6f}} = -\frac{1}{\sqrt{6}}\left(\frac{-2A}{t^2}\right) \quad \text{[Using (4.28)]}$$

$$\Rightarrow \dot{\varphi}^2 = \sqrt{\frac{2}{3}}\frac{A}{t^2} \quad \ldots\ldots\ldots (4.30)$$

Therefore,

$$\dot{\varphi} = -\frac{A_1}{t} \quad \ldots\ldots\ldots (4.31)$$

Where, $A_1 = \left(\sqrt{\frac{2}{3}}A\right)^{1/2}$ & negative sign is taken for convenience.

So that from (4.31) we obtain,

$$\varphi = -A_1 \ln t + A_2 \quad \ldots\ldots\ldots (4.32)$$

$$\Rightarrow \ln t = \frac{A_2 - \varphi}{A_1}$$

$$\Rightarrow t = e^{\frac{A_2 - \varphi}{A_1}}$$

$$\therefore \frac{1}{t} = e^{-\frac{(A_2-\varphi)}{A_1}} = e^{-\frac{A_2}{A_1}+\frac{\varphi}{A_1}} = e^{-\frac{A_2}{A_1}} e^{\frac{\varphi}{A_1}} = K_0 e^{\frac{\varphi}{A_1}} \quad \ldots\ldots\ldots (4.33)$$

Where, $K_0 = e^{-\frac{A_2}{A_1}}$ = *Constant*

Now using equation (4.33) we find from equation (4.29),

$$V(\varphi)=(\frac{A^2}{2}-\frac{A}{\sqrt{6}})K_0^2 e^{\frac{2\varphi}{A_1}}+ABK_0 e^{\frac{\varphi}{A_1}}+\frac{B^2}{2} \quad \ldots\ldots\ldots (4.34)$$

Equation (4.34) gives the φ dependence of the potential which in more compact form can be recasted as,

$$V(\varphi)=Ce^{\frac{2\varphi}{A_1}}+De^{\frac{\varphi}{A_1}}+\frac{B^2}{2} \quad \ldots\ldots\ldots (4.35)$$

Where, $C=\left(\frac{A^2}{2}-\frac{A}{\sqrt{6}}\right)K_0^2$ = *Constant* & $D = AB\,K_0$ = *Constant*

In this model we choose the starting time of inflation as $t_i=10^{-43}$ second i.e. just after tunneling and inflation ends at $t_f=10^{33}$ second. Particle production in inflationary period is assumed to be negligible and ignored.

Now from equation (4.27) we can find the e-folding during inflation.

$$N=\ln\frac{R_f}{R_i}=\ln\{e^{\frac{B}{\sqrt{6}}(t_f-t_i)}\}+\frac{A}{\sqrt{6}}\ln(\frac{t_f}{t_i})$$

ie. $$N=\frac{B}{\sqrt{6}}(t_f-t_i)+\frac{A}{\sqrt{6}}\ln(\frac{t_f}{t_i}) \quad \ldots\ldots\ldots (4.36)$$

Now we take B = 10^{-17} *sec* … … (4.37) [for inflation to take place $A>\sqrt{6}$ & B > 0]

& $$\left.\begin{aligned} t_f &= 10^{-33}\,\text{sec} \\ t_i &= 10^{-43}\,\text{sec} \end{aligned}\right\} \quad \ldots\ldots\ldots (4.38)$$

From equation (4.37) & (4.38) we obtain from (4.36),

$$N\sim\frac{10A}{\sqrt{6}}\ln 10=\frac{10A\times 2.3025}{2.4494}$$

$$\therefore N=9.40A \quad \ldots\ldots\ldots (4.39)$$

For inflation to take place $A > \sqrt{6}$.

If we choose,   A = 7.5

Then from (4.39) the result is, $N = 9.40 \times 7.5$

$$ie. \quad N = 70.5 \quad \ldots \ldots \ldots (4.40)$$

Therefore the e-folding one obtains is 70.5, which is perfectly satisfactory.

4.4 Graceful exit and starting of radiation era:

It was assumed in previous section that inflation starts at $t_i = 10^{-43}\sec$ and stops at $t_f = 10^{-33}\sec$. The mechanism by which inflation stops is like this. It was postulated in before, that there were some hanged up fields for which $\dot{\psi}_i$ and $\chi'(\psi_i)$ were negligible so that they did not contribute to the field equations. When inflation starts the inflation field decays. During the period of inflation particle production due to decaying inflation field is assumed to be negligible and not taken into account. But all of the hanged up fields interact among themselves and produce new particles with significant negative energy density around the time $10^{-34}\sec$.The effect of these negative energy density particles is to stop inflation at $t_f = 10^{-33}\sec$. The newly born fields created by these particles are denoted by ψ_{E_i}.

The equation of state of dark energy is $\omega = \frac{P}{\rho}$ where ω **≈ -1**. For dark matter we assume the same equation of state as dark energy but a different negative value of ω. Now since ω is negative, there exist two possibilities i) P > 0, ρ < o or ii) P < 0, ρ > 0. Generally for dark energy the second possibility is accepted.

High energy physics assert that many forms of exotic particles form around the time $t = 10^{-34}$ second. The natures of the particles depend on the theory concerned and their natures are not very important for our purpose. We take it for granted that many forms of exotic particles are formed around the time 10^{-34} sec. In analogy with dark energy equations of state we take the equation of state of these particles $\omega = \frac{P}{\rho}$ as with ω negative. However, we take the first possibility discussed before i.e. we take P **> 0** and ρ **< 0** for these exotic particles. And appearance of a large negative energy density field is enough to stop inflation at 10^{-33} sec. Because creation of a large number of exotic particles with properties P>0 and ρ<o will certainly decrease the energy density and create a situation for which an overall condition 3P + ρ> 0 would appear if we take $-1 \le \omega \le -\frac{1}{3}$ for these particles, as it turns out that 3P + ρ> 0 for these large number of exotic particles. As a result inflation must stop.

The assumption of negative energy density particles is perfectly consistent with the Null energy condition and Strong energy condition.

The appearance of new negative energy density due to creation of new particles does not alter equation (4.1) though they contribute to the Lagrangian from this time $(t \sim 10^{-33}\,\text{sec})$. The reasons are, the inflation field φ has no appreciable interactions ψ_i with the fields or with the newly born ψ_{Ei} fields at the time of graceful exit.

But the Friedmann equation assumes a new form from the time of graceful exit. Considering the appearance of negative energy density particles we find that Friedmann equation assumes its new form at the time of graceful exit:

$$\frac{\dot{R}^2}{R^2} = \frac{1}{3}(\rho_\varphi - \sum \rho_{E_i}) \quad \ldots \ldots \ldots (4.41)$$

Where, $\sum \rho_{E_i} = \sum \frac{1}{2} \dot{\Psi}_{E_i}^2 + \sum V(\Psi_{E_i})$

We neglect further variations of $\dot{\Psi}_{E_i}$ and $V(\Psi_{E_i})$ and they do not interact further among themselves.

Here $\sum \rho_{E_i}$ is the energy density of the new exotic particles formed. The negative sign indicates that the energy densities of the exotic particles are negative.

At the time of graceful exit the universe enters into a decelerated phase. It is known that the conditions of accelerated phase is 3P+ ρ<0 and that of decelerated phase is 3P+ ρ>0. We can therefore assume that the creation of new energy density due to newly born particles create an overall situation where an overall condition like 3P+ ρ >0 holds from the time of graceful exit. The foregoing discussions illustrate the mechanism of graceful exit. An accelerated expansion reduces to a time half power law at the time of graceful exit i.e. at $t \sim 10^{-33}\,\text{sec}$. So from this time radiation era starts. We can exactly calculate value of $\sum \rho_{E_i}$ at the time of graceful exit using (4.30) and (4.31) and taking $R \sim t^{1/2}$ .This is, however unnecessary for our purpose.

4.5 Cosmological constant and dark matter/energy problem:

After graceful exit the expansion of universe continues and the inflation field φ goes on decaying. The energy density gradually increases. We assume that particles are produced in this phase with properties $P \geq 0$, ρ>o as well as P<0, ρ>o .For the second type of particles if we assume an equation of state $\omega = \frac{P}{\rho}$ with $-1 \leq \omega \leq -\frac{1}{3}$ then 3P + ρ < 0 for these particles. All energy conditions permit this. We take it for granted that these type of particles are produced more than the first type in matter dominated phase. Now $\Sigma\rho > 0$, since $\rho > 0$ for both type of particles. The overall effect is the appearance of a positive energy density denoted by $\Sigma\rho_i$ .Thus total energy density of all created particles after graceful exit upto present moment is represented by $\Sigma\rho_i$.

With this idea we can now write the Friedmann equation at present epoch:

$$\frac{\dot{R}^2}{R^2} = \frac{1}{3}(\rho_\varphi + \sum \rho_i - \sum \rho_{E_i}) \quad \ldots \ldots \ldots (4.41)$$

& $$\frac{\dot{R}^2}{R^2} = \frac{1}{3}(\rho_\varphi - \sum \rho_{E_i}) \quad \ldots \ldots \ldots (4.42)$$

Equation (4.41) follows from equation (4.42) by introducing the term $\sum \rho_i$ in R.H.S of equation (4.42).

Here, ρ_φ is the energy density of the inflation field.

ie. $\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)$

And, $\sum \rho_i$ = energy density of the created particles after graceful exit upto present epoch.

And $\sum \rho_{E_i}$ = energy density of exotic particles created just before the time of graceful exit.

It is difficult to calculate $\sum \rho_i$ but one can safely assume that $\sum \rho_i$ is much less than $\sum \rho_{E_i}$ so that we can write:

$$\sum \rho_i - \sum \rho_{E_i} = -\sum \rho_* \quad \ldots \ldots \ldots (4.43)$$

Then equation (4.41) can be recasted as,

$$\rho_\varphi - \sum \rho_* = \frac{3\dot{R}^2}{R^2} = 3H_0^2 \quad \dots\dots\dots (4.44)$$

Where H_0 is the present value of Hubble constant. Using the present value of H_0 as,

$$H_0 = 2.27 \times 10^{-18}\, Sec^{-1} \quad \dots\dots\dots (4.45)$$

We find from (4.44) the present value of $\rho_\varphi - \sum \rho_*$ as,

$$\rho_\varphi - \sum \rho_* = 3 \times (2.27)^2 \times 10^{-36} = 1.54 \times 10^{-35} \sec^{-2} \quad \dots\dots\dots (4.46)$$

Now we define cosmological constant as the energy density of the inflation field (i.e. ρ_φ) as,

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \quad \dots\dots\dots (4.47)$$

From equation,

$$V(\varphi) = w(t) = \frac{f(t)}{2} + \frac{\dot{f}(t)}{2\sqrt{6f}} \qquad \& \qquad \dot{\varphi}^2 = \frac{-\dot{f}(t)}{\sqrt{6f}}$$

So equation (4.47) becomes,

$$\rho_\varphi = \frac{1}{\sqrt{6}}\frac{A}{t^2} + \left(\frac{A^2}{2} - \frac{A}{\sqrt{6}}\right)\frac{1}{t^2} + \frac{AB}{t} + \frac{B^2}{2}$$

$$\Rightarrow \rho_\varphi = \frac{1}{\sqrt{6}}\frac{A}{t^2} + \left(\frac{A^2}{2t^2} - \frac{A}{\sqrt{6}t^2}\right) + \frac{AB}{t} + \frac{B^2}{2}$$

$$\Rightarrow \rho_\varphi = \frac{A^2}{2t^2} + \frac{AB}{t} + \frac{B^2}{2} \quad \dots\dots\dots (4.48)$$

Now taking $A=7.5$ & $B=10^{-17}$ as earlier, we find,

$$\rho_\varphi = \frac{(7.5)^2}{2t^2} + \frac{7.5\times 10^{-17}}{t} + \frac{(10^{-17})^2}{2}$$

$$\Rightarrow \rho_\varphi = \frac{28.12}{t^2} + \frac{7.5\times 10^{-17}}{t} + 0.5\times 10^{-34} \quad \dots\dots\dots (4.49)$$

Now at $t = 10^{-43}$sec *ie.* at Planck epoch

$$\Rightarrow \rho_{\varphi_{Planck}} = \rho_\varphi(t = 10^{-43}\sec) = \frac{28.12}{(10^{-43})^2} + \frac{7.5\times 10^{-17}}{10^{-43}} + 0.5\times 10^{-34}$$

$$= 28.12\times10^{86} + 7.5\times10^{26} + 0.5\times10^{-34}$$

$$\approx 2.81\times10^{87}\sec^{-2} \qquad \ldots\ \ldots\ \ldots\ (4.50)$$

At present i.e. at $t = 4.4\times10^{17}\sec$

$$\rho_{\varphi_{\Pr esent}} = \rho_{\varphi}(t = 4.4\times10^{17}\sec) = \frac{28.12}{(4.4)^2}\times10^{-34} + \frac{7.5\times10^{-34}}{4.4} + 0.5\times10^{-34}$$

$$= 1.45\times10^{-34} + 1.70\times10^{-34} + 0.5\times10^{-34}$$

$$\approx 3.65\times10^{-34}\sec^{-2} \qquad \ldots\ \ldots\ \ldots\ (4.51)$$

Then using (4.50) & (4.51) we get,

$$\frac{\rho_{\varphi_{Planck}}}{\rho_{\varphi_{\Pr esent}}} = \frac{\rho_{\varphi}(t = 10^{-43}\sec)}{\rho_{\varphi}(t = 4.4\times10^{17}\sec)} = \frac{2.81\times10^{87}}{3.65\times10^{-34}} = 7.69\times10^{120} \qquad \ldots\ \ldots\ \ldots\ (4.52)$$

Equation (4.51) gives the present value of cosmological constant and equation (4.52) exactly accounts for the so called discrepancy of 120 orders of magnitude of the value of cosmological constant.

Since L.H.S of (4.44) represents effective vacuum energy density at present, so more precise present value of cosmological constant is given by (4.46) and equals $1.54\times10^{-35}\sec^{-2}$.

Then using this value we find from equation (4.50)

$$\frac{\rho_{\varphi_{Planck}}}{\rho_{\varphi_{\Pr esent}}} = \frac{\rho_{\varphi}(t = 10^{-43}\sec)}{\rho_{\varphi}(t = 4.4\times10^{17}\sec)} = \frac{2.81\times10^{87}}{1.54\times10^{-35}} = 1.82\times10^{122} \qquad \ldots\ \ldots\ \ldots\ (4.53)$$

This equation gives more precise ratio of cosmological constant at Planck epoch and at present epoch. Now we indentify $\sum\rho_*$ defined by equation (4.43) is the energy density of dark matter/energy and calculate its present value. The negative sign before $\sum\rho_*$ in (4.43) indicates that energy density of dark matter/ energy is negative.

Using (4.46) and (4.51) we find the present value of energy density of dark matter/energy as,

$$\rho_{\varphi_{\text{Present}}} - (\rho_{\varphi_{\text{Present}}} - \sum \rho_*) = 3.65 \times 10^{-34} - 1.54 \times 10^{-35}$$

$$\Rightarrow \sum \rho_* = 36.5 \times 10^{-35} - 1.54 \times 10^{-35}$$

$$\therefore \sum \rho_* = 34.96 \times 10^{-35} \sec^{-2} \quad \dots \dots \dots (4.54)$$

Now using (4.51) and (4.53) the present ratio of $\sum \rho_*$ and ρ_φ is obtained as,

$$\frac{\sum \rho_*}{\rho_\varphi} = \frac{34.96 \times 10^{-35}}{3.65 \times 10^{-34}} = \frac{34.96}{36.5} = 0.9578 \quad \dots \dots \dots (4.55)$$

In view of this equation we can safely conclude that 95.78% energy density of the inflation field is diminished by the presence of negative energy density of dark matter/energy and the rest 4.22% represent ordinary matter energy, since for ordinary matter / energy $\rho>0$. Thus the present energy density budget of the universe finds its correct accounting, 95.78% corresponds to dark matter and energy and 4.22% corresponds to ordinary matter and energy. However there is a basic difference in the nature of the above energy densities. The energy density of inflation i.e. vacuum energy density is positive, while the energy density of dark matter/energy is negative. The present, energy density of ordinary matter-energy equals present vacuum energy density less the magnitude of present energy density of dark matter/energy. And as energy densities of exotic particles were taken negative, it turns out that constituents of dark matter/energy are exotic particles as energy density of dark matter/energy is also negative.

4.6 Matter domination and present accelerated state of the universe:

It was explained in previous sections that the mechanism of graceful exit is due to formation of some kinds of particles due to interaction of the hanged up fields themselves.

Now during the course of evolution, after graceful exit the energy density slowly increases due to further formation of new particles. Unlike exotic particles energy density, these particles have positive energy densities. So that they add up with inflation energy density ρ_φ . Cooling also increases of the energy density of the universe. And due to this overall increase of energy density, the universe gradually enters into matter dominated phase, when formation of matter takes place.

Present accelerated phase is due to further continuation of above features i.e. formation of more and more positive energy density particles together with cooling etc. It was assumed in before that particles produced after graceful exit has the property $\Sigma\ \rho_i > 0$ in matter dominated phase more particles are produced with property $\rho > 0$, $P < 0$ than particles with property $\rho > 0, P \geq 0$. The equation of state of the particles with property $\rho > 0$, $P < 0$ is such that $3P + \rho < 0$. Particles with $\rho > 0$, $P \geq 0$ are ordinary matter /radiation, whereas particles with $\rho > 0$, $P < 0$ along with $3P + \rho < 0$ probably represent unstable particles which have vacuum like properties. Now in matter dominated phase as more and more particles are produced with property $\rho > 0$, $P < 0$, $3P + \rho < 0$, a situation is gradually reached for which $\Sigma 3P + \Sigma\rho < 0$. And acceleration of the universe starts right from the moment when $\Sigma 3P + \Sigma\rho$ become negative. Such a situation still continues for which we observe our universe accelerating presently. It is once again mentioned that particles produced in various phases after graceful exit has properties $\rho > 0$, $P < 0$ as well as $\rho > 0$, $P \geq 0$, whereas for exotic particles which were formed just before graceful exit $\rho < 0$, $P > 0$.

4.7 Conclusion.

A variety of cosmological models were proposed in last three decades to solve the major problems of cosmology. Among these are the Coleman-Weinberg SU model, models by P_i and *Shafi* and *Vilenkin* and many other models. All the above models were either a failure or partially successful to explain few features only. And all models so far proposed failed to explain the mysterious cosmological constant problem.

Also no model has yet predicted the existence of dark matter and energy.

The present work solves the mysterious cosmological constant problem i.e. the discrepancy of 120 or more precisely 122 orders of the measured value of cosmological constant and predicts the existence of dark matter and energy. The work removes the ambiguity of definition of cosmological constant by clearly defining it as scalar field energy density or vacuum energy density and not the energy density of dark matter/energy. Further, this model gives extremely accurate estimate of present values of vacuum energy density as well as of energy density of dark matter/energy. It also solves flatness and horizon problem, gives a satisfactory estimate of e-folding which is necessary to solve horizon and flatness problems and of course trivially monopole problem**[3]**.

Finally this work also supplies the explanation for the present state of acceleration of the universe.

CHAPTER -5

THE SCALAR FIELD POTENTIAL

5.1 Introduction :

In this chapter we will study cosmological scaling solutions in spatially-flat isotropic model. We assume that scaling solutions describe a perfect fluid with equation of state $p_M = (\gamma - 1)\rho_M, (w_M = \gamma - 1)$ and scalar field φ with the potential V(φ). We then derive exact form of the scalar field potential**[41].**

5.2 Approximate Features of Scalar Field Dynamics:

The fine-tuning problem, (the Planck energy density, $\rho_{P1} \approx 10^{72}$ GeV^4 and the observed value of the dark energy density, ($\rho_\Lambda \approx 0.7 \times \rho_C \approx 10^{-48}$ GeV^4) implies that $\rho_\Lambda / \rho_{P1} \sim 10^{-120}$. Thus ρ_Λ needs to be fine-tuned to the level of one part in 10^{120} from the Planck epoch, in order to match the present universe. The fine-tuning problem is associated with the cosmological constant which induces the exploration of cosmological dynamics of a variety of scalar field models such as quintessence, phantoms, tachyons and K-essence. Scalar field can mimic dark energy at late times including kinematics in the past. Dark energy should have important properties allowing it to relieve the fine-tuning and solving coincidence problems without interfering with the thermal history of universe.

The energy density of scalar field may be larger or smaller than the background (radiation/matter) energy density ρ_M. In case it is larger than the background density, the scalar field density ρ_φ should scale faster than ρ_M . In this case the scalar field energy density overreaches the background and becomes sub-dominant to it (see Fig.i). It's beside course of evolution crucially depending on the form of the scalar field potential. In order to obtain viable dark energy models, we require that the energy density of the scalar field remains insignificant during radiation and matter dominant epochs and occurs only at late times to give accelerating universe. The cosmological solutions where the scalar field energy density follows that of radiation or matter which satisfies this condition is called *scaling solutions*. The scaling solutions are characterized by the relation**[41].**

$$\frac{\rho_\varphi}{\rho_M} = Constant \quad \ldots\ldots\ldots (5.1)$$

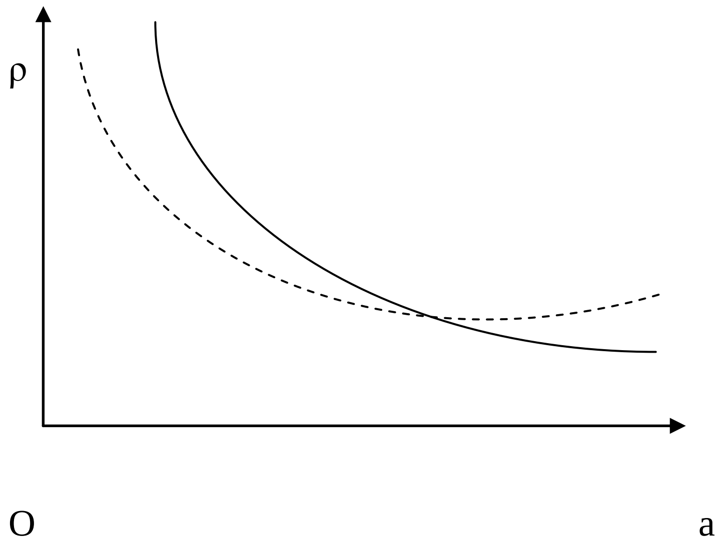


Figure (i): *Evolution of scalar field energy density versus the scale factor. (ρ_φ indicate the dot line & ρ_M indicate the plane line)*

5.3 Cosmological Scaling Solutions:

A flat universe contains a homogeneous and isotropic barotropic perfect fluid and a scalar field φ, with the potential $V(\varphi)$, density ρ and pressure p, satisfies the Friedmann equation,

$$H^2 = \frac{8\pi G}{3}(\rho_\varphi + \rho_M) \quad \ldots\ldots\ldots (5.2)$$

Where energy density and pressure in FRW background

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \quad \ldots \ldots \ldots (5.3)$$

$$P_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \quad \ldots \ldots \ldots (5.4)$$

So we can rewrite equation (5.2),

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi) + \rho_M\right] \quad \ldots \ldots \ldots (5.5)$$

Where H is the *Hubble* parameter.

From energy conservative tensor,

$$\dot{\rho} + 3\frac{\dot{R}}{R}(P + \rho) = 0$$

The fluid equation for matter is,

$$\dot{\rho}_M = -3H(P_M + \rho_M) \quad \ldots \ldots \ldots (5.6)$$

We have field equation of scalar field density,

$$\dot{\rho}_\varphi = -3H(P_\varphi + \rho_\varphi) \quad \ldots \ldots \ldots (5.7)$$

And the total scalar field energy density,

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \quad \ldots \ldots \ldots (5.8)$$

From equation (5.7),

$$\dot{\rho}_\varphi = -3H(P_\varphi + \rho_\varphi)$$

$$\Rightarrow \frac{d\rho_\varphi}{dt} = -3H(\rho_\varphi + P_\varphi)$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right] = -3H\left[\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right) + \left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right)\right]$$

$$\Rightarrow \frac{d}{dt}(\frac{1}{2}\dot{\varphi}^2) + \frac{d}{dt}(V(\varphi)) = -3H\dot{\varphi}^2$$

$$\Rightarrow \dot{\varphi}\ddot{\varphi} + \frac{dV}{d\varphi}\frac{d\varphi}{dt} = -3H\dot{\varphi}^2$$

$$\Rightarrow \dot{\varphi}\ddot{\varphi} + \frac{dV}{d\varphi}\dot{\varphi} = -3H\dot{\varphi}^2$$

$$\Rightarrow \ddot{\varphi} + \frac{dV}{d\varphi} = -3H\dot{\varphi}$$

$$\therefore \ddot{\varphi} = -3H\dot{\varphi} - \frac{dV(\varphi)}{d\varphi} \quad \ldots \ldots \ldots (5.9)$$

Which is the Klein-Gordon equation.

The dust perfect fluid has equation of state $p_M = (\gamma - 1)\rho_M, (w_M = \gamma - 1)$ we assume,

$$\rho_M = DR^{-m} \quad \ldots \ldots \ldots (5.10)$$

Similarly, we can assume scaling solution for scalar field,

$$\rho_\varphi = KR^{-n} \quad \ldots \ldots \ldots (5.11)$$

Where m and n are index of exponents of the scale factor of matter (m = 3γ) and of scalar field. We assume in first part of this work that $\rho_\varphi << \rho_M$ at earlier time. The constant D and K are their density values at present time. We let $n < m$ so that the scalar field can dominate at late time. From fluid equation, the rate of field density change is,

$$\dot{\rho}_\varphi = -3H(P_\varphi + \rho_\varphi)$$

$$\Rightarrow \dot{\rho}_\varphi = -3H[\frac{1}{2}\dot{\varphi}^2 + V(\varphi) + \frac{1}{2}\dot{\varphi}^2 - V(\varphi)]$$

$$\therefore \dot{\rho}_\varphi = -3H\dot{\varphi}^2 \quad \ldots\ldots\ldots (5.12)$$

Dividing the fluid equation of the scalar field by ρ_φ then we get,

$$\frac{\dot{\rho}_\varphi}{\rho_\varphi} = \frac{-3H\dot{\varphi}^2}{\rho_\varphi} \quad \ldots\ldots\ldots (5.13)$$

From equation (5.11) we get,

$$\rho_\varphi = KR^{-n}$$

$$\Rightarrow \dot{\rho}_\varphi = -nR^{-n-1}\dot{R}K = -n\frac{KR^{-n}\dot{R}}{R}$$

$$\therefore \frac{\dot{\rho}_\varphi}{\rho_\varphi} = -n\frac{KR^{-n}\dot{R}/R}{KR^{-n}} = -n\frac{\dot{R}}{R} \quad \ldots\ldots\ldots (5.14)$$

Equating equation (5.13) & (5.14) we get,

$$-n\frac{\dot{R}}{R} = \frac{-3H\dot{\varphi}^2}{\rho_\varphi}$$

$$\Rightarrow nH = \frac{3H\dot{\varphi}^2}{\rho_\varphi}$$

$$\Rightarrow \frac{n}{3} = \frac{\dot{\varphi}^2}{\rho_\varphi}$$

$$\therefore \frac{n}{6} = \frac{\dot{\varphi}^2/2}{\rho_\varphi} \quad \ldots\ldots\ldots (5.15)$$

We see that for a scaling assumption, the ratio between the scalar field kinetic energy density $(\frac{1}{2}\dot{\varphi}^2)$ and its total energy density (ρ_φ) is constant. If the kinetic energy dominates, then n = 6 or if the field kinetic term is negligible, n = 0. Hence scaling behavior for the scalar field energy density lies in a range $0 \le n \le 6$.

5.4 The potential construction:

With the scaling solution of ρ_φ and ρ_M, we substitute them into the Friedmann equation. Let the perfect fluid with $\rho_M \propto R^{-m}$ dominate before the scalar field, the flat Friedmann equation reads,

$$H^2 = \frac{8\pi G}{3}\rho_M$$

$$\Rightarrow (\frac{\dot{R}}{R})^2 = \frac{\kappa^2}{3} DR^{-m} \quad \dots\dots\dots (5.16)$$

Where $\kappa^2 = 8\pi G$ is the scale factor of the matter density $\rho_M = DR^{-m}$ we can find,

$$\frac{\dot{R}}{R} = \kappa\sqrt{\frac{D}{3}}R^{-\frac{m}{2}}$$

$$\Rightarrow \dot{R} = \kappa\sqrt{\frac{D}{3}}R^{-\frac{m}{2}}R = \kappa\sqrt{\frac{D}{3}}R^{-\frac{m}{2}+1}$$

$$\Rightarrow \frac{dR}{dt} = \kappa\sqrt{\frac{D}{3}}R^{-\frac{m}{2}+1}$$

$$\Rightarrow \frac{dR}{R^{-\frac{m}{2}+1}} = \kappa\sqrt{\frac{D}{3}}dt$$

Integrating,

$$\Rightarrow \int \frac{dR}{R^{-\frac{m}{2}+1}} = \kappa\sqrt{\frac{D}{3}}\int dt$$

$$\Rightarrow \frac{R^{\frac{m}{2}}}{m/2} = \kappa\sqrt{\frac{D}{3}}t + const.$$

$$\Rightarrow R^{\frac{m}{2}} = \frac{m}{2}\kappa\sqrt{\frac{D}{3}}t + const.$$

$$\Rightarrow R = \left(\frac{m}{2}\kappa\sqrt{\frac{D}{3}}\right)^{\frac{2}{m}} t^{\frac{2}{m}} + const \; \dots \dots \dots \; (5.17)$$

Let the constant be zero & defining $J \equiv \left(\frac{m}{2}\kappa\sqrt{\frac{D}{3}}\right)^{\frac{2}{m}}$ then we get,

$$R = Jt^{\frac{2}{m}} \quad \dots \dots \dots \; (5.18)$$

Differentiating,

$$\Rightarrow \dot{R} = J(\frac{2}{m})t^{\frac{2}{m}-1} \; \dots \dots \dots \; (5.19)$$

Dividing equation (5.19) by (5.18) we get,

$$\Rightarrow \frac{\dot{R}}{R} = \frac{J\left(\frac{2}{m}\right)t^{\frac{2}{m}-1}}{Jt^{\frac{2}{m}}} = \frac{2}{m}t^{-1} \quad \dots \dots \dots \; (5.20)$$

Putting this value in Klein-Gordon equation (5.9),

$$\ddot{\varphi} = -3(\frac{\dot{R}}{R})\dot{\varphi} - \frac{dV}{d\varphi}$$

Therefore,

$$\ddot{\varphi} = -3(\frac{2}{mt})\dot{\varphi} - \frac{dV}{d\varphi}$$

$$\Rightarrow \ddot{\varphi} = -\frac{6}{mt}\dot{\varphi} - \frac{dV}{d\varphi} \quad \dots \dots \dots \; (5.21)$$

Considering scaling behavior of the ρ_φ from equation (5.11), substitute it into equation (5.15) we get,

$$\frac{\dot{\varphi}^2/2}{\rho_\varphi} = \frac{n}{6}$$

$$\Rightarrow \dot{\varphi}^2 = \frac{n}{3}\frac{K}{R^n}$$

$$\Rightarrow \dot{\varphi} = \pm\sqrt{\frac{nK}{3}}R^{-\frac{n}{2}} = \pm\sqrt{\frac{nK}{3}}(Jt^{\frac{2}{m}})^{-\frac{n}{2}} \qquad \text{[By using (5.18)]}$$

$$\Rightarrow \dot{\varphi} = \pm\sqrt{\frac{nK}{3}}\left[\left(\frac{m\kappa}{2}\sqrt{\frac{D}{3}}\right)^{\frac{2}{m}} t^{\frac{2}{m}}\right]^{-\frac{n}{2}} \qquad \left[\because J = \left(\frac{m\kappa}{2}\sqrt{\frac{D}{3}}\right)^{\frac{2}{m}}\right]$$

$$\Rightarrow \dot{\varphi} = \pm\sqrt{\frac{nK}{3}}\left(\frac{m\kappa}{2}\sqrt{\frac{D}{3}}t\right)^{-\frac{n}{m}}$$

$$\therefore \dot{\varphi} = \pm\sqrt{\frac{nK}{3}}\left(\frac{m\kappa}{2}\sqrt{\frac{D}{3}}\right)^{-\frac{n}{m}} t^{-\frac{n}{m}} \qquad \ldots\ldots\ldots (5.22)$$

Now taking, $A \equiv \pm\sqrt{\frac{nK}{3}}\left(\frac{m\kappa}{2}\sqrt{\frac{D}{3}}\right)^{-\frac{n}{m}}$ we get,

$$\therefore \dot{\varphi} = At^{-\frac{n}{m}} \qquad \ldots\ldots\ldots (5.23)$$

<u>Case (i):</u> **when m = n**

When m=n ten equation (5.23) we get ,

$$\dot{\varphi} = At^{-\frac{n}{n}} = At^{-1}$$

Integrating, $$\varphi = A\ln(t) + \varphi_0 \quad \ldots\ldots\ldots (5.24)$$

And, we have ,

$$\dot{\varphi} = At^{-1}$$

$$\Rightarrow \ddot{\varphi} = -At^{-2}$$

Hence, also $$\varphi = A\ln(t) + \varphi_0$$

$$\Rightarrow A\ln(t) = \varphi - \varphi_0$$

$$\Rightarrow \ln(t) = \frac{\varphi - \varphi_0}{A}$$

$$\Rightarrow t = \exp\left(\frac{\varphi - \varphi_0}{A}\right)$$

Using equation (5.21) we have,

$$\frac{dV}{d\varphi} = -\frac{6}{m}\frac{1}{t}\dot{\varphi} - \ddot{\varphi} = -\frac{6}{m}\frac{1}{t}(At^{-1}) - (-At^{-2}) = -\frac{6}{m}\frac{A}{t^2} + \frac{A}{t^2}$$

$$\Rightarrow \frac{dV}{d\varphi} = \frac{A}{t^2}(1 - \frac{6}{m}) \quad \ldots\ldots\ldots (5.25)$$

We define, $A \equiv \frac{1}{\lambda'}$ and constant $\lambda' \equiv \frac{\lambda}{2}$ we obtain

$$\frac{dV}{d\varphi} = \frac{A}{\left(\exp\frac{\varphi - \varphi_0}{A}\right)^2}\left(1 - \frac{6}{m}\right)$$

$$\Rightarrow \frac{dV}{d\varphi} = \frac{1}{\lambda'}(1 - \frac{6}{m})\left[\exp\frac{-2(\varphi - \varphi_0)}{1/\lambda'}\right]$$

$$\Rightarrow \frac{dV}{d\varphi} = \frac{1}{\lambda'}(1-\frac{6}{m})\exp[-2\lambda'(\varphi-\varphi_0)]$$

$$\Rightarrow dV = \frac{1}{\lambda'}(1-\frac{6}{m})\exp[-2\lambda'(\varphi-\varphi_0)]d\varphi$$

Integrating,

$$\int dV = \frac{1}{\lambda'}(1-\frac{6}{m})\int \exp[-2\lambda'(\varphi-\varphi_0)]d\varphi$$

$$\Rightarrow V(\varphi) = \frac{1}{\lambda'}(1-\frac{6}{m})\frac{1}{(-2\lambda')}\exp[-2\lambda'(\varphi-\varphi_0)] + V_0$$

$$\Rightarrow V(\varphi) = \frac{1}{2\lambda'^2}(\frac{6}{m}-1)\exp[-2\lambda'(\varphi-\varphi_0)] + V_0$$

$$\Rightarrow V(\varphi) = \frac{2}{\lambda^2}(\frac{6}{m}-1)\exp[-\lambda(\varphi-\varphi_0)] + V_0 \quad \dots \dots \dots (5.26)$$

Where, $\lambda = \frac{2}{A}$ & $\frac{2}{\lambda^2} = \frac{A^2}{2}$ & let $V_0 = 0$.

The potential has an exponential form and can be comparable to the potential found in Lucchin & Matarrese,

$$V(\varphi) = (\frac{\sigma}{t_0})^2\left(\frac{3p-1}{2}\right)\exp\left(\frac{-2}{\sigma}(\varphi-\varphi_0)\right) \quad \dots \dots \dots (5.27)$$

As we express $\lambda = \frac{2}{\sigma}$, $3p = \frac{6}{m}$ & $t_0 = 1$.

Note that *A*, λ can be either positive or negative. The constants are linked as

$$\lambda = \frac{2}{\sigma} = 2\lambda' = \frac{2}{A} \quad \dots \dots \dots (5.28)$$

<u>Case (ii):</u> when m $\neq$ n:

When m $\neq$ n then, $$\frac{d\varphi}{dt} = At^{-\frac{n}{m}}$$

$$\Rightarrow d\varphi = At^{-\frac{n}{m}}dt$$

Integrating, $$\Rightarrow \int d\varphi = A\int t^{-\frac{n}{m}}dt$$

$$\Rightarrow \varphi = A\frac{t^{-\frac{n}{m}+1}}{1-\frac{n}{m}} + \varphi_0$$

$$\Rightarrow \varphi = \frac{A}{(1-\frac{n}{m})}t^{(1-\frac{n}{m})} + \varphi_0 \quad \ldots\ldots\ldots (5.29)$$

$$\Rightarrow \dot{\varphi} = \frac{A}{\left(1-\frac{n}{m}\right)}t^{\left(1-\frac{n}{m}\right)-1}\left(1-\frac{n}{m}\right) + 0$$

$$\Rightarrow \dot{\varphi} = At^{-\frac{n}{m}} \quad \ldots\ldots\ldots (5.30)$$

$$\Rightarrow \ddot{\varphi} = -\frac{n}{m}At^{-1-\frac{n}{m}}$$

$$\therefore \ddot{\varphi} = -\frac{n}{m}At^{-\left(1+\frac{n}{m}\right)} \quad \ldots\ldots\ldots (5.31)$$

From equation (5.29) we have,

$$\varphi-\varphi_0=\frac{A}{\left(1-\frac{n}{m}\right)}t^{\left(1-\frac{n}{m}\right)}$$

$$\Rightarrow t^{\frac{m-n}{m}}=\frac{\left(1-\frac{n}{m}\right)}{A}(\varphi-\varphi_0)$$

$$\Rightarrow t=\left[\frac{1}{A}(1-\frac{n}{m})(\varphi-\varphi_0)\right]^{\frac{m}{m-n}} \quad \ldots\ldots\ldots (5.32)$$

So from equation (5.21) we get,

$$\frac{dV}{d\varphi}=-\frac{6}{mt}\dot{\varphi}-\ddot{\varphi}$$

$$\Rightarrow \frac{dV}{d\varphi}=-\frac{6}{mt}At^{\frac{-n}{m}}-\{\frac{-n}{m}At^{-(1+\frac{n}{m})}\}$$

$$\Rightarrow \frac{dV}{d\varphi}=-\frac{6A}{m}t^{\frac{-n}{m}-1}+\frac{nA}{m}t^{-1-\frac{n}{m}}$$

$$\Rightarrow \frac{dV}{d\varphi}=A(\frac{n}{m}-\frac{6}{m})t^{\frac{-n}{m}-1}$$

$$\Rightarrow dV=A(\frac{n-6}{m})t^{-1-\frac{n}{m}}d\varphi$$

$$\Rightarrow dV=A(\frac{n-6}{m})t^{-1-\frac{n}{m}}d\varphi$$

$$\Rightarrow dV=A(\frac{n-6}{m})t^{-1-\frac{n}{m}}\frac{d\varphi}{dt}dt$$

$$\Rightarrow dV = A(\frac{n-6}{m})t^{-1-\frac{n}{m}}\dot{\varphi}dt$$

$$\Rightarrow dV = A(\frac{n-6}{m})t^{-1-\frac{n}{m}}At^{-\frac{n}{m}}dt$$

Integrating,

$$\Rightarrow \int dV = A^2(\frac{n-6}{m})\int t^{-1-\frac{2n}{m}}dt$$

$$\Rightarrow V = A^2\left(\frac{n-6}{m}\right)\frac{t^{-\frac{2n}{m}}}{\frac{-2n}{m}} + t_0$$

$$\Rightarrow V = A^2\left(\frac{6-n}{m}\right)\frac{m}{2n}t^{-\frac{2n}{m}} + t_0$$

$$\Rightarrow V = A^2\left(\frac{6-n}{2n}\right)t^{-\frac{2n}{m}} + t_0$$

Where,

$$t^{\frac{-2n}{m}} = \left[\frac{1}{A}(1-\frac{n}{m})(\varphi-\varphi_0)\right]^{\frac{m}{m-n}\times\frac{-2n}{m}}$$

$$\Rightarrow t^{\frac{-2n}{m}} = \left[\frac{1}{A}(1-\frac{n}{m})(\varphi-\varphi_0)\right]^{\frac{-2n}{m-n}}$$

And t_0 is the constant of integration yields,

$$V(\varphi) = A^2(\frac{3}{n}-\frac{1}{2})\left[(1-\frac{n}{m})(\frac{\varphi-\varphi_0}{A})\right]^{-\frac{2n}{m-n}} + t_0$$

$$\therefore V(\varphi) = A^2(\frac{3}{n} - \frac{1}{2})\left[(1 - \frac{n}{m})(\frac{\varphi - \varphi_0}{A})\right]^{\frac{2n}{n-m}} + t_0 \quad \dots \dots \dots (5.33)$$

Thus the potential can be written down as a power-law form. The act that n < m makes the exponent negative & the scalar field grows with time **[41].**

5.5 The potential reconstruction at the Early time :

It is possible to reconstruct the scalar field potential in the case of n < m . We use to present value of the scale factor R_0 = 1 & the present density parameter $\Omega_{M,0}$ the density parameter,

$$\Omega_{M,0} = \frac{\rho_{M,0}}{\rho_{C,0}} = \frac{D}{\rho_{C,0}} \quad \dots \dots \dots (5.34)$$

Where, $D = \rho_{M,0}$ & ρ_C is the present critical density $\rho_C = \frac{3H_0^2}{8\pi G}$

$$\therefore D = \rho_{M,0} = \Omega_{M,0}\rho_{C,0} = \Omega_{M,0}\frac{3H_0^2}{8\pi G} = \frac{3H_0^2\Omega_{M,0}}{8\pi G} \quad \dots \dots \dots (5.35)$$

Since $\Omega_{M,0} + \Omega_{\varphi,0} = 1$ & we assumed that, $\rho_\varphi = KR^{-n}$ hence,

$$\Omega_{\varphi,0} = 1 - \Omega_{M,0} \quad \dots \dots \dots (5.36)$$

$$\& \qquad \rho_{\varphi,0} = \Omega_{\varphi,0}\rho_{C,0}$$

$$\Rightarrow K = \frac{3H_0^2}{8\pi G}(1 - \Omega_{M,0}) \quad \text{Where} \quad K = \rho_{\varphi,0} \quad \dots \dots \dots (5.37)$$

From equation (5.15) we can directly write down,

$$\frac{\dot{\varphi}^2}{2} = \rho_\varphi(\frac{n}{6})$$

$$\Rightarrow \dot{\varphi}^2 = \rho_\varphi(\frac{2n}{6})$$

$$\Rightarrow \dot{\varphi}^2 = \frac{n}{3} \times KR^{-n} \quad \ldots\ldots\ldots (5.38)$$

We have,

$$H = \frac{\dot{R}}{R} = \frac{dR}{dt}\frac{1}{R}$$

$$\Rightarrow HR\frac{1}{dR} = \frac{1}{dt}$$

$$\Rightarrow \frac{d\varphi}{dt} = HR\frac{d\varphi}{dR} \quad \ldots\ldots\ldots (5.39)$$

Which is the time derivative of the field.

This equation can be re written as,

$$\frac{d\varphi}{dR} = \frac{1}{HR}\frac{d\varphi}{dt} \quad \ldots\ldots\ldots (5.40)$$

$$\Rightarrow \left(\frac{d\varphi}{dR}\right)^2 = \frac{1}{H^2R^2}\left(\frac{d\varphi}{dt}\right)^2 = \frac{1}{H^2R^2}\dot{\varphi}^2 \quad \ldots\ldots\ldots (5.41)$$

A spatially flat universe containing a perfect fluid and a scalar field φ, with the potential $V(\varphi)$, satisfies the Friedmann equation,

$$H^2 = \frac{8\pi G}{3}(\rho_M + \rho_\varphi) \quad \ldots\ldots\ldots (5.42)$$

Putting the values of ρ_M & ρ_φ in this equation we have,

$$H^2 = \frac{8\pi G}{3}(DR^{-m} + KR^{-n})\backslash$$

$$\Rightarrow H^2 = \frac{8\pi G}{3}\left[\frac{3H_0^2\Omega_{M,0}}{8\pi G}R^{-m} + \frac{3H_0^2(1-\Omega_{M,0})}{8\pi G}R^{-n}\right] \text{ [From(5.35) \& (5.37)]}$$

$$\Rightarrow H^2 = \frac{8\pi G}{3}\times\frac{3H_0^2}{8\pi G}[\Omega_{M,0}R^{-m} + (1-\Omega_{M,0})R^{-n}]$$

$$\Rightarrow H^2 = H_0^2[\Omega_{M,0}R^{-m} + (1-\Omega_{M,0})R^{-n}]$$

Now from equation (5.41) we get,

$$(\frac{d\varphi}{dR})^2 = \frac{1}{H^2R^2}\dot{\varphi}^2$$

$$\Rightarrow (\frac{d\varphi}{dR})^2 = \frac{1}{R^2H_0^2[\Omega_{M,0}R^{-m} + (1-\Omega_{M,0})R^{-n}]}\frac{Kn}{3}R^{-n}$$

$$\Rightarrow (\frac{d\varphi}{dR})^2 = \frac{K}{3H_0^2}\frac{n}{[\Omega_{M,0}R^{n-m+2} + (1-\Omega_{M,0})R^2]}$$

$$\Rightarrow \frac{d\varphi}{dR} = \sqrt{\frac{K}{3H_0^2}}\frac{\sqrt{n}}{\sqrt{[\Omega_{M,0}R^{n-m+2} + (1-\Omega_{M,0})R^2]}} \quad \ldots\ldots\ldots (5.43)$$

From equation (5.37) we get,

$$K = \frac{3H_0^2}{8\pi G}(1-\Omega_{M,0})$$

$$\Rightarrow \frac{K}{3H_0^2} = \frac{1}{8\pi G}(1-\Omega_{M,0})$$

$$\Rightarrow \sqrt{\frac{K}{3H_0^2}} = \frac{1}{\sqrt{8\pi G}}\sqrt{(1-\Omega_{M,0})}$$

So from (5.43) we get,

$$\frac{d\varphi}{dR}=\sqrt{\frac{(1-\Omega_{M,0})}{8\pi G}}\,\frac{\sqrt{n}}{\sqrt{[\Omega_{M,0}R^{n-m+2}+(1-\Omega_{M,0})R^2]}}$$

$$\Rightarrow d\varphi=\sqrt{\frac{(1-\Omega_{M,0})}{8\pi G}}\,\frac{\sqrt{n}}{\sqrt{[\Omega_{M,0}R^{n-m+2}+(1-\Omega_{M,0})R^2]}}\,dR$$

Integrating,

$$\Rightarrow \int d\varphi=\sqrt{n}\sqrt{\frac{(1-\Omega_{M,0})}{8\pi G}}\int\frac{dR}{\sqrt{[\Omega_{M,0}R^{n-m+2}+(1-\Omega_{M,0})R^2]}}$$

$$\Rightarrow \varphi=\sqrt{n}\sqrt{\frac{(1-\Omega_{M,0})}{8\pi G}}\,\frac{1}{\sqrt{(1-\Omega_{M,0})}}\int\frac{dR}{R\sqrt{(\frac{\Omega_{M,0}}{1-\Omega_{M,0}})R^{n-m}+1}}$$

$$\Rightarrow \varphi=2\sqrt{\frac{n}{8\pi G}}\,\frac{1}{m-n}\,Sinh^{-1}\left(\sqrt{\frac{1-\Omega_{M,0}}{\Omega_{M,0}}}R^{\frac{m-n}{2}}\right)+\varphi_0 \qquad \ldots\ldots\ldots (5.44)$$

Again we know,

$$\rho_\varphi=\frac{1}{2}\dot{\varphi}^2+V(\varphi)$$

$$\Rightarrow V(\varphi)=\rho_\varphi-\frac{1}{2}\dot{\varphi}^2=KR^{-n}-\frac{1}{2}\left(\frac{Kn}{3}R^{-n}\right)$$

$$\Rightarrow V(\varphi)=KR^{-n}\left(1-\frac{n}{6}\right)=\frac{3H_0^2(1-\Omega_{M,0})R^{-n}}{8\pi G}\left(1-\frac{n}{6}\right)$$

$$\therefore V(\varphi)=\frac{3H_0^2}{8\pi G}(1-\Omega_{M,0})\left(1-\frac{n}{6}\right)R^{-n} \qquad \ldots\ldots\ldots (5.45)$$

From equation (5.44) we get,

$$2\sqrt{\frac{n}{8\pi G}}\,\frac{1}{m-n}\,Sinh^{-1}\left(\sqrt{\frac{1-\Omega_{M,0}}{\Omega_{M,0}}}R^{\frac{m-n}{2}}\right)=\varphi-\varphi_0$$

$$\Rightarrow Sinh^{-1}\left(\sqrt{\frac{1-\Omega_{M,0}}{\Omega_{M,0}}}R^{\frac{m-n}{2}}\right)=\frac{1}{2}\sqrt{\frac{8\pi G}{n}}(m-n)(\varphi-\varphi_0)$$

$$\Rightarrow \sqrt{\frac{1-\Omega_{M,0}}{\Omega_{M,0}}}R^{\frac{m-n}{2}}=Sinh\left[\frac{\sqrt{8\pi G}}{2}\frac{(m-n)}{\sqrt{n}}(\varphi-\varphi_0)\right]$$

$$\Rightarrow R^{\frac{m-n}{2}}=\sqrt{\frac{\Omega_{M,0}}{1-\Omega_{M,0}}}Sinh\left[\frac{\sqrt{8\pi G}}{2}\frac{(m-n)}{\sqrt{n}}(\varphi-\varphi_0)\right]$$

$$\Rightarrow R=\left(\frac{\Omega_{M,0}}{1-\Omega_{M,0}}\right)^{\frac{1}{m-n}}\left\{Sinh\left[\frac{\sqrt{8\pi G}}{2}\frac{(m-n)}{\sqrt{n}}(\varphi-\varphi_0)\right]\right\}^{\frac{2}{m-n}}$$

$$\Rightarrow R^{-n}=\left(\frac{\Omega_{M,0}}{1-\Omega_{M,0}}\right)^{\frac{-n}{m-n}}\left\{Sinh\left[\frac{\sqrt{8\pi G}}{2}\frac{(m-n)}{\sqrt{n}}(\varphi-\varphi_0)\right]\right\}^{\frac{-2n}{m-n}}$$

$$\therefore R^{-n}=\left(\frac{1-\Omega_{M,0}}{\Omega_{M,0}}\right)^{\frac{n}{m-n}}\left\{Sinh\left[\frac{\sqrt{8\pi G}}{2}\frac{(m-n)}{\sqrt{n}}(\varphi-\varphi_0)\right]\right\}^{\frac{-2n}{m-n}}$$

Now putting this value in equation (5.45) we get,

$$\therefore V(\varphi)=\frac{3H_0^2}{8\pi G}(1-\Omega_{M,0})\left(1-\frac{n}{6}\right)\left(\frac{1-\Omega_{M,0}}{\Omega_{M,0}}\right)^{\frac{n}{m-n}}\left\{Sinh\left[\frac{\sqrt{8\pi G}}{2}\frac{(m-n)}{\sqrt{n}}(\varphi-\varphi_0)\right]\right\}^{\frac{-2n}{m-n}}$$

… … … (5.46)

The slope of the potential for a tracker condition is,

$$\Gamma = \frac{V''V}{(V')^2} \qquad \dots\dots\dots (5.47)$$

Here,

$$V(\varphi) = \frac{3H_0^2}{8\pi G}(1-\Omega_{M,0})\left(1-\frac{n}{6}\right)\left(\frac{1-\Omega_{M,0}}{\Omega_{M,0}}\right)^{\frac{n}{m-n}}\left\{Sinh\left[\frac{\sqrt{8\pi G}}{2}\frac{(m-n)}{\sqrt{n}}(\varphi-\varphi_0)\right]\right\}^{\frac{-2n}{m-n}}$$

Let, $$P = \frac{3H_0^2}{8\pi G}(1-\Omega_{M,0})\left(1-\frac{n}{6}\right)\left(\frac{1-\Omega_{M,0}}{\Omega_{M,0}}\right)^{\frac{n}{m-n}}$$

$$\therefore V(\varphi) = P\left\{Sinh[\frac{\sqrt{8\pi G}}{2}\frac{(m-n)}{\sqrt{n}}(\varphi-\varphi_0)]\right\}^{\frac{-2n}{m-n}}$$

$$\Rightarrow V'(\varphi) = P[2\sqrt{n}\sqrt{8\pi G}\left\{Sinh[\frac{\sqrt{8\pi G}}{2}\frac{(m-n)}{\sqrt{n}}(\varphi-\varphi_0)]\right\}^{\frac{-m-n}{m-n}}$$

$$\{Cosh(\frac{\sqrt{8\pi G}}{2}\frac{(m-n)}{\sqrt{n}}(\varphi-\varphi_0))\}]$$

$$\Rightarrow V''(\varphi) = P[16\pi G(m+n)\{Sinh[\frac{\sqrt{8\pi G}}{2}\frac{(m-n)}{\sqrt{n}}(\varphi-\varphi_0)]\}^{\frac{-2n}{m-n}} \times$$

$$Cosh^2[\frac{\sqrt{8\pi G}}{2}\frac{(m-n)}{\sqrt{n}}(\varphi-\varphi_0)]+16P\pi G(m-n)\{Sinh[\frac{\sqrt{8\pi G}}{2}\frac{(m-n)}{\sqrt{n}}(\varphi-\varphi_0)]\}^{\frac{-2n}{m-n}}$$

Now from equation (5.47) we get,

$$\Gamma = 1+\frac{(m-n)}{2n}\{Sech[\frac{\sqrt{8\pi G}}{2}\frac{(m-n)}{\sqrt{n}}\varphi]\}^2 > 1 \qquad \dots\dots\dots (5.48)$$

When n = 6 the potential is zero. The slope and the amplitude of V(φ) is φ-dependent. When different barotrpoic fluid dominates, Potential acquires different slopes. We should include three fluids and the same time in the analysis. That is

$$\varphi = \int \sqrt{\frac{\Omega_{\varphi,0}}{8\pi G}} \frac{\sqrt{n}dR}{\sqrt{\Omega_{r,0}R^{n-2} + \Omega_{d,0}R^{n-1} + \Omega_{\varphi,0}R^{2}}} + \varphi_0 \quad \dots \dots \dots (5,49)$$

$$V(\varphi) = \frac{3H_0^2}{8\pi G}\Omega_{\varphi,0}(1 - \frac{n}{6})R^{-n} \quad \dots \dots \dots (5.50)$$

Which is impossible to find the solution and analytically. One needs to employ numerical integration for the potential. Otherwise, between the radiation and dust needs to be done numerically. Because the potential cannot be solved analytically with simple function of field **[41].**

CHAPTER - 6

SCALAR FIELD COSMOLOGY IN PHASE SPACE

6.1 Introduction:

The standard cosmological model based on the spatially homogeneous and isotropic Friedmann-Lemaitre-Robertson-Walker (FLRW) **[17]** metric is very successful at describing many observations at different scales. It has become normal to include an inflationary epoch in this model during the early universe. Although there is no direct proof that inflation actually occurred, and other scenarios should still be considered, the 1992 discovery of temperature fluctuations in the cosmic microwave background by the COBE satellite provided evidence of a nearly scale-invariant spectrum of primordial density perturbations of the kind predicted by inflationary scenarios. In addition, the study of these temperature fluctuations initiated by COBE ushered in an era of "precision cosmology" continued with later cosmic microwave background experiments, most notably WMAP and PLANCK. Most models of early universe inflation are based on scalar fields, and those based on quadratic quantum corrections to the Einstein-Hilbert action ("Starobinsky inflation") can be reduced to the study of scalar field degrees of freedom. A second revolution in cosmology occurred in 1998 with the discovery, obtained by studying type Ia supernovae, that the current expansion of the universe is accelerated. In the context of general relativity, on which the standard Λ-cold dark matter model is based, this acceleration can only be explained with a cosmological constant Λ of extremely fine-tuned, but non-vanishing, magnitude, or with a very exotic fluid having pressure P and density ρ related by the equation of state $P \approx -\rho$, and dubbed "dark energy". Most models of dark energy are based on a scalar field φ (also known as "quintessence") rolling in a flat section of its potential $V(\varphi)$. Alternative scenarios, seeking to replace the Einstein-Hilbert action ("f(R)" or "modified" gravity), can again be reduced to the dynamics of a scalar field degree of freedom. Both inflation and quintessence models mandate a general understanding of scalar field dynamics in general-relativistic cosmology. Furthermore, a scalar field provides the simplest field theory of matter, and although no fundamental classical scalar field has been discovered in nature so far (except possibly for quintessence), they do provide a toy model useful for understanding many basic theoretical features of more realistic field theories, without the extra details and complications. As such, scalar field theory also constitutes an excellent pedagogical tool used in most relativity textbooks. In this paper we approach the spatially homogeneous and isotropic cosmology of scalar fields minimally coupled to gravity from the phase space point of view. Although dynamical system methods have been widely used in cosmology since the 1960s and this type of analysis has been performed for non-minimally coupled scalar fields and general scalar-tensor or f(R) gravity we could not find in the literature a complete and self-contained analysis for the simpler case of relativity with a minimally coupled scalar field, apart from specific scenarios corresponding to particular choices of the scalar field potential $V(\varphi)$ .By contrast, here we do not commit to any particular scenario, and at most, we make general assumptions on properties of the potential (such as boundedness or monotonicity), refraining from choosing specific forms of the function $V(\varphi)$. Given that there is no preferred scenario of inflation or quintessence, general considerations are valuable. Since the relevant equations, which reduce to ordinary differential equations (ODEs) in this case, are still non-linear and not amenable to exact solution, the phase space view becomes important in gaining a qualitative understanding of the solutions without actually solving the field equations. It is generally believed that in order to say anything about the phase space and the qualitative behavior of the solutions of the equations, one must first fully specify the scenario of inflation or quintessence being studied. While this is certainly true if one wants a complete qualitative picture of the dynamics, many aspects of the phase space portrait are common to most, if not all, scenarios and the study of these aspects, without committing to any particular scenario or potential V (φ), is a necessary preliminary for more detailed analyses of specific models. The purpose of this paper is to discuss these general features, specifically the geometry of the phase space, the existence, nature, and stability of the fixed points, and the late-time behavior of the solutions, without specifying the form of the scalar field potential energy density, and instead making some generic assumptions on its behavior (boundedness, presence of asymptotes, etc.) **[17]**.

6.2 Background:

We consider a scalar field minimally coupled to the space-time curvature as the only source of gravity in the Einstein field equations. This assumption is fully justified in inflationary scenarios of the early universe, and only approximately justified in quintessence models of the late universe. In the latter case, scalar field dark energy is present along with a dust fluid, which combines to determine the dynamics of the universe. However, observations suggest that dark energy comes to dominate the dynamics very quickly, starting from redshifts $z \sim 0.5$, thus we can once again neglect the dust fluid and other forms of energy in the late regimes. In short, there is plenty of motivation to study scalar field cosmology.

The Lagrangian density of a scalar field φ minimally coupled to the spacetime curvature is,

$$L^{(\varphi)} = -\frac{1}{2}\nabla^{\alpha}\varphi\nabla_{\alpha}\varphi - V(\varphi) \qquad \dots\dots\dots (6.1)$$

Where V(φ) is the scalar field potential. The action for gravity and the scalar field is,

$$S = \int d^4x\sqrt{-g}[\frac{R}{16\pi G} + L^{(\varphi)}(\varphi, g_{\mu\vartheta})] \equiv S_{(g)} + S_{(\varphi)} \quad \dots\dots\dots (6.2)$$

Where $g_{\mu\vartheta}$the spacetime metric, g is is its determinant, and R is its Ricci scalar. The action (6.2) is also the action for general scalar-tensor gravity in vacuo, after performing a conformal transformation to the Einstein frame. The variation of the scalar field action $S_{(\varphi)} = \int d^4x\sqrt{-g}L^{(\varphi)}$ gives the stress-energy tensor**[17].**

$$T_{\mu\vartheta}^{(\varphi)} = -\frac{2\delta L^{(\varphi)}}{\sqrt{-g}\delta g^{\mu\vartheta}} = \nabla^{\alpha}\varphi\nabla_{\alpha}\varphi - \frac{1}{2}g_{\mu\vartheta}\nabla^{\alpha}\varphi\nabla_{\alpha}\varphi - g_{\mu\vartheta}V(\varphi) \quad \dots\dots\dots (6.3)$$

A spatially homogeneous and isotropic universe is described by the FLRW line element,

$$ds^2 = -dt^2 + R^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + Sin^2\theta d\varphi^2)] \quad \dots\dots\dots (6.4)$$

in comoving coordinates (t, r, θ,ϕ), where R(t) is the scale factor and k is the curvature index. The Einstein equations,

$$R_{\mu\vartheta} - \frac{1}{2}g_{\mu\vartheta}R = -8\pi G T_{\mu\vartheta} \quad \dots\dots\dots (6.5)$$

(Where $R_{\mu\upsilon}$ is the Ricci tensor and $R \equiv R^{\mu}{}_{\mu}$) reduce to ODEs for the scale factor and matter degrees of freedom. It is customary to approximate the matter content of the universe with a single perfect fluid with four-velocity $u^{\mu} = \delta^{0\mu}$ in comoving coordinates, energy density ρ, pressure P, and energy momentum tensor

$$T_{\mu\vartheta} = (p+\rho)U_{\mu}U_{\vartheta} - pg_{\mu\vartheta} \quad \dots\dots\dots (6.6)$$

The pressure and energy density are usually related by a barotropic equation of state P = P(ρ), often of the form P = ωρ where the constant ω is called the "equation of state parameter". The Einstein equations (6.5) in the presence of a single perfect fluid reduce to,

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) \quad \ldots\ldots\ldots (6.7)$$

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{R^2} \quad \ldots\ldots\ldots (6.8)$$

$$\dot{\rho} + \frac{3\dot{R}}{R}(p + \rho) = 0 \quad \ldots\ldots\ldots (6.9)$$

Where an over dot denotes differentiation with respect to the comoving time t. Eqs. (6.7) and (6.8) are called the acceleration equation and the Hamiltonian constraint, respectively, and the Klein-Gordon equation (6.9) is nothing but the covariant conservation equation $\nabla^{\vartheta} T_{\mu\vartheta} = 0$ (when $\varphi \neq$ const.). The Klein-Gordon equation is not independent of equations (6.7) and (6.8) and can be derived from them. Excellent pedagogical analyses of the phase space of a FLRW universe coupled to a perfect fluid are available in the literature.

In a FLRW universe, a gravitating scalar field must necessarily depend only on the comoving time, φ = φ(t), in order to respect the space-time symmetries. Therefore, its gradient $\nabla_\mu \varphi$ is time-like (or null but trivial if φ =const.). In regions where $\nabla^\alpha \varphi \, \nabla_\alpha \varphi < 0$, we can introduce the four-vector

$$u_\mu = \frac{\nabla_\mu \varphi}{\sqrt{\left|\nabla^\alpha \varphi \nabla_\alpha \varphi\right|}} \quad \ldots\ldots\ldots (6.10)$$

With $u_\mu u^\mu = -1$, and the scalar field is equivalent to a perfect fluid with stress-energy tensor of the form (6.6) and energy density and pressure,

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \quad \ldots\ldots\ldots (6.11)$$

$$P = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \quad \ldots.\ldots.\ldots (6.12)$$

One can define the effective equation of state parameter,

$$\omega(\varphi, \dot{\varphi}) \equiv \frac{P}{\rho} = \frac{\dot{\varphi}^2 - 2V(\varphi)}{\dot{\varphi}^2 + 2V(\varphi)} \quad \ldots\ldots\ldots (6.13)$$

The Einstein-Friedmann equations (6.7), (6.8) & (6.9) become,

$$\frac{\ddot{R}}{R} = -\frac{8\pi G}{3}[\dot{\varphi}^2 - V(\varphi)] \quad \ldots\ldots\ldots (6.14)$$

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right] - \frac{k}{R^2} \quad \ldots\ldots\ldots (6.15)$$

$$\ddot{\varphi} + 3\frac{\dot{R}}{R}\dot{\varphi} + \frac{dV}{d\varphi} = 0 \quad \ldots\ldots\ldots (6.16)$$

In the following it will be useful to rewrite these equations in terms of the Hubble parameter as $H = \frac{\dot{R}}{R}$ as,

$$\dot{H} = -H^2 - \frac{8\pi G}{3}[\dot{\varphi}^2 - V(\varphi)] + \frac{k}{R^2} = \frac{-4\pi G\dot{\varphi}^2}{3} + \frac{k}{R^2} \quad \ldots\ldots\ldots (6.17)$$

$$H^2 = \frac{8\pi G}{3}[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)] - \frac{k}{R^2} \quad \ldots\ldots\ldots (6.18)$$

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0 \quad \ldots\ldots\ldots (6.19)$$

Where a prime denotes differentiation with respect to φ. These equations can also be derived from an effective Lagrangian or Hamiltonian. The equations of scalar field cosmology are non-linear and few exact solutions are known for particular choices of the potential V (φ). We would like to discuss the dynamics of the variables R(t) and φ(t) in as much depth as possible without choosing a specific form of V (φ). Before we begin, let us note that

• For V (φ) = 0 the scalar field is equivalent to a fluid with stiff equation of state P = ρ, which does not seem to be very relevant for inflation and late-time acceleration (although it is relevant for matter at nuclear densities in the core of neutron stars, and possibly near the Big Bang singularity).

• For V (φ) = V_0 = const. the potential reduces to a pure cosmological constant Λ. The scalar field stress-energy tensor (6.3) reduces to

$$T_{\mu\vartheta} = \frac{-\Lambda}{8\pi G} g_{\mu\vartheta} - \delta_\mu \varphi \delta_\vartheta \varphi - \frac{1}{2} g_{\mu\vartheta} \delta^\alpha \varphi \delta_\alpha \varphi \qquad \dots\dots\dots (6.20)$$

With $\Lambda = 8\pi G V_0$. Further, for $\varphi = \varphi_0$ = *Constant*, one recovers the stress energy tensor of a pure cosmological constant.

6.3 Phase space:

Equations (6.14) and (6.16) describe the evolution of R(t) and φ(t) (remember that there are only two independent equations in the set (6.14), (6.15),(6.16) if φ is not constant). Equation (6.15) is a first order constraint (contrary to Equations (6.14) and (6.16) which are of second order). The phase space is, therefore, a 4-dimensional space $(R, \dot{R}, \varphi, \dot{\varphi})$, but the Hamiltonian constraint (6.15) forces the orbits of the solutions to live on a 3-dimensional hypersurface, introducing a relation between the four variables. For example, one can use the constraint to express $\dot{\varphi}$ in terms of the other three variables, $\dot{\varphi} = \dot{\varphi}(R, \dot{R}, \varphi)$

For particular choices of the scalar field potential and especially for k ≠ 0, one can change variables to functions of $(R, \dot{R}, \varphi, \dot{\varphi})$ which can lead to exact solutions or to simpler calculations. In general, however, these new variables do not have an immediate or clear physical meaning and are to be regarded as a mere mathematical trick to perform calculations. Often the results of these calculations cannot be translated explicitly or easily in terms of the variables $(R, \dot{R}, \varphi, \dot{\varphi})$ However, current observations seem to indicate that we live in a spatially flat (k = 0) universe, which is much simpler to analyze than the k ≠ 0 case. This is the situation that we consider in the following**[17].**

6.4 Spatially flat FLRW scalar field cosmologies:

The description of the phase space greatly simplifies for k = 0 as, in this case, the scale factor R(t) appears in the dynamical equations only through the combination $\frac{\dot{R}}{R} \equiv H$,the Hubble parameter, which is a physical observable obtained by fitting theoretical models to cosmological data. Since φ is the only matter field in the theory, it is natural from the field theory point of view to choose it as another dynamical variable. By choosing H and φ as dynamical variables, the phase space reduces to the 3-dimensional space $(H, \varphi, \dot{\varphi})$, but the orbits of the solutions of equations (6.17),(6.18),(6.19) with k = 0 are forced to move on a 2-dimensional subset of the phase space by the Hamiltonian constraint (6.18).

Let us examine the structure of the "energy surface" on which the orbits are forced to move. We choose to eliminate $\dot{\varphi}$ by expressing it in terms of the other variables (H, φ) in eq. (6.18) with k = 0, which can then be viewed formally as a quadratic algebraic equation for $\dot{\varphi}$ and solved, obtaining

$$\dot{\varphi} = \pm\sqrt{\frac{3H^2}{4\pi G} - 2V(\varphi)} \qquad \dots\dots\dots (6.21)$$

For certain choices of the potential V (φ), an arbitrary choice of values of the pair (H, φ) could make the argument of the square root on the right hand side negative. Therefore, in general, there can be a region of the phase space forbidden to the orbits of the dynamical solutions,

$$F \equiv \{(H, \varphi, \dot{\varphi}) : 3H^2 < 8\pi G V(\varphi)\} \qquad \dots\dots\dots (6.22)$$

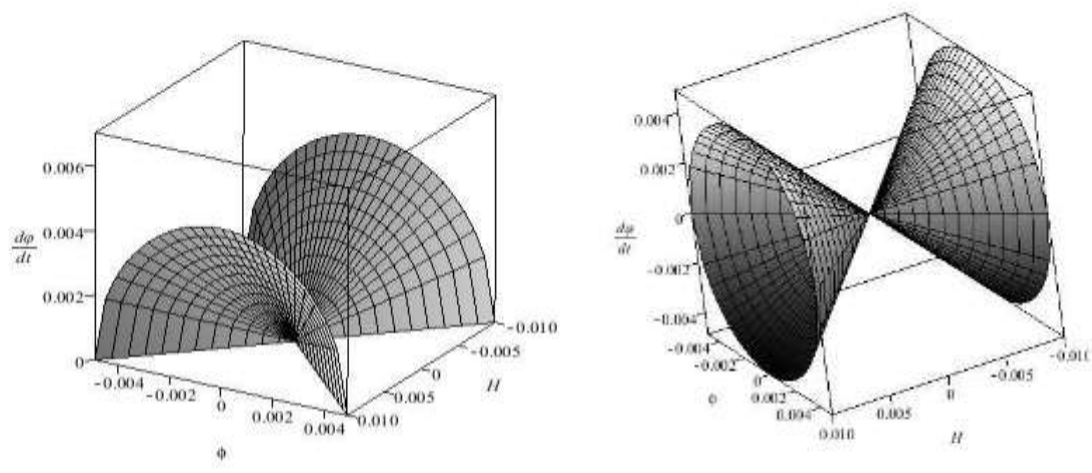
("Forbidden region"). This region may or may not exist depending on the form of V (φ). There are two portions of the phase space region accessible to the dynamics (the "energy surface" corresponding to vanishing effective Hamiltonian), corresponding to the two signs of the right hand side of eq. (6.1). These sets are symmetric with respect to the $\dot{\varphi}$ = 0 plane of the $(H, \varphi, \dot{\varphi})$ space. We call these two subsets of the energy surface "upper sheet" and "lower sheet", corresponding to the positive and negative sign, respectively.

In the upper sheet φ is always increasing $(\dot{\varphi} > 0)$ while on the lower sheet φ is always decreasing. The two sheets are either disconnected, or always join on the plane $\dot{\varphi}$ = 0, on the boundary of the forbidden region

$$B \equiv \delta F = \{(H, \varphi, \dot{\varphi}) : \dot{\varphi} = 0 \Leftrightarrow 3H^2 = 8\pi G V(\varphi)\} \qquad \dots\dots\dots (6.23)$$

Figures (i) and (ii) show the upper and lower sheet for the example potential V (φ) = $m^2\varphi^2/2$. The dynamics of the spatially curved (k ≠ 0) scalar field universe are confined to either side of the "energy surface" corresponding to k = 0 in the phase space-this fact can be deduced by reducing the constraint to,

$$\dot{\varphi} = \pm\sqrt{\frac{3H^2}{4\pi G} - 2V(\varphi) + \frac{3k}{4\pi G R^2}} \qquad \ldots\ldots\ldots (6.24)$$



(i) (ii)

Figure (i): *The upper sheet corresponding to the positive sign in eq. (6.21), for the quadratic potential V (φ) = $m^2\varphi^2/2$ (in arbitrary units).*

Figure (ii): *The surface describing the Hamiltonian constraint eq. (6.21) for the quadratic potential V (φ) = $m^2\varphi^2/2$ (in arbitrary units). The upper and lower sheets join at the $\dot{\varphi}$ = 0 plane to form a cone.*

Trajectories corresponding to k > 0 would exist above the k = 0 upper sheet (i.e., for larger values of $\dot{\varphi}$ than those corresponding to the k = 0 upper sheet) and below the k = 0 lower sheet (i.e., for lower values of $\dot{\varphi}$). Trajectories corresponding to k < 0 would exist between each k = 0 sheet (i.e., for values of $\dot{\varphi}$ comprised between those given by eq. (6.30). Trajectories in a region corresponding to k > 0 cannot cross the k = 0 sheet and move to regions corresponding to k < 0, and *vice-versa*. Such dynamical transitions between different topologies of the universe are forbidden (the topology of space-time is not ruled by the dynamics).

Finally, the lower dimension of the "energy surface" leads one to believe that chaos is impossible in the dynamical system under study. This statement is not trivial given that the standard results on the absence of chaos in a two-dimensional phase space are proven for a plane, not for a curved surface or for a subset of a higher-dimensional phase space obtained by gluing two 2-dimensional sheets. However, it is not difficult to reduce this situation to the standard case, as has been shown for scalar-tensor gravity. The theory of a minimally coupled scalar field in Einstein gravity is contained in this reference as a special case **[17].**

6.5 Equilibrium points:

Having chosen H and φ as dynamical variables, the equilibrium points of the dynamical system (when they exist) are, by definition, of the form $(\dot{H},\dot{\varphi})=(0,0)$ and $(\ddot{H},\ddot{\varphi})=(0,0)$ or (H, φ) = (H_0, φ_0) = (const., const.), and they must all lie in the $\dot{\varphi}$ = 0 plane, and therefore, on the boundary *B* of the forbidden region (if this region exists). These equilibrium points are de-Sitter spaces with a constant scalar field. When they exist, they are the only de Sitter spaces possible in this theory. In fact, eq. (6.17) with k = 0 reduces to $\dot{H}=-4\pi G\dot{\varphi}^2$, and a de Sitter space with H = constant necessarily has φ =const. as well. A degenerate case is H_0 = 0, which corresponds to *Minkowski* space. de Sitter spaces are important in cosmology because they are usually attractors in inflation and quintessence models.

For φ =const. $L^{(\varphi)}$ & $T^{(\varphi)}_{\mu\vartheta}$ reduce to $L^{(\varphi)} = -V_0 \equiv -V(\varphi_0)$ and $T^{(\varphi)}_{\mu\vartheta} = V_0 g_{\mu\vartheta}$ i.e. , to a pure cosmological constant term with $\Lambda = 8\pi G V_0$ **[17].**

The necessary and sufficient conditions for the existence of de Sitter fixed points are easily obtained from eqs. (6.17),(6.18),(6. 19) with k = 0

$$H_0^2 = \frac{8\pi G}{3} V_0 \qquad \dots \dots \dots (6.25)$$

$$V_0' = 0 \qquad \dots \dots \dots (6.26)$$

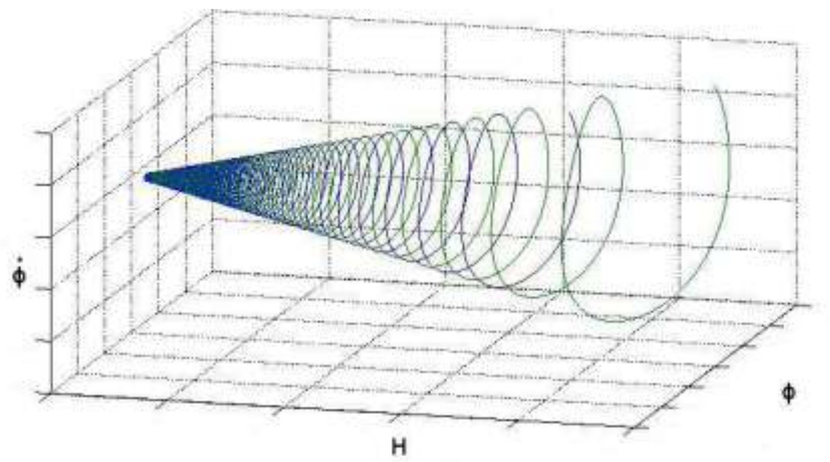


Figure (iii): *Trajectories converging to a Minkowski fixed point* $(H, \varphi, \dot{\varphi})$ *= (0, 0, 0) for the previous example*

$$V(\varphi) = m^2\varphi^2/2.$$

Which obviously require $V_0 \geq 0$ (Minkowski space is obtained for V_0 = 0). Equation (6.26) expresses the condition that V (φ) has an extremum or a point with horizontal tangent at φ_0. Fig.(iii) shows two trajectories, corresponding to different initial conditions, converging to a Minkowski space attractor point for the example of the V (φ) = $m^2\varphi^2/2$ potential.

Attractors (or repellors) could exist as an asymptotic limit in R or φ. To check for these we must search for fixed points with infinite values of the variables.

6.6 Fixed points at infinity with a Poincar´e projection:

Fixed points at infinity can be found by adopting polar coordinates (r, θ) with

$$H = \bar{r} Cos\theta \quad , \qquad \varphi = \bar{r} Sin\theta \qquad \ldots \ldots \ldots (6.27)$$

and the standard Poincar´e transformation $\bar{r} \to r$ with

$$\bar{r} \equiv \frac{\sqrt{r}}{1-r} \qquad \ldots \ldots \ldots (6.28)$$

Which maps infinity onto the circle of radius r = 1. Since,

$$\frac{d}{dt}(H) = \frac{d}{dt}(\bar{r} Cos\theta) = \frac{d}{dt}\left(\frac{\sqrt{r}}{1-r} Cos\theta\right)$$

$$\Rightarrow \dot{H} = \frac{1+r}{2\sqrt{r}(1-r)^2}\dot{r} Cos\theta - \frac{\sqrt{r}}{1-r}\dot{\theta} Sin\theta \qquad \ldots \ldots \ldots (6.29)$$

& $$\frac{d}{dt}(\varphi) = \frac{d}{dt}(\bar{r} Sin\theta) = \frac{d}{dt}\left(\frac{\sqrt{r}}{1-r} Sin\theta\right)$$

$$\Rightarrow \dot{\varphi} = \frac{1+r}{2\sqrt{r}(1-r)^2}\dot{r} Sin\theta + \frac{\sqrt{r}}{1-r}\dot{\theta} Cos\theta \qquad \ldots \ldots \ldots (6.30)$$

fixed points (H, φ) = (const., const.) correspond to (r, θ) = (const., const.) thanks to the linear independence of the sine and cosine functions. The dynamical system (6.17), (6.18) & (6.19) becomes,

we have from (6.18),

$$H^2 = \frac{8\pi G}{3}[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)] - \frac{k}{R^2}$$

$$\Rightarrow \frac{r}{(1-r)^2} Cos^2\theta = \frac{4\pi G}{3}(\dot{\varphi}^2 + 2V)$$

$$\Rightarrow \frac{r}{(1-r)^2} Cos^2\theta = \frac{4\pi G}{3}\left[\left\{\frac{\dot{r}(1+r)}{2\sqrt{r}(1-r)^2} Sin\theta + \frac{r}{(1-r)}\dot{\theta} Cos\theta\right\}^2 + 2V\right]$$

… … … (6.31)

Now from (6.17) we get,

$$\dot{H} = -4\pi G\dot{\varphi}^2$$

$$\Rightarrow \frac{\dot{r}(1+r)}{2\sqrt{r}(1-r)^2} Cos\theta - \frac{\sqrt{r}}{(1-r)}\dot{\theta} Sin\theta = -4\pi G\left[\frac{\dot{r}(1+r)}{2\sqrt{r}(1-r)^2} Sin\theta + \frac{\sqrt{r}}{(1-r)}\dot{\theta} Cos\theta\right]^2$$

… … … (6.32)

Again, from (6.30) we get,

$$\dot{\varphi} = \frac{1+r}{2\sqrt{r}(1-r)^2}\dot{r} Sin\theta + \frac{\sqrt{r}}{1-r}\dot{\theta} Cos\theta$$

$$\Rightarrow \dot{\varphi} = \frac{1}{2}\dot{r}(1+r)r^{\frac{-1}{2}}(1-r)^{-2} Sin\theta + r^{\frac{1}{2}}(1-r)^{-1}\dot{\theta} Cos\theta$$

$$\Rightarrow \ddot{\varphi} = \frac{1}{2}[\ddot{r}(1+r)r^{\frac{-1}{2}}(1-r)^{-2} + \dot{r}^2 r^{\frac{-1}{2}}(1-r)^{-2} + \dot{r}(1+r)(\frac{-1}{2})r^{\frac{-3}{2}}(1-r)^{-2} + \dot{r}^2(1+r)r^{\frac{-1}{2}}(-2)(1-r)^{-3}(-1)] Sin\theta + \frac{1}{2}\dot{r}(1+r)r^{\frac{-1}{2}}(1-r)^{-2}\dot{\theta} Cos\theta + \frac{1}{2}r^{\frac{-1}{2}}\dot{r}(1-r)^{-1}\dot{\theta} Cos\theta + \sqrt{r}(-1)(1-r)^{-2}(-\dot{r})\dot{\theta} Cos\theta + \sqrt{r}(1-r)^{-1}\ddot{\theta} Cos\theta - \sqrt{r}(1-r)^{-1}\dot{\theta}^2 Sin\theta$$

$$\Rightarrow \ddot{\varphi} = [\frac{\ddot{r} + \ddot{r}r + \dot{r}^2}{2\sqrt{r}(1-r)^2} - \frac{r\dot{r}^2 + \dot{r}^2}{4r^{\frac{3}{2}}(1-r)^2} + \frac{r\dot{r}^2 + \dot{r}^2}{\sqrt{r}(1-r)^3}] Sin\theta + [\frac{1+r}{2\sqrt{r}(1-r)^2} + \frac{1}{2\sqrt{r}(1-r)} + \frac{\sqrt{r}}{(1-r)^2}]\dot{r}\dot{\theta} Cos\theta + \frac{\sqrt{r}}{1-r}(\ddot{\theta} Cos\theta - \dot{\theta}^2 Sin\theta)$$

$$\Rightarrow \ddot{\varphi} = \left[\frac{\ddot{r} + \ddot{r}r + \dot{r}^2}{2\sqrt{r}(1-r)^2} - \frac{r\dot{r}^2 + \dot{r}^2}{4r^{\frac{3}{2}}(1-r)^2} + \frac{r\dot{r}^2 + \dot{r}^2}{\sqrt{r}(1-r)^3} \right] Sin\theta + \frac{r + r\dot{r}}{\sqrt{r}(1-r)^2} \dot{\theta} Cos\theta + \frac{\sqrt{r}}{1-r}(\ddot{\theta} Cos\theta - \dot{\theta}^2 Sin\theta)$$

Again, from the equation, $\ddot{\varphi} + 3H\dot{\varphi} + V' = 0$

$$\Rightarrow [\frac{\ddot{r} + \ddot{r}r + \dot{r}^2}{2\sqrt{r}(1-r)^2} - \frac{r\dot{r}^2 + \dot{r}^2}{4r^{\frac{3}{2}}(1-r)^2} + \frac{r\dot{r}^2 + \dot{r}^2}{\sqrt{r}(1-r)^3}] Sin\theta + \frac{r + r\dot{r}}{\sqrt{r}(1-r)^2} \dot{\theta} Cos\theta + \frac{\sqrt{r}}{1-r}(\ddot{\theta} Cos\theta - \dot{\theta}^2 Sin\theta) =$$

$$-3[\frac{r}{(1-r)^2} \dot{\theta} Cos^2\theta + \frac{\dot{r}(1+r)}{2(1-r)^3} Sin\theta Cos\theta] - V'$$

… … … (6.33)

Setting $(\dot{H}, \dot{\varphi}) = (0,0)$ & $(\ddot{H}, \ddot{\varphi}) = (0,0)$ yields,

$$\frac{rCos^2\theta}{(1-r)^2} = \frac{8\pi G}{3} V_0 \qquad \text{… … … (6.34)}$$

$$V_0' = 0 \qquad \text{… … … (6.35)}$$

Where $\varphi_0 = \varphi(r_0, \theta_0)$. In order to satisfy equation (6.34) in the limit r → 1 we must have either,

(i) Cos θ = 0, corresponding to H → 0, φ → ± ∞, and V (φ → ±∞) = 0 (this situation includes potentials V (φ) with compact support).

(ii) Cos θ = ±1, corresponding to H → ±∞, φ → 0, and V (φ → 0) = ∞ (i.e., V has a vertical asymptote at φ = 0. This is the case of the potentials $V(\varphi) \propto \frac{1}{\varphi^\alpha}$, α > 0 used in many quintessence models).

(iii) $\theta \neq 0, \pm\pi$, $\pm\pi/2$, which allows $H \rightarrow \pm\infty$, $\varphi \rightarrow \pm\infty$, and $V(\varphi \rightarrow \pm\infty) = \infty$. This case includes unbounded monotonic potentials such as $V(\varphi) = V_0 e^{\pm\alpha\varphi}$

Fixed points corresponding to any of these situations must have a potential that asymptotically satisfies equation (6.35) as well as the stated conditions.

6.7 Holographic Dark Energy (HDE):

Holographic dark energy (HDE) models have got a lot of enthusiasm recently, because they link the dark energy density to the cosmic horizon, a global property of the universe, and have a close relationship to the spacetime foam. For a recent review on different HDE models and their consistency check with observational data .There are also a number of theoretical motivations leading to the form of HDE, among which some are motivated by holography and others from other principles of physics. A fairly comprehensive motivation is worthwhile to mention that in the literature, various models of HDE have been investigated via considering different system's IR cutoff. In the presence of interaction between dark energy and dark matter, the simple choice for IR cutoff could be the Hubble radius, $L = H^{-1}$ which can simultaneously drive accelerated expansion and solve the coincidence problem. Besides, it was argued that for an accelerating universe inside the event horizon the generalized second law does not satisfy, while the accelerating universe enveloped by the Hubble horizon satisfies the generalized second law. This implies that the event horizon in an accelerating universe might not be a physical boundary from the thermodynamical point of view. Thus, it looks that Hubble horizon is a convenient horizon for which satisfies all of our accepted principles in a flat Friedmann-Robertson-Walker (FRW) universe. There has been a lot of interest in recent years in establishing a connection between holographic/age graphic energy density and scalar field models of dark energy. These studies lead to reconstruct the potential and the dynamics of the scalar fields according to the evolution of the holographic/age graphic energy density **[58]**.

In this paper, by choosing the Hubble radius $L = H^{-1}$ as system's IR cutoff, we implement the connection between the holographic dark energy and scalar fields models including the quintessence, tachyon, K-essence and dilaton energy density in a flat FRW universe.

6.8 HDE with Hubble radius as IR cut-off:

For the flat FRW universe, the first Friedmann equation is,

$$H^2 = \frac{8\pi G}{3}(\rho_D + \rho_M)$$

$$\Rightarrow H^2 = \frac{1}{3M_p^2}(\rho_D + \rho_M) \dots \dots \dots (6.36)$$

Where $\rho_\varphi \,\&\, \rho_M$ are the energy density of dark matter and dark energy, respectively. Taking the interaction between dark matter and dark energy into account, the continuity equation maybe written as,

$$\dot{\rho}_M + 3H\rho_M = Q \quad \dots \dots \dots (6.37)$$

$$\dot{\rho}_D + 3H\rho_D(1 + w_D) = -Q \quad \dots \dots \dots (6.38)$$

Where $w_D = \frac{P_D}{\rho_D}$ is the EoS parameter of HDE, and Q stands for the interaction term. It is important to note that the continuity equations imply that the interaction term should be a function of a quantity with units of inverse of time (a first and natural choice can be the Hubble factor H) multiplied with the energy density. Therefore, the interaction term could be in any of the following forms:

(i) $Q \propto H\rho_D$ (ii) $Q \propto H\rho_M$ (iii) $Q \propto H(\rho_D + \rho_M)$

$$\left.\begin{array}{lll} \Gamma = 3b^2 H & \text{for} & Q \propto H\rho_D \\ \Gamma = 3b^2 Hu & \text{for} & Q \propto H\rho_M \\ \Gamma = 3b^2 H(1+u) & \text{for} & Q \propto H(\rho_M + \rho_D) \end{array}\right\} \quad \ldots\ldots\ldots(6.39)$$

It should be noted that the ideal interaction term must be motivated from the theory of quantum gravity. In the absence of such a theory, we rely on pure dimensional basis for choosing an interaction Q. The freedom of choosing the specific form of the interaction term Q stems from our incognizance of the origin and nature of dark energy as well as dark matter. Moreover, a microphysical model describing the interaction between the dark components of the universe is not available nowadays. We introduce, as usual, the fractional energy densities are,

$$\Omega_M = \frac{\rho_M}{3M_P^2 H^2} \quad , \quad \Omega_D = \frac{\rho_D}{3M_P^2 H^2} \quad , \quad \Omega_K = \frac{K}{R^2 H^2} \quad \ldots\ldots\ldots (6.40)$$

We assume the HDE density has the form,

$$\rho_D = 3C^2 M_P^2 H^2 \quad \ldots\ldots\ldots (6.41)$$

Where c^2 is a constant and we have set the Hubble radius $L = H^{-1}$ as system's IR cutoff. Inserting Eq. (6.40) in Eq. (6.36) immediately yields,

$$u = \frac{1-C^2}{C^2} \qquad \ldots\ \ldots\ \ldots\ (6.42)$$

Where $u = \frac{P_M}{\rho_D}$ is the energy density ratio. From Eq. (6.42) we see that the ratio of the energy densities is a constant; thus the coincidence problem can be alleviated.

Taking the time derivative of equation (6.41) after using Friedmann equation (6.36) we get,

$$\dot{\rho}_D = -3C^2 H\rho_D(1+u+w_D) \qquad \ldots\ \ldots\ \ldots\ (6.43)$$

Combining this equation with equation (6.38) after using relation $Q = \Gamma\rho_D$ we obtain,

$$-3C^2 H\rho_D(1+u+w_D) + 3H\rho_D(1+w_D) = -\Gamma\rho_D$$

$$\Rightarrow -3H\rho_D\{C^2(1+u+w_D) - (1+w_D)\} = -\Gamma\rho_D$$

$$\Rightarrow C^2(1+u+w_D) - (1+w_D) = \frac{\Gamma}{3H}$$

$$\Rightarrow \frac{1}{1+u}(1+u+w_D) - (1+w_D) = \frac{\Gamma}{3H} \qquad \text{[by eq}^n\text{ (6.42)]}$$

$$\Rightarrow 1 + \frac{w_D}{1+u} - 1 - w_D = \frac{\Gamma}{3H}$$

$$\Rightarrow \left(\frac{1}{1+u} - 1\right)w_D = \frac{\Gamma}{3H}$$

$$\Rightarrow \left(\frac{-u}{1+u}\right)w_D = \frac{\Gamma}{3H}$$

$$\Rightarrow w_D = -\frac{\Gamma}{3H}\left(1+\frac{1}{u}\right) \quad \ldots\ldots\ldots (6.44)$$

Again by using (6.42) we get,

$$w_D = -\frac{\Gamma}{3H}\left(\frac{1}{1-C^2}\right) \quad \ldots\ldots\ldots (6.45)$$

Thus we have three expressions for EoS parameter depending on the interaction rate

$$\left.\begin{array}{lll} w_D = \frac{-b^2}{1-C^2} & \text{for} & \Gamma = 3b^2 H \\ w_D = \frac{-b^2}{C^2} & \text{for} & \Gamma = 3b^2 Hu \\ w_D = \frac{-b^2}{C^2(1-C^2)} & \text{for} & \Gamma = 3b^2 H(1+u) \end{array}\right\} \quad \ldots\ldots (6.46)$$

Therefore for constant parameters c and b the EoS parameter is also a constant for three cases. In the absence of interaction, $b^2 = 0$, we encounter dust with $w_D = 0$. For the choice $L = H^{-1}$ an interaction is the only way to have an EoS different from that for dust. Since in what follows the analysis is similar for three cases, hereafter we consider only the first case, namely $w_D = \frac{-b^2}{1-C^2}$. In this case, the condition $w_D < 0$ is achieved provided $c^2 < 1$. Besides for $c^2 > 1 - 3b^2$ we have $w_D < -1/3$. Thus this model can describe the accelerated expansion if $1 - 3b^2 < c^2 < 1$. Moreover, w_D can cross the phantom line ($w_D < -1$) provided $c^2 > 1 - b^2$ **[58]**.

6.9 Correspondence with Scalar field models:

In this section we implement a correspondence between interacting HDE by taking Hubble radius as an IR cutoff, and various scalar field models, by comparing the holographic density with the corresponding scalar field model density and also equating the equations of state for this model with the equations of state parameter of interacting HDE obtained in (6.45).

6.10 Reconstructing holographic quintessence model:

In order to establish the correspondence between HDE and quintessence scalar field, we assume the quintessence scalar field model of dark energy is the effective underlying theory. The energy density and pressure of the quintessence scalar field are given by,

$$\rho_{\varphi} = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \quad \ldots \ldots \ldots (6.47)$$

$$P_{\varphi} = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \quad \ldots \ldots \ldots (6.48)$$

Thus the potential and the kinetic energy term can be written as,

$$V(\varphi) = \frac{1-w_{\varphi}}{2}\rho_{\varphi} \quad \ldots. \ldots \ldots (6.49)$$

& $$\dot{\varphi}^2 = (1+w_{\varphi})\rho_{\varphi} \quad \ldots \ldots \ldots (6.50)$$

Where $w_{\varphi} = \frac{P_{\varphi}}{\rho_{\varphi}}$ In order to implement the correspondence between HDE and quintessence scalar field, we identify $\rho_{\varphi} = \rho_D$ and $w_{\varphi} = w_D$.

Inserting equation (6.41) & (6.45) in equation (6.50) we get,

$$\dot{\varphi}^2 = \left[1 + \left\{-\frac{\Gamma}{3H}\left(\frac{1}{1-C^2}\right)\right\}\right]3C^2M_P^2H^2$$

$$\Rightarrow \dot{\varphi}^2 = \left[1 - \frac{3b^2H}{3H}\left(\frac{1}{1-C^2}\right)\right]3C^2M_P^2H^2 \qquad [\because \Gamma = 3b^2H\,]$$

$$\Rightarrow \dot{\varphi}^2 = (1 - \frac{b^2}{1-C^2})3C^2M_P^2H^2$$

$$\Rightarrow \dot{\varphi} = \sqrt{3(1-\frac{b^2}{1-C^2})}CM_P H$$

$$\Rightarrow \dot{\varphi} = \sqrt{3(1-\frac{b^2}{1-C^2})}CM_P \frac{\dot{R}}{R} \quad \dots\dots\dots (6.51)$$

Integrating,

$$\varphi(R) = \sqrt{3\left(1-\frac{b^2}{1-C^2}\right)}CM_P \ln R \quad \dots\dots\dots (6.52)$$

Where we have set φ(R_0 = 1) = 0 for simplicity. Next we want to obtain the scale factor as a function of t. Taking the time derivative of Eq. (6.36) and using (6.45) we find,

$$H^2 = \frac{1}{3M_P^2}(\rho_D + \rho_M)$$

Now differentiating,

$$2H\dot{H} = \frac{1}{3M_P^2}(\dot{\rho}_D + \dot{\rho}_M)$$

$$\Rightarrow 2H\dot{H} = \frac{1}{3M_P^2}[Q - 3H\rho_M - Q - 3H\rho_D(1+w_D)] \quad \text{[Using (6.37) \& (6.38)]}$$

$$\Rightarrow 2H\dot{H} = \frac{1}{3M_P^2}(-3H)[\rho_M + \rho_D(1+w_D)]$$

$$\Rightarrow 2H\dot{H} = \frac{-H}{M_P^2}\left[\rho_M + \rho_D\left(1-\frac{b^2}{1-C^2}\right)\right]$$

$$\Rightarrow \dot{H} = \frac{-1}{2M_P^2}\left[\rho_M + \left(\rho_D - \frac{\rho_D b^2}{1-C^2}\right)\right]$$

$$\therefore \frac{\dot{H}}{H^2} = \frac{\frac{-1}{2M_P^2}\left[\rho_M + \rho_D - \frac{\rho_D b^2}{1-C^2}\right]}{\frac{1}{3M_P^2}(\rho_M + \rho_D)} = \frac{-3}{2}\left\{1 - \frac{\rho_D b^2}{(1-C^2)(\rho_M + \rho_D)}\right\}$$

$$\Rightarrow \frac{\dot{H}}{H^2} = \frac{-3}{2}\left\{1 - \frac{\rho_D}{(\rho_M + \rho_D)}\frac{b^2}{(1-C^2)}\right\}$$

$$\therefore \frac{\dot{H}}{H^2} = \frac{-3}{2}\left(1 - \frac{b^2 C^2}{1-C^2}\right)$$

Again we have,

$$\frac{\dot{H}}{H^2} = \frac{-3}{2}K \qquad [\because K = 1 - \frac{b^2C^2}{1-C^2}]$$

$$\Rightarrow \frac{dH}{dt}\frac{1}{H^2} = \frac{-3}{2}K$$

$$\Rightarrow \frac{dH}{H^2} = \frac{-3}{2}Kdt$$

Integrating,

$$-\frac{1}{H} = \frac{-3}{2}Kt$$

$$\Rightarrow H = \frac{2}{3}\frac{1}{Kt} = \frac{\dot{R}}{R} \qquad \ldots\ldots\ldots (6.53)$$

$$\Rightarrow \frac{dR}{R} = \frac{2}{3K}\frac{1}{t}dt$$

$$\Rightarrow \int \frac{dR}{R} = \frac{2}{3K}\int \frac{1}{t}dt$$

$$\Rightarrow \ln R = \frac{2}{3K}\ln t$$

$$\therefore R = t^{\frac{2}{3K}} \qquad \ldots\ldots\ldots (6.54)$$

Hence equation (6.52) can be re-written as,

$$\varphi(t) = \sqrt{3(1-\frac{b^2}{1-C^2})CM_P \ln t^{\frac{2}{3K}}}$$

$$\Rightarrow \varphi(t) = \frac{2}{3K} CM_P \sqrt{3(1-\frac{b^2}{1-C^2})} \ln t \qquad \ldots\ldots\ldots (6.55)$$

Next we obtain the potential as a function of φ .Combining equation (6.45) with equation (6.49) we reach,

$$V(\varphi) = \frac{1-w_\varphi}{2}\rho_\varphi = \frac{1}{2}\left\{1-\left(\frac{-b^2}{1-C^2}\right)\right\} 3C^2 M_P^2 H^2$$

$$\therefore V(\varphi) = \frac{3}{2}\left(1+\frac{b^2}{1-C^2}\right) C^2 M_P^2 H^2$$

Using equation (6.53) & (6.55) we obtain the explicit expansion for potential,

$$V(\varphi) = \frac{3C^2 M_P^2}{2}[1+\frac{b^2}{1-C^2}]\frac{4}{9K^2} t^{-2}$$

$$\Rightarrow V(\varphi) = \frac{2C^2 M_P^2}{3K^2}[1+\frac{b^2}{1-C^2}]t^{-2} \qquad \ldots\ldots\ldots (6.56)$$

From (6.55) we have,

$$\sqrt{3\left(1-\frac{b^2}{1-C^2}\right)} \ln t = \frac{3K\varphi}{2CM_P}$$

$$\Rightarrow \ln t = \frac{1}{\sqrt{3\left(1-\frac{b^2}{1-C^2}\right)}} \frac{3K\varphi}{2CM_P}$$

$$\Rightarrow \ln t = \frac{3K}{2CM_P}\left\{3\left(1-\frac{b^2}{1-C^2}\right)\right\}^{\frac{-1}{2}} \varphi$$

$$\Rightarrow t = \exp\left[\frac{3K}{2CM_P}\left(3-\frac{3b^2}{1-C^2}\right)^{\frac{-1}{2}}\varphi\right]$$

$$\Rightarrow t^{-2} = \exp\left[-2.\frac{3K}{2CM_P}\left(3-\frac{3b^2}{1-C^2}\right)^{\frac{-1}{2}}\varphi\right]$$

$$\therefore t^{-2} = \exp\left[\frac{-3K}{CM_P}\left(3-\frac{3b^2}{1-C^2}\right)^{\frac{-1}{2}}\varphi\right]$$

Now from (6.56) we get,

$$V(\varphi) = \frac{2C^2M_P^2}{3K^2}\left[1+\frac{b^2}{1-C^2}\right]\exp\left[\frac{-3K}{CM_P}(3-\frac{3b^2}{1-C^2})^{\frac{-1}{2}}\varphi\right] \quad \ldots (6.57)$$

Let us discuss the condition for which the scale factor and hence the obtained potential, leads to the acceleration expansion at the present time. Requiring $\ddot{R} > 0$ for the present time, leads to k < 2/3, which can be translated into $c^2 > (1+3b^2)^{-1}$. Note that the condition k < 2/3 valid only for the late time where we have a dark energy dominated universe. In general k depends on c, and for the matter dominated epoch where c is no longer a constant, then k is also not a constant and varies with time. The obtained exponential potential here is well-known in the literature for the quintessence scalar field **[58].**

In addition to the fact that exponential potentials can give rise to an accelerated expansion, they possess cosmological scaling solutions in which the field energy density ρ_φ is proportional to the matter energy density ρ_m. Exponential potentials were used in one of the earliest models which could accommodate a period of acceleration today within it, the loitering universe.

CHAPTER -7

INFLATION IN HOMOGENOUS & ISOTROPIC SPACE-TIME

7.1 Introduction:

In this chapter we introduce the inflationary scenario of the early universe. We will take the cosmological principle and the Einstein equations as our starting point and from there on derive the conditions needed for inflation. We will then see that inflation may solve the flatness and horizon problem of standard Big Bang cosmology, and see how the physics may be described by a single scalar field**[29]**.

7.2 Friedmann Equations:

The dynamics of FRW space-time is characterized by the evolution of the scale factor *R(t)*, which is related to the energy-momentum density of the universe by the Einstein equation .Without a cosmological constant term $\Lambda g_{\mu\vartheta}$ they read

$$G_{\mu\vartheta} \equiv R_{\mu\vartheta} - \frac{1}{2} R g_{\mu\vartheta} = 8\pi G T_{\mu\vartheta} \quad \dots \dots \dots (7.1)$$

G is Newtonian constant which we shall express in terms of the reduced Plank mass M_p which is defined by $M_p^{-2} = 8\pi G$ in units $\hbar$ = c = 1. We shall often work in units were also M_p = 1. The symmetries of FRW space-time reduces the Einstein equations to just two coupled ordinary differential equations called the Friedmann equations .To see this consider first the Einstein tensor $G_{\mu\vartheta} = R_{\mu\vartheta} - \frac{1}{2} R g_{\mu\vartheta}$. The Ricci tensor $R_{\mu\vartheta}$ and the scalar curvature R are given by contractions of the Riemannian curvature tensor $R^{\rho}_{\mu\sigma\vartheta}$.

$$R_{\mu\vartheta} = R^{\rho}_{\mu\sigma\vartheta} = \delta_{\rho}\Gamma^{\rho}_{\mu\vartheta} - \delta_{\vartheta}\Gamma^{\rho}_{\mu\rho} + \Gamma^{\rho}_{\mu\vartheta}\Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\vartheta\sigma} \quad , \; R = g^{\mu\vartheta} R_{\mu\vartheta} \quad \dots \dots \dots (7.2)$$

$\Gamma^{\rho}_{\mu\vartheta}$ is the Christoffel connection which is related to the metric by ,

$$\Gamma^{\rho}_{\mu\vartheta} = \frac{1}{2} g^{\rho\sigma} (\delta_{\mu} g_{\vartheta\sigma} + \delta_{\vartheta} g_{\sigma\mu} - \delta_{\sigma} g_{\mu\vartheta})$$

Inserting the FRW metric

$$ds^2 = g_{\mu\vartheta} dx^{\mu} dx^{\vartheta} = -dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + Sin^2\theta d\varphi^2)\right]$$

the scalar curvature become

$$R_{00} = -\frac{3\ddot{R}}{R}\ ,\ R_{i0} = 0,\ R_{ij} = \left[\frac{\ddot{R}}{R} + 2H^2 + \frac{2k}{R^2}\right] g_{ij}\ ,\ R = 6\left[\frac{\ddot{R}}{R} + H^2 + \frac{k}{R^2}\right] \quad \dots\dots\dots (7.3)$$

Where g_{ij} is the spatial part of the FRW metric .We shall model the energy-momentum of the universe by a perfect fluid

$$T_{\mu\vartheta} = (p + \rho) U_{\mu} U_{\vartheta} - p g_{\mu\vartheta} \quad \dots\dots\dots (7.4)$$

ρ (t) and p(t) are the energy density and pressure respectively and U^{μ} is the four velocity of the fluid. The fluid is at rest in co-moving co-ordinates such that the cosmological principle is respected U^{μ} =(1,0,0,0) ,hence the energy momentum tensor takes the form

$$T^{\mu}_{\vartheta} = diag[-\rho(t), p(t), p(t), p(t)] \quad \dots\dots\dots (7.5)$$

Inserting this in the Einstein equation we obtain the Friedmann equations which are the two promised differential equations

$$H^2 = \frac{\rho}{3M_p^2} - \frac{k}{R^2} \quad \dots\dots\dots (7.6)$$

$$\dot{H} + H^2 = \frac{\ddot{R}}{R} = -\frac{1}{6M_p^2}(\rho + 3p) \quad \dots\dots\dots (7.7)$$

Oftentimes the first equation will be called the Friedmann equation while the second equation will be called the acceleration.

We considered the Einstein equations without an explicit cosmological constant term $\Lambda g_{\mu\vartheta}$.This term may be included by redefining/decomposing the energy density and pressure

$$\rho \to \tilde{\rho} + M_p^2\Lambda \quad , \quad p \to \tilde{p} - M_p^2\Lambda \qquad \ldots\ldots\ldots (7.8)$$

The tilde has temporarily been introduced to denote the contributions from matter and radiation. This leads to the notion of a vacuum energy $\rho_{vac} = M_p^2\Lambda$ with negative pressure $\rho_{vac} = -p_{vac}$.

This Friedmann equation then read

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{\rho}{3M_p^2} - \frac{k}{R^2} + \frac{\Lambda}{3} \qquad \ldots\ldots\ldots (7.9)$$

$$\frac{\ddot{R}}{R} = -\frac{(\rho + 3p)}{6M_p^2} + \frac{\Lambda}{3} \qquad \ldots\ldots\ldots (7.10)$$

A universe dominated by a cosmological constant provides the simplest example of Inflation. We will return to this point shortly.

<u>7.3 Standard Big-Bang Model :</u>

Now we briefly review the basics of the Standard hot Big Bang model, in which the universe is in a thermal radiation dominated state at the earliest times. We start by solving the Friedmann equation for the simple cases where the universe is dominated by either matter, radiation, curvature or a cosmological constant**[29]**. To do this, we first consider the time component of energy momentum conservation $\nabla_\mu T^\mu_0 = 0$

$$\dot{\rho} + 3H(\rho + p) = 0 \quad \dots\dots\dots (7.11)$$

We also define the equation of state parameter ω=P/ρ and consider it to be constant for simplicity.

Now from equation (7.10) We get ,

$$\dot{\rho} + 3H\rho(1 + \frac{p}{\rho}) = 0$$

$$\text{Or, } \dot{\rho} + 3H\rho(1 + \omega) = 0$$

$$\text{Or, } \frac{\dot{\rho}}{\rho} = -3H(1 + \omega)$$

$$\text{Or, } \frac{\dot{\rho}}{\rho} = -3\frac{\dot{R}}{R}(1 + \omega)$$

Integrating,

$$\text{Or, } \ln \rho = -3(1 + \omega)\ln R + \ln \rho_0$$

$$\text{Or, } \ln \rho = \ln R^{-3(1+\omega)} + \ln \rho_0$$

$$\therefore \rho = \rho_0 R^{-3(1+\omega)} \quad \dots\dots\dots (7.12)$$

The values of w for the different types of stress-energy are listed in table (i)

	ω	ρ(R)	R(t)	R(τ)
Radiation	1/3	R^{-4}	$t^{\frac{1}{2}}$	τ
Matter	0	R^{-3}	$t^{\frac{2}{3}}$	τ^2
Curvature	-1/3	R^{-2}	t^1	
Λ	-1	R^0	e^{Ht}	$-\frac{1}{H\tau}$

Table (i): *FRW solutions for a universe dominated by radiation, matter, curvature and a cosmological constant. Solutions in terms of conformal time* $d\tau = \frac{dt}{R}$ *are included.*

Where at present time t_0 , the scalar factor has been normalized to unity R (t_0) =1 .

Inserting this in the Friedmann equation

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{\rho}{3M_p^2} - \frac{k}{R^2} + \frac{\Lambda}{3}$$

For k =Λ =0 we get ,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{\rho}{3M_p^2}$$

$$\Rightarrow \frac{\dot{R}^2}{R^2} = \frac{1}{3M_p^2}\rho_0 R^{-3(1+\omega)} \qquad [\because \rho = \rho_0 R^{-3(1+\omega)}]$$

$$\Rightarrow \dot{R}^2 = \frac{1}{3M_p^2}\rho_0 R^{-3(1+\omega)} R^2 = \frac{1}{3M_p^2}\rho_0 R^{-3-3\omega+2}$$

$$\Rightarrow \dot{R} = \sqrt{\frac{\rho_0}{3}}\frac{1}{M_p} R^{\frac{-(1+3\omega)}{2}}$$

$$\Rightarrow \frac{dR}{dt} = \sqrt{\frac{\rho_0}{3}}\frac{1}{M_p} R^{\frac{-(1+3\omega)}{2}}$$

$$\Rightarrow R^{\frac{(1+3\omega)}{2}} dR = \sqrt{\frac{\rho_0}{3}}\frac{1}{M_p} dt$$

Integrating,

$$\frac{R^{\frac{(1+3\omega)}{2}+1}}{\frac{(1+3\omega)}{2}+1} = \sqrt{\frac{\rho_0}{3}}\frac{1}{M_p} t$$

$$\Rightarrow R^{\frac{3(1+\omega)}{2}} = \sqrt{\frac{\rho_0}{3}}\frac{1}{M_p}t.\frac{3(1+\omega)}{2}$$

$$\Rightarrow R(t) = t^{\frac{2}{3(1+\omega)}}$$

[Considering $\sqrt{\frac{\rho_0}{3}}\frac{1}{M_p} = 1$ & Neglecting the constant]

$$\therefore R(t) = t^{\frac{2}{3(1+\omega)}} \quad \dots\dots\dots (7.13)$$

Again $$R(t_0) = t_0^{\frac{2}{3(1+\omega)}} \quad \dots\dots\dots (7.14)$$

Dividing (7.13) & (7.14) we get ,

$$\frac{R(t)}{R(t_0)} = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+\omega)}}$$

$$\therefore R(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+\omega)}}$$

Again, we have,

$$H = \frac{\dot{R}(t)}{R(t)}$$

Integrating,

$$Ht + c = \ln R(t) \quad \dots\dots\dots (7.15)$$

But if for t_0 ,

$$R(t_0) = R_0$$

$$\therefore \ln[R(t_0)] = Ht_o + c$$

$$\Rightarrow \ln R_0 = Ht_o + c \quad \dots\dots\dots (7.16)$$

Now subtracting (7.16) from (7.15) we get ,

$$\ln R(t) - \ln R(t_0) = H(t - t_0)$$

$$\Rightarrow \ln \frac{R(t)}{R(t_0)} = H(t - t_0)$$

$$\Rightarrow \frac{R(t)}{R(t_0)} = e^{H(t-t_0)}$$

$$\Rightarrow R(t) = R(t_0) e^{H(t-t_0)}$$

$$\therefore R(t) = e^{H(t-t_0)} \quad [\because R(t_0) = 1] \quad \text{... … …} \; (7.17)$$

From the Friedmann equation we also find that the curvature contribution may be treated as a fictitious energy with $\rho_k = \frac{-3M_p^2 k}{R^2}$ & $\omega = \frac{-1}{3}$.The solutions for the different types of stress-energy may then be listed as in table (i).

If more than one species contribute to the energy density, ρ and p denote the sum of all components,

$$\rho \equiv \sum \rho_i \quad , \quad p \equiv \sum p_i \quad , \quad \omega = \frac{p_i}{\rho_i}$$

If the species are non-interacting the scaling laws applies throughout the expansion such that a flat universe k = 0 initially will be dominated by radiation. The energy density of radiation scales both with a volume factor R^{-3} and redshift of wavelength R^{-1} which combines to give R^{-4}. Hence matter which only scales with volume R^{-3} will eventually become the dominant constituent. At later times the evolution will be dominated by vacuum energy which does not scale at all. Note also that the scaling law for matter and radiation implies infinite energy density and temperature at an initial singularity $R \to 0$ for $t \to 0$. This leads to the notion of a hot Big Bang at some finite time t = 0 in the past. We have arrived at the hot Big Bang picture of the universe: A cosmological singularity at finite time in the past, followed by a hot radiation dominated phase, which gradually cools as the universe expands. At later times matter will be the dominant constituent and eventually vacuum energy**[29]**.

The Friedmann equation $H^2 = \frac{\rho}{3M_p^2} - \frac{k}{R^2}$ provides a time dependent critical energy density ρ_c for which the universe is spatially flat for K=0

$$H^2 = \frac{\rho_c}{3M_p^2}$$

$$\therefore \rho_c = 3M_p^2 H^2 = \frac{3H^2}{8\pi G} \quad \dots\dots\dots (7.18)$$

It is convenient to express the actual energy density ρ as a fraction of the critical value by defining the density parameter $\Omega \equiv \frac{\rho}{\rho_c}$. The Friedmann equation then takes the form,

$$H^2 = \frac{\rho}{3M_p^2} - \frac{k}{R^2}$$

$$\Rightarrow \frac{H^2}{H_0^2} = \frac{\rho}{3M_p^2 H_0^2} - \frac{k}{R^2 H_0^2}$$

$$\Rightarrow \frac{H^2}{H_0^2} = \frac{\rho_0 R^{-3(1+\omega)}}{\rho_{c_0}} + \frac{\rho_k R^2}{3M_p^2 H_0^2} R^{-2}$$

$$\Rightarrow \frac{H^2}{H_0^2} = \frac{\rho_0}{\rho_{c_0}} R^{-3(1+\omega)} + \frac{\rho_{k_0}}{\rho_{c_0}} R^{-3(1+\omega)}$$

$$\Rightarrow \frac{H^2}{H_0^2} = \Omega_{i,0} R^{-3(1+\omega_i)} + \Omega_{k,0} R^{-3(1-\frac{1}{3})} \quad [\because \omega = \frac{-1}{3}]$$

$$\therefore \left(\frac{H}{H_0} \right)^2 = \Omega_{i,0} R^{-3(1+\omega_i)} + \Omega_{k,0} R^{-2} \quad \ldots \ldots \ldots (7.19)$$

Which implies a consistency relation at present time ,

$$\sum_i \Omega_{i,0} + \Omega_{k,0} = 1 \quad \ldots \ldots \ldots (7.20)$$

According to observations of the CMB and large-scale structure, the present day universe is flat, dominated by dark energy, and has a considerable amount of dark matter and only traces of baryonic matter and radiation.

$$\Omega_b = 0.0499(22) \ , \ \Omega_{DM} = 0.265(11) \ , \ \Omega_\Lambda = 0.685^{+0.017}_{-0.016} \ , \ \Omega_k \cong 0 \ \ldots \ldots \ldots (7.21)$$

The universe went from being radiation dominated to matter dominated to $\frac{R_0}{R_{eq}} \approx 3\times10^3$ the CMB was emitted to $\frac{R_0}{R_{rec}} \approx 1100$ & dark energy became the dominant constituent at $\frac{R_0}{R_\Lambda} \approx \frac{1}{2}$ where the scale factor at present time R_0, have been included explicitly.

This concludes our brief review of the standard Hot Big Bang model.

7.4 Flatness problem:

The flatness problem comes from considering the Friedmann equations in a universe with matter and radiation, but no vacuum energy. To state and quantify the problem we rewrite the Friedmann equations in terms of the critical density $\Omega \equiv \frac{\rho}{\rho_c}$

We have ,

$$H^2 = \frac{8\pi G\rho}{3} - \frac{k}{R^2}$$

$$\Rightarrow 1 = \frac{8\pi G\rho}{3H^2} - \frac{k}{H^2R^2}$$

$$\Rightarrow 1 = \frac{\rho}{3H^2 / 8\pi G} - \frac{k}{H^2R^2}$$

$$\Rightarrow 1 = \frac{\rho}{\rho_c} - \frac{k}{H^2R^2}$$

$$\Rightarrow 1 = \Omega - \frac{k}{H^2R^2}$$

$$\Rightarrow \Omega - 1 = \frac{k}{H^2R^2} \quad \ldots\ldots\ldots (7.22)$$

Again we know,

$$\frac{\ddot{R}}{R} = \frac{-4\pi G}{3}(\rho + 3p)$$

$$\Rightarrow \frac{\ddot{R}}{R} = -\frac{1}{2}.\frac{8\pi G}{3}\rho(1 + 3\frac{p}{\rho})$$

$$\Rightarrow \frac{\ddot{R}}{R} = -\frac{1}{2}.\frac{8\pi G H^2}{3H^2}\rho(1 + 3\omega)$$

$$\Rightarrow \frac{\ddot{R}}{R} = -\frac{1}{2}H^2\frac{1}{(3H^2 / 8\pi G)}\rho(1 + 3\omega)$$

$$\Rightarrow \frac{\ddot{R}}{R} = -\frac{1}{2}H^2\frac{\rho}{\rho_c}(1 + 3\omega)$$

$$\therefore \frac{\ddot{R}}{R} = -\frac{1}{2}H^2\Omega(1 + 3\omega) \quad \ldots\ldots\ldots (7.23)$$

Now , From equation (7.22) & (7.23) we obtain ,

$$\Omega - 1 = \frac{k}{R^2 H^2} = \frac{k}{R^2 (\frac{\dot{R}}{R})^2} = \frac{k}{\dot{R}^2} = k\dot{R}^{-2}$$

Differentiating we get ,

$$d\Omega = -2k\dot{R}^{-3}\ddot{R} \quad \ldots \ldots \ldots (7.24)$$

Again ,

$$\frac{\ddot{R}}{R} = -\frac{1}{2} H^2 \Omega(1+3\omega)$$

$$\Rightarrow \frac{\ddot{R}}{R} = -\frac{1}{2} \frac{\dot{R}}{R^2}^{2} \Omega(1+3\omega)$$

$$\Rightarrow \frac{-2\ddot{R}R^2}{R\dot{R}^2} = \Omega(1+3\omega)$$

$$\Rightarrow \frac{-2\ddot{R}R^2}{R\dot{R}^2} \times \frac{k}{R^2 H^2} = \Omega(1+3\omega)(\Omega-1)$$

[Multiplying by eq^n (7.22) on both side]

$$\Rightarrow \frac{-2\ddot{R}R}{\dot{R}^2} \times k\dot{R}^{-2} = \Omega(1+3\omega)(\Omega-1)$$

$$\Rightarrow -2k\ddot{R}\dot{R}^{-3}.\frac{R}{\dot{R}} = \Omega(1+3\omega)(\Omega-1)$$

$$\Rightarrow d\Omega.\frac{1}{d(\ln R)} = \Omega(1+3\omega)(\Omega-1)$$

$$\therefore \frac{d\Omega}{d(\ln R)} = \Omega(1+3\omega)(\Omega-1)$$

A flat universe Ω=1 therefore remains flat at all times. This is an unstable fixed point if the strong energy condition $1+3w>0$ is satisfied (valid for radiation $\omega = \frac{1}{3}$ and $\omega=0$)

$$1+3\omega > 0 \qquad \Rightarrow \frac{d|\Omega - 1|}{d \ln R} > 0$$

Any deviation from flatness is amplified by the subsequent expansion; hence the flatness of the universe at present time $\Omega_0 \cong 1$ represents an initial fine tuning problem. This is referred to as the flatness problem of standard Big Bang cosmology in which the universe is initially dominated by radiation and later matter. On the other hand if 1 + 3ω< 0 (valid for example for a cosmological constant ω = -1), the universe evolves towards flatness:

$$1+3\omega < 0 \qquad \Rightarrow \frac{d|\Omega - 1|}{d \ln R} < 0 \quad \text{… … … (7.25)}$$

From (7.23) we see that this leads to accelerated expansion. The flatness problem may therefore be solved by introducing a period of accelerated expansion prior to radiation domination. The inflationary paradigm does exactly that. We may also state the flatness problem and its solution in terms of the co-moving Hubble scale $(RH)^{-1}$. From the Friedmann equation we infer the following behavior

$$\frac{d}{dt}(RH)^{-1} < 0 \rightarrow \text{Expansion towards flatness} \quad \text{… … … (7.26)}$$

$$\frac{d}{dt}(RH)^{-1} > 0 \rightarrow \text{Expansion away from flatness} \quad \text{… … … (7.27)}$$

The first condition applies to matter and radiation while the second applies to a cosmological constant. A shrinking comoving Hubble scale may be taken as the defining feature of inflation, it implies accelerated expansion since

$$\frac{d}{dt}(RH)^{-1} = \frac{-\ddot{R}}{(RH)^2} \qquad \text{…. … … (7.28)}$$

7.5 Horizon problem:

The isotropy of the CMB pose another problem in standard Big Bang cosmology called the horizon problem. The problem arises since the surface of last scattering consists of many $\approx 10^4$ causally disconnected patches as illustrated. It is highly unlikely that each patch, independently of the others should produce the same spectrum of black body radiation to make the CMB appear isotropic today. To be a bit more precise we consider particle horizons R_H (t) which are the distance light can travel between the initial singularity and time t. Photons travel along null paths which for radial trajectories in a flat universe are characterized by dr = dt/R. The comoving distance light can travel between times t_1 and t_2 is then

$$\Delta r = \int_{t_1}^{t_2} \frac{dt}{R(t)} = \frac{n}{1-n} H_0^{-1} (R_2^{\frac{1-n}{n}} - R_1^{\frac{1-n}{n}}) \quad \text{for} \quad R(t) \propto t^n \quad , \quad n < 1$$

…. … … (7.29)

Thus the comoving horizon size at time t is $R_H(t) \sim H_0^{-1} R(t)^{\frac{1-n}{n}}$ .At present time $R_H(t) \sim H_0^{-1}$ and we see explicitly that the Hubble scale H^{-1} provides a good estimate for the size and age of the observable universe if its constituents are matter and radiation.

When we look at the CMB we are observing the universe at scale factor $R_{rec} \cong 1/1100$.

Today the comoving distance to a point on the surface of last scattering is then well approximated by the horizon size $\Delta r \sim H_0^{-1}$. At recombination the comoving horizon size of such a point is $R_H(t_{rec}) \sim H_0^{-1}\sqrt{a_{rec}} \sim 10^{-2} H_0^{-1}$, were we assumed that the universe is matter dominated from t_{rec} until present time. Hence widely separated points on the surface of last scattering have non overlapping horizons at the time of recombination. So far we have compared the radius of two spheres. By including area and volume factors we find that the surface of last scattering consist of ~ 10^4 disconnected patches and ~ 10^6 disconnected volumes at the time of recombination**[29]**.

The horizon problem may be solved by introducing an early period of inflation prior to radiation domination. To see this and for later convenience we switch to conformal time τ defined by

$$d\tau = \frac{dt}{R}$$

The FRW metric is then conformally related to a static Minkowsky metric,

$$ds^2 = R(\tau)^2[-d\tau^2 + dr^2] \quad \dots \dots \dots (7.30)$$

Where we again restricted ourselves to radial propagation in a flat universe for the sake of simplicity (Generalization to curved spatial slides is straightforward). Conformal time allows us to draw light cones and infer casual relationships in a manner similar to that of special relativity. With these coordinates the particle horizon is conveniently given by the age of the universe in conformal time:

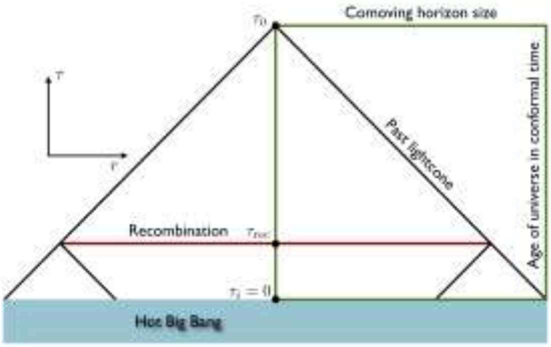
$$\tau = \int_0^t \frac{dt'}{R(t')} = \int_0^R d\ln R(\frac{1}{RH}) \quad \dots \dots \dots (7.31)$$

The size is the width of the past light cone projected onto the surface τ = 0 defined by the initial singularity. The integral has been written in terms of the commoving Hubble scale $(RH)^{-1}$ which is a more useful scale in inflationary cosmology than the particle horizon. We shall follow standard conventions and call $(RH)^{-1}$ the horizon. As we have seen it is about the size of the particle horizon during matter and radiation domination, but this does not hold in general. We classify co-moving length scales λ with associated wave number k according to their size relative to the horizon

$$\frac{k}{RH} << 1 \quad \Rightarrow \quad \text{scale } \lambda \text{ inside the horizon}$$

$$\frac{k}{RH} >> 1 \quad \Rightarrow \quad \text{scale } \lambda \text{ outside the horizon}$$

If a scale is larger than the horizon size causal physics cannot affect it. In standard Big Bang cosmology $d/dt\ (RH)^{-1} > 0$ such that scales which are outside the horizon at earlier times, such as the CMB scale cf. the horizon problem, may enter the horizon at later times. It is now clear that the horizon problem may be solved by an early period of inflation in which $d/dt\ (RH)^{-1} < 0$. In this scenario the CMB scale may initially be inside the comoving horizon such that causal physics can equilibrate it. However during Inflation the scale exits the horizon. When inflation ends the standard hot Big Bang commences and the comoving horizon size starts growing such that the CMB scale eventually reenters the horizon. In this scenario τ will get most of its contribution from early times and will be much larger than the estimate RH^{-1} provided by standard Big Bang cosmology**[29]**.

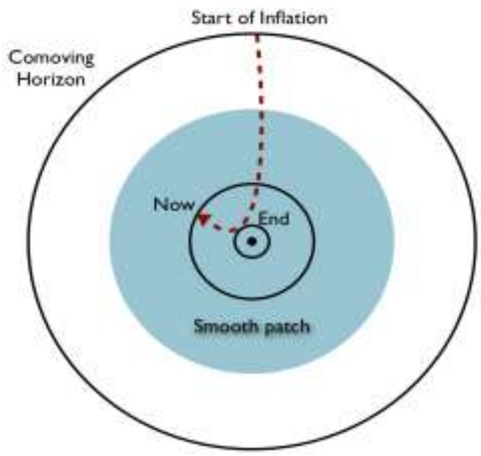


Figure(i): *Conformal diagram of standard Big Bang cosmology. The past light cone at the surface of last scattering does not overlap. This is the source of the horizon problem in standard Big Bang cosmology. In the text we estimated the surface of last scattering to consist of ~10^4 causally disconnected patches.*

The inflationary paradigm may be visualized by the conformal diagram in Fig(ii), as we now explain. In standard hot Big Bang cosmology the universe is dominated by radiation early on such that there is an initial singularity R $(\tau_i \equiv 0) =$ 0. However in the inflationary paradigm we assume that prior to radiation domination, there is a period of inflation d/dt $(RH)^{-1} < 0$. For the purpose of this discussion we assume that the universe is dominated by a cosmological constant in this period. This is the simplest case of inflation. Then $R \propto e^{Ht}$ and in conformal time the scale factor evolves as

$$R(\tau) = -\frac{1}{H\tau} \qquad \ldots\ldots\ldots (7.32)$$

Hence the initial singularity is pushed to the infinity past in conformal time, R → 0



Figure(ii): *Evolution of the comoving Hubble radius 1/RH in a universe which undergoes a period of inflation prior to radiation domination. The comoving Hubble radius shrinks dramatically during inflation. This allows the present day horizon to lie within a "smooth patch" that was well inside the horizon at the start of inflation. This solves the flatness and horizon problems.*

For $\tau \to -\infty$ thereby allowing past light cones to overlap. Note that the scale factor becomes infinite at $\tau = 0$. This is because we have assumed pure de Sitter space with H = constant. In this case inflation lasts forever, with $\tau = 0$ corresponding to the infinite future $t \to \infty$. In more realistic models, inflation ends at some finite time which is characterized by the breakdown as an approximation valid during inflation. In these models τ = 0 does not correspond to the initial singularity but a transition from inflation to radiation dominated expansion called reheating.

7.6 Inflation from a Scalar field:

In the preceding sections we introduced the inflationary paradigm as a solution to the flatness and horizon problems of the standard hot Big Bang model. We considered a simple model in which inflation is driven by a cosmological constant. This is not a realistic model since the universe stays dominated by the cosmological constant at all times such that inflation never ends. In order to transition from inflation to radiation domination the vacuum-like energy during inflation must be time dependent. This is traditionally modeled by introducing a single scalar field φ the inflation. We start by

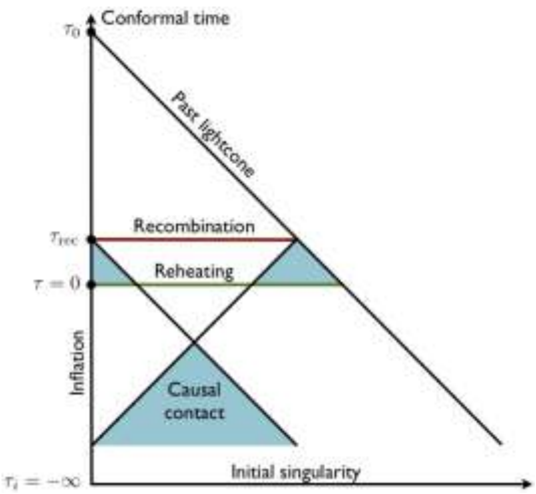


Figure (iii): *Conformal diagram of inflationary cosmology. Inflation pushes the initial singularity to the infinity past in conformal time, thereby allowing past light cones at recombination to overlap. Inflation ends in a reheating phase at τ ~ 0. During reheating the vacuum like energy of the inflationary sector is converted to other sectors.*

Considering the action of a scalar field with a minimal coupling to gravity

$$S = \int d^4x \sqrt{-g} [\frac{1}{2} M_p^2 R - \frac{1}{2} \delta_\mu \varphi \delta_\vartheta \varphi - V(\varphi)] \quad \ldots \ldots \ldots (7.33)$$

Where V(φ) is the potential energy associated with the field. Later we will consider the more general case of non-minimally coupled theories in which we add the term $^1/_2 R\xi\varphi^2$ to the action. The field is split into a classical homogeneous background φ (t) and fluctuations δφ (t, x)

φ (t, x)= φ (t)+ δφ (t, x)

The near isotropy of the CMB suggest that we may treat δφ(t,x) as small perturbations which evolve on a classical homogenous background solution given by φ (t) and the FRW metric. In this chapter we are only concerned with the evolution of the homogeneous background while fluctuations are considered later. The energy momentum tensor is

$$T_{\mu\vartheta} = \delta_\mu \varphi \delta_\vartheta \varphi - g_{\mu\vartheta} \left(\frac{1}{2} \delta^\mu \varphi \delta_\mu \varphi + V(\varphi) \right) \quad \ldots \ldots \ldots (7.34)$$

For the homogeneous background it is of the perfect fluid form (7.5) with

$$\rho = \frac{1}{2} \dot{\varphi}^2 + V(\phi)$$

$$P = \frac{1}{2} \dot{\varphi}^2 - V(\phi)$$

If the potential energy of the field is dominant $V(\varphi) >> \dot{\varphi}^2$ we recover the vacuum like behavior of inflation characterized by negative pressure ω < P/ρ accelerated expansion ω < -1/3 and hence shrinking horizon.

CHAPTER -8

INFLATION IN NON-MINIMALLY COUPLED THEORIES & INFLATION VIA MODIFIED GRAVITY

8.1 Introduction:

We consider the more general case where an explicit non-minimal coupling term $\xi\varphi^2R$ is added to the action. This leads to several interesting consequences which we explore. In particular it leads to lowering of the tensor-to-scalar ratio r, a feature which is favored by current experiments. It also alleviates the problem of tiny values for the inflation self-coupling. A coupling of this type is in general allowed by all symmetries of the scalar field sector and gravity. In fact the coupling is inevitable, as renormalization of a scalar field in curved space-time requires introduction of divergent counter terms of this type**[29]**.

Next we consider another approach where inflation is driven directly by the gravitational part of the action. This requires one to go beyond standard Einstein gravity and consider modified versions, for example in the context of f (R)-theories

8.2 Minimal coupling :

We are mainly interested in models with a non-minimal coupling to gravity. However, to appreciate what the non-minimal coupling term does, we first consider the case where the inflation is minimally coupled to gravity,

$$S=\int d^4x\sqrt{-g}[\frac{1}{2}M_p^2R-\frac{1}{2}g^{\mu\vartheta}\delta_\mu\varphi\delta_\vartheta\varphi-\frac{\lambda}{4}\varphi^4]$$

For quartic potential $V(\varphi)=\frac{\lambda}{4}\varphi^4$

However if we apply the slow-roll condition,

$$3H\dot{\varphi}+V'(\varphi)\approx 0$$

$$\Rightarrow \dot{\varphi} \approx \frac{-V'(\varphi)}{3H} \propto \varphi$$

$$\therefore \dot{\varphi}^2 \propto \varphi^2$$

Hence the potential energy $V(\varphi) = \frac{\lambda}{4}\varphi^4 \Rightarrow V(\varphi) \propto \varphi^4$ grows much faster than the kinetic energy $\dot{\varphi}^2 \propto \varphi^2$ as long as the field is far enough out on the potential, the slow-roll approximation are self-consistent.

We apply the slow-roll approximation & obtain from

$$\varepsilon_v = \frac{M_p^2}{2}\left(\frac{V'(\varphi)}{V}\right)^2 \quad \dots\dots\dots (8.1)$$

$$\eta_v = M_p^2 \frac{V''(\varphi)}{V} \quad \dots\dots\dots (8.2)$$

Here,

$$V(\varphi) = \frac{\lambda}{4}\varphi^4$$

$$\Rightarrow V'(\varphi) = \frac{\lambda}{4}.4.\varphi^3 = \lambda\varphi^3$$

$$\Rightarrow V''(\varphi) = 3\lambda\varphi^2$$

From (8.1),

$$\varepsilon_v = \frac{M_p^2}{2}\left(\frac{V'(\varphi)}{V}\right)^2$$

$$\Rightarrow \varepsilon_V = \frac{M_p^2}{2}\left(\frac{4}{\varphi}\right)^2 = \frac{M_p^2}{2}\cdot\frac{16}{\varphi^2}$$

$$\therefore \varepsilon_V = 8\left(\frac{M_p}{\varphi}\right)^2 \qquad \ldots\ldots\ldots (8.3)$$

From (8.2) ,

$$\eta_V = M_p^2 \frac{V''(\varphi)}{V} = M_p^2 \frac{12}{\varphi^2}$$

$$\therefore \eta_V = 12\left(\frac{M_p}{\varphi}\right)^2$$

So we get ,

$$\therefore \eta_H \cong \eta_V - \varepsilon_V = 4\left(\frac{M_p}{\varphi}\right)^2$$

Inflation ends when $\varepsilon_V(\varphi_{end}) \cong 1 \Rightarrow \varphi_{end} \cong \sqrt{8}M_p$

Assuming that the pivot scale K_* crossed the horizon $N_* = 60$ e-folds before the end of inflation yields,

$$N_* = \ln\frac{R_{end}}{R_*} \cong \frac{1}{M_p^2}\int_{\varphi_{end}}^{\varphi} \frac{V}{V'} d\varphi \qquad \Leftrightarrow \qquad K = R_* H_*$$

$$\varphi_* \approx (\sqrt{8N_* + 1})M_p \cong 22M_p$$

Hence we obtain inflation at super planckian field values. To stay out of the domain of quantum gravity the self-coupling λ needs to be small such that the energy density can be much less than the Planck density. In fact, matching the potential to the observed value of scalar perturbations

$$A_S = \frac{1}{24\pi^2 M_P^4}\frac{V_*}{\varepsilon_{V_*}} \qquad \Leftrightarrow \frac{V_*}{\varepsilon_{V_*}} \cong (0.0269M_P)^4$$

Requires that λ is extremely small

$$\frac{V_*}{\varepsilon_{V_*}} \cong (0.0269M_P)^4 \Leftrightarrow \lambda \sim 10^{-13}$$

The presence of an extremely small parameter is generic to minimally coupled models are represents a fine turning problem.

The values of r & η_S are estimated from

$$\eta_S - 1 \equiv \frac{d \ln \Delta_R^2}{d \ln K} = 2\eta_{V_*} - 6\varepsilon_{V_*} \quad \& \quad r \equiv \frac{\Delta_n^2}{\Delta_R^2} \cong 6\varepsilon_{V_*}$$

& r ~ 0.26 , η_S ~ 0.95

The high value of r places the model well outside the 99.7% CL regiuon in the (η_S ,r) plane as measured by Planck & is effectively ruled out.

8.3 Non-minimal coupling :

Adding a non-minimal coupling term changes the picture .The action now reads,

$$S_J = \int d^4x\sqrt{-g}\left[\frac{M_p^2 + \xi\varphi^2}{2}R - \frac{1}{2}g^{\mu\vartheta}\delta_\mu\varphi\delta_\vartheta\varphi - \frac{\lambda}{4}\varphi^4\right] \quad \ldots\ldots\ldots \quad (8.4)$$

We use our previous findings U(χ) = Ω^{-4} V(φ) where $\Omega^{-4} \sim \frac{M_P^4}{\xi^2\varphi^4}$.

$$\chi \cong \sqrt{6}M_P \ln\frac{\sqrt{\xi}\varphi}{M_P} \quad \text{For} \quad \varphi >> \frac{M_P}{3} \quad \text{and} \quad V(\varphi) = \frac{\lambda}{4}\varphi^4$$

To find the Einstein frame potential in the large field inflationary region $\varphi >> \frac{M_P}{\xi}$.

It is useful to express the potential in terms of both the original field φ, defined in the Jordan frame & the canonical Einstein frame field χ.

$$U(\chi(\varphi)) = \Omega^{-4} \; V(\varphi) \cong \frac{M_P^4}{\xi^2\varphi^4} \cdot \frac{\lambda}{4} . \varphi^4 (1 - \frac{M_P^2}{\xi\varphi^2})^2$$

$$\cong \frac{M_P^4 \lambda}{4\xi^2} (1 - \frac{M_P^2}{\xi\varphi^2})^2$$

$$\cong \frac{M_P^4 \lambda}{4\xi^2} \{1 - \exp[\ln(\frac{\sqrt{\xi}\varphi}{M_P})^{-2}]\}^2$$

$$\cong \frac{M_P^4 \lambda}{4\xi^2} \{1 - \exp[-2\ln(\frac{\sqrt{\xi}\varphi}{M_P})]\}^2$$

$$\cong \frac{M_P^4 \lambda}{4\xi^2} \{1 - \exp[\frac{-2}{\sqrt{6}M_P} \sqrt{6}M_P \ln(\frac{\sqrt{\xi}\varphi}{M_P})]\}^2$$

$$\therefore U(\chi(\varphi)) \cong \frac{M_P^4 \lambda}{4\xi^2} \left(1 - \exp\left[\frac{-2\chi}{\sqrt{6}M_P}\right]\right)^2$$

The slow-roll parameters are

$$\varepsilon_V = \frac{1}{2} M_P^2 (\frac{U'(\chi)}{U(\chi)})^2 \; , \quad \eta_V = M_P^2 \frac{U''(\chi)}{U(\chi)} \; , \quad N_* = \frac{1}{M_P^2} \int_{\chi_{end}}^{\chi_*} \frac{U(\chi)}{U'(\chi)} d\chi \qquad \dots \dots \dots (8.5)$$

Now,

$$U(\chi) \cong \frac{M_P^4 \lambda}{4\xi^2} \left(1 - \exp[\frac{-2\chi}{\sqrt{6}M_P}]\right)^2$$

$$\Rightarrow U(\chi) \cong \frac{M_P^4 \lambda}{4\xi^2} \left(1 - e^{\frac{-2\chi}{\sqrt{6}M_P}}\right)^2$$

$$\Rightarrow U(\chi) = \frac{M_P^4 \lambda}{4\xi^2} \left(\frac{e^{\frac{2\chi}{\sqrt{6}M_P}} - 1}{e^{\frac{2\chi}{\sqrt{6}M_P}}}\right)^2$$

$$\therefore U(\chi) = \frac{M_P^4 \lambda}{4\xi^2}\left(e^{\frac{2\chi}{\sqrt{6}M_P}} - 1\right)^2 e^{\frac{-4\chi}{\sqrt{6}M_P}}$$

Differentiating,

$$U'(\chi) = \frac{dU(\chi)}{d\chi} = \frac{M_P^4 \lambda}{4\xi^2}.2\left(1 - e^{\frac{-2\chi}{\sqrt{6}M_P}}\right) e^{\frac{-2\chi}{\sqrt{6}M_P}}\left(\frac{2}{\sqrt{6}M_P}\right)$$

$$\Rightarrow U'(\chi) = \frac{M_P^3 \lambda}{\sqrt{6}\xi^2}\left(1 - \frac{1}{e^{\frac{2\chi}{\sqrt{6}M_P}}}\right) e^{\frac{-2\chi}{\sqrt{6}M_P}}$$

$$\Rightarrow U'(\chi) = \frac{M_P^3 \lambda}{\sqrt{6}\xi^2}\left(\frac{e^{\frac{2\chi}{\sqrt{6}M_P}} - 1}{e^{\frac{2\chi}{\sqrt{6}M_P}}}\right) e^{\frac{-2\chi}{\sqrt{6}M_P}}$$

$$\therefore U'(\chi) = \frac{M_P^3 \lambda}{\sqrt{6}\xi^2}\left(e^{\frac{2\chi}{\sqrt{6}M_P}} - 1\right) e^{\frac{-4\chi}{\sqrt{6}M_P}}$$

Again, $$U'(\chi) = \frac{M_P^3 \lambda}{\sqrt{6}\xi^2}\left(e^{\frac{-2\chi}{\sqrt{6}M_P}} - e^{\frac{-4\chi}{\sqrt{6}M_P}}\right)$$

Differentiating,

$$U''(\chi) = \frac{M_P^3 \lambda}{\sqrt{6}\xi^2}\frac{-2}{\sqrt{6}M_P}\left[e^{\frac{-2\chi}{\sqrt{6}M_P}} + \frac{4}{\sqrt{6}M_P}e^{\frac{-4\chi}{\sqrt{6}M_P}}\right]$$

$$\Rightarrow U''(\chi) = \frac{M_P^3 \lambda}{\sqrt{6}\xi^2} \cdot \frac{2e^{\frac{-2\chi}{\sqrt{6}M_P}}}{\sqrt{6}M_P}\left(2e^{\frac{-2\chi}{\sqrt{6}M_P}} - 1\right)$$

$$\Rightarrow U''(\chi) = \frac{M_P^2\lambda}{3\xi^2} e^{\frac{-2\chi}{\sqrt{6}M_P}} \left(\frac{2}{e^{\frac{2\chi}{\sqrt{6}M_P}}} - 1 \right)$$

$$\Rightarrow U''(\chi) = \frac{M_P^2\lambda}{3\xi^2} e^{\frac{-2\chi}{\sqrt{6}M_P}} \left(\frac{2 - e^{\frac{2\chi}{\sqrt{6}M_P}}}{e^{\frac{2\chi}{\sqrt{6}M_P}}} \right)$$

$$\therefore U''(\chi) = \frac{M_P^2\lambda}{3\xi^2} \left(2 - e^{\frac{2\chi}{\sqrt{6}M_P}} \right) e^{\frac{-4\chi}{\sqrt{6}M_P}}$$

$$\therefore \varepsilon_V = \frac{1}{2} M_P^2 (\frac{U'(\chi)}{U(\chi)})^2$$

$$\Rightarrow \varepsilon_V = \frac{1}{2} M_P^2 \left[\frac{\frac{M_P^6\lambda^2}{6\xi^4} (e^{\frac{2\chi}{\sqrt{6}M_P}} - 1)^2 e^{\frac{-8\chi}{\sqrt{6}M_P}}}{\frac{M_P^8\lambda^2}{16\xi^4} (e^{\frac{2\chi}{\sqrt{6}M_P}} - 1)^4 e^{\frac{-8\chi}{\sqrt{6}M_P}}} \right]$$

$$\Rightarrow \varepsilon_V = \frac{1}{2} M_P^2 \times \frac{M_P^6\lambda^2}{6\xi^4} \times \frac{16\xi^4}{M_P^8\lambda^2} \left(e^{\frac{2\chi}{\sqrt{6}M_P}} - 1 \right)^{2-4} = \frac{4}{3} \left(e^{\frac{2\chi}{\sqrt{6}M_P}} - 1 \right)^{-2}$$

$$\therefore \varepsilon_V \cong \frac{4}{3} e^{\frac{-4\chi}{\sqrt{6}M_P}}$$ [Neglecting the constant term]

Now,

$$\eta_V = M_P^2 \frac{U''(\chi)}{U(\chi)}$$

$$\Rightarrow \eta_V = M_P^2 \times \left[\frac{\frac{M_P^2\lambda}{3\xi^2}(2-e^{\frac{2\chi}{\sqrt{6}M_P}})e^{\frac{-4\chi}{\sqrt{6}M_P}}}{\frac{M_P^4\lambda}{4\xi^2}(e^{\frac{2\chi}{\sqrt{6}M_P}}-1)^2 e^{\frac{-4\chi}{\sqrt{6}M_P}}} \right]$$

$$\Rightarrow \eta_V = M_P^2 \times \frac{M_P^2\lambda}{3\xi^2} \times \frac{4\xi^2}{M_P^4\lambda} \frac{(2-e^{\frac{2\chi}{\sqrt{6}M_P}})}{(e^{\frac{2\chi}{\sqrt{6}M_P}}-1)^2}$$

$$\Rightarrow \eta_V \cong \frac{4}{3} \frac{(-e^{\frac{2\chi}{\sqrt{6}M_P}})}{e^{\frac{4\chi}{\sqrt{6}M_P}}}$$ [Neglecting the constant term]

$$\therefore \eta_V \cong -\frac{4}{3} e^{\frac{-2\chi}{\sqrt{6}M_P}}$$

So finally we get,

$$\therefore \varepsilon_V \cong \frac{4}{3} e^{\frac{-4\chi}{\sqrt{6}M_P}}$$

$$\therefore \eta_V \cong -\frac{4}{3} e^{\frac{-2\chi}{\sqrt{6}M_P}}$$

At this point the conformal transformation to the Einstein frame and the subsequent field-redefinition has served its purpose. It allowed us to do the standard slow-roll approximation and find the slow-roll conditions in a simple way. We now reinsert the field redefinition.

$\chi \cong \sqrt{6}M_P \ln \frac{\sqrt{\xi}\varphi}{M_P}$ For $\varphi >> \frac{M_P}{3}$

$$\Rightarrow \frac{\chi}{\sqrt{6}M_P} = \ln(\frac{M_P}{\sqrt{\xi}\varphi})^{-1}$$

$$\Rightarrow \ln(e^{-\frac{\chi}{\sqrt{6}M_P}}) = \ln(\frac{M_P}{\sqrt{\xi}\varphi})$$

$$\Rightarrow e^{-\frac{\chi}{\sqrt{6}M_P}} = \frac{M_P}{\sqrt{\xi}\varphi} \qquad \therefore e^{-\frac{2\chi}{\sqrt{6}M_P}} = \frac{M_P^2}{\xi\varphi^2}$$

$\chi \cong \sqrt{6}M_P \ln\frac{\sqrt{\xi}\varphi}{M_P}$ for $\varphi >> \frac{M_P}{3}$ to express the results in terms of φ

$$\therefore \varepsilon_V \cong \frac{4}{3}\frac{M_P^4}{\xi^2\varphi^4} \quad \& \quad \eta_V \cong -\frac{4}{3}\frac{M_P^2}{\xi\varphi^2}$$

The field value at the end of inflation is ,

$$\varepsilon_V \cong \frac{4}{3}\frac{M_P^4}{\xi^2\varphi^4} \quad , \quad \eta_V \cong -\frac{4}{3}\frac{M_P^2}{\xi\varphi^2}$$

The field value at the end of inflation is ,

$$\varepsilon_V \sim 1 \Rightarrow 1 \cong \frac{4}{3}\frac{M_P^4}{\xi^2\varphi^4} \Rightarrow \varphi^4 \cong \frac{4}{3}\frac{M_P^4}{\xi^2}$$

$$\Rightarrow \varphi_{end} \cong \frac{\sqrt{2}}{\sqrt[4]{3}}\frac{M_P}{\sqrt{\xi}} \cong 1.07\frac{M_P}{\sqrt{\xi}}$$

Erasing also the number of e-folds (8.5) in terms of the Jordan frame field φ we get ,

$$N_* = \frac{1}{M_P^2}\int_{\varphi_{end}}^{\varphi_*} \frac{U}{dU\big/d\varphi}\left(\frac{d\chi}{d\varphi}\right)^2 \quad \ldots\ldots\ldots (8.6)$$

Here ,

$$U(\chi) = \frac{M_P^4\lambda}{4\xi^2}\left(1-\frac{M_P^2}{\xi\varphi^2}\right)^2$$

$$\Rightarrow \frac{dU}{d\varphi} = \frac{M_P^4\lambda}{4\xi^2}\times 2\left(1-\frac{M_P^2}{\xi\varphi^2}\right)\left(-\frac{M_P^2}{\xi}\right)(-2)\frac{1}{\varphi^3}$$

$$\Rightarrow \frac{dU}{d\varphi} = \frac{M_P^6\lambda}{\xi^3}\left(1-\frac{M_P^2}{\xi\varphi^2}\right)\frac{1}{\varphi^3}$$

Again,

$$\chi \cong \sqrt{6}M_P \ln\frac{\sqrt{\xi}\varphi}{M_P}$$

$$\frac{d\chi}{d\varphi} = \sqrt{6}M_P\frac{M_P}{\sqrt{\xi}\varphi}\frac{\sqrt{\xi}}{M_P} = \sqrt{6}M_P\frac{1}{\varphi}$$

$$\therefore \left(\frac{d\chi}{d\varphi}\right)^2 = 6M_P^2\frac{1}{\varphi^2}$$

Now,

$$\therefore N_* = \frac{1}{M_P^2}\int_{\varphi_{end}}^{\varphi_*} \frac{\frac{M_P^4\lambda}{4\xi^2}\left(1-\frac{M_P^2}{\xi\varphi^2}\right)^2}{\frac{M_P^6\lambda}{\xi^3}\left(1-\frac{M_P^2}{\xi\varphi^2}\right)\frac{1}{\varphi^3}} 6M_P^2\frac{1}{\varphi^2}\,d\varphi$$

$$\Rightarrow N_* = \frac{1}{M_P^2}\frac{M_P^4\lambda}{4\xi^2}\cdot\frac{6M_P^2\xi^3}{M_P^6\lambda}\int_{\varphi_{end}}^{\varphi_*}\left(1-\frac{M_P^2}{\xi\varphi^2}\right)\varphi\, d\varphi$$

$$\Rightarrow N_* = \frac{3\xi}{2M_P^2}\int_{\varphi_{end}}^{\varphi_*}(\varphi - \frac{M_P^2}{\xi\varphi})d\varphi$$

$$\Rightarrow N_* = \frac{3\xi}{2M_P^2}\int_{\varphi_{end}}^{\varphi_*}\varphi d\varphi$$ [Neglecting the 2nd term]

$$\Rightarrow N_* = \frac{3\xi}{4M_P^2}(\varphi_*^2 - \varphi_{end}^2)$$

$$\Rightarrow \varphi_*^2 = N_* \frac{4M_P^2}{3\xi}$$ [Neglecting the 2nd term]

$$\Rightarrow \varphi_*^2 = 60 \times \frac{4M_P^2}{3\xi}$$ [For N* = 60]

$$\Rightarrow \varphi_*^2 = \frac{80M_P^2}{\xi}$$

$$\therefore \varphi_* \sim \frac{9M_P}{\sqrt{\xi}}$$ [For N* = 60]

At first sight the large field approximation $\varphi >> \frac{M_P}{\sqrt{\xi}}$ seems inconsistent since φ_* is only about one order of magnitude larger than $\frac{M_P}{\sqrt{\xi}}$. Exact analytical solution to

$$S_E = \int d^4x\sqrt{-g}[\frac{1}{2}M_p^2R - \frac{1}{2}g^{\mu\vartheta}\delta_\mu\varphi\delta_\vartheta\varphi - U(\chi)]$$

do exist. However, they approach the large field approximation rapidly and the approximation is in fact quite good. To generate the proper value of scalar perturbations A_S we match the potential to $\frac{V_*}{\varepsilon_{V_*}} \cong (0.0269M_P)^4$ .In the previous section we found that this condition require the self-coupling to be extremely small $\lambda \sim 10^{-13}$. However, with the non-minimal coupling as an additional parameter the condition instead yields a relation between λ and ξ

$$\frac{U_*}{\varepsilon_{V_*}} = (0.0269 M_P)^4 \Rightarrow \xi \sim 48000\sqrt{\lambda}$$

Hence the problem of the tiny inflation self-coupling is alleviated, at the price however, of a large non-minimal coupling to gravity, which begs for a fundamental explanation. Note also that the initial assumption ξ >> 1 is self-consistent for sensible values of λ.

The values of r and ns may be estimated from

$$r \equiv \frac{\Delta_n^2}{\Delta_R^2} \cong 6\varepsilon_{V_*} \quad \& \quad n_S - 1 \equiv \frac{d \ln \Delta_R^2}{d \ln K} = 2\eta_{V_*} - 6\varepsilon_{V_*}$$

& r ~ 0.26 , $\eta_S = 0.95$

This lies well inside the 95% CL region as determined by Planck.

8.4 The Chaotic inflation in slow-roll approximation:

The chaotic inflation model is defined by the potential,

$$V(\varphi) = \frac{1}{2} m^2 \varphi^2 \quad \ldots \ldots \ldots (8.7)$$

Which is exactly analogous to the harmonic oscillator potential

$$V(x) = \frac{1}{2} kx^2$$

Using $V_0(\varphi) = \frac{1}{2} m^2 \phi_0{}^2$ the slow-roll solution,

$$R(t) \approx R_0 e^{t\sqrt{\frac{8\pi G v_0}{3}}}$$

becomes,

$$R(t) = R_0 e^{t\sqrt{\frac{8\pi G}{3}\times\frac{1}{2}m^2\varphi_0^2}}$$

$$\Rightarrow R(t) = R_0 e^{m\varphi t\sqrt{\frac{4\pi G}{3}}}$$

$$\therefore R(t) = R_0 e^{\frac{m}{M_p}\varphi t\sqrt{\frac{4\pi}{3}}} \qquad [\text{Where } G \equiv \frac{1}{M_p^2}]$$

Having specified V (φ) in equation (8.7) we can solve the equation $\sqrt{24\pi G V_0}\dot{\varphi} + V' = 0$ which is ,

$$\dot{\varphi} = -\frac{1}{\sqrt{24\pi G}} \cdot \frac{V'}{\sqrt{V_0}}$$

$$\Rightarrow \dot{\varphi} = -\frac{1}{\sqrt{24\pi G}} \cdot \frac{V'}{\sqrt{\frac{1}{2}m^2\varphi_0^2}}$$

$$\Rightarrow \dot{\varphi} = -\frac{1}{\sqrt{12\pi G}} \cdot \frac{V'}{m\varphi_0}$$

$$\Rightarrow \dot{\varphi} = -\frac{1}{\sqrt{12\pi G}} \cdot \frac{m^2\varphi}{m\varphi_0} \qquad [\because V' = m^2\varphi]$$

$$\Rightarrow \frac{\dot{\varphi}}{\varphi} = -\frac{1}{\sqrt{12\pi G}} \cdot \frac{m}{\varphi_0}$$

$$\Rightarrow \frac{\dot{\varphi}(t)}{\varphi(t)} = -\frac{1}{\sqrt{12\pi G}} \cdot \frac{m}{\varphi_0}$$

$$\Rightarrow \frac{d}{dt}\ln[\varphi(t)] = -\frac{m}{\sqrt{12\pi G}\varphi_0}$$

Integrating,

$$\ln\varphi(t) = -\frac{mt}{\varphi_0\sqrt{12\pi G}} + \ln\varphi_0$$

$$\Rightarrow \ln\varphi(t) = \ln\left(e^{-\frac{mt}{\varphi_0\sqrt{12\pi G}}} \right) + \ln\varphi_0$$

$$\Rightarrow \varphi(t) = \varphi_0 e^{-\frac{mt}{\varphi_0\sqrt{12\pi G}}}$$

$$\Rightarrow \varphi(t) = \varphi_0 e^{-\frac{mM_p t}{\varphi_0\sqrt{12\pi}}} \qquad [\because G \equiv \frac{1}{M_p^2}]$$

$$\Rightarrow \varphi(t) = \varphi_0\left(1 - \frac{mM_p t}{\varphi_0\sqrt{12\pi}} - \ldots\ldots\ldots \right)$$

$$\therefore \varphi(t) = \varphi_0 - \frac{mM_p t}{\sqrt{12\pi}} \qquad \ldots\ldots\ldots (8.8)$$

For m << M_p or for short times we see that $\varphi(t)$ will be approximately constant or slowly rolling. We see that $\varphi(t)$ is a decaying exponential on time .Thus there will be a “half-time” or “life-time” associated with slow-roll which is define as ,

$$\tau = \frac{\varphi_0\sqrt{12\pi}}{mM_p} \qquad \text{[by assuming } \varphi(t)=0]$$

When $t = \tau$ we see that $\varphi = \frac{\varphi_0}{e}$ .That is the amplitude is reduced by factor $\frac{1}{e}$.

We expect that the slow-roll approximation will be valid for $t < \tau$

We can obtain $\varphi(t)$ slightly differently.

Let's not assume $V=V_0$ =constant .but only that $\dot{\varphi}^2 << 2V$ & $\ddot{\varphi} << 0$.

Then instead of $\sqrt{24\pi G V_0}\dot{\varphi} + V' = 0$ we can write ,

$$\sqrt{24\pi G V}\,\dot{\varphi} + V' = 0$$

Which for chaotic inflation becomes ,

$$\sqrt{24\pi G.\frac{1}{2}m^2\varphi^2}\,\dot{\varphi} + m^2\varphi = 0$$

$$\Rightarrow \sqrt{12\pi G}m\varphi\dot{\varphi} + m^2\varphi = 0$$

$$\Rightarrow \sqrt{12\pi G}m\varphi\dot{\varphi} = -m^2\varphi$$

$$\Rightarrow \dot{\varphi} = \frac{-m}{\sqrt{12\pi G}}$$

Which has the solution ,

$$\varphi(t) = \varphi_0 - \frac{mt}{\sqrt{12\pi G}}$$

$$\Rightarrow \varphi(t) = \varphi_0 - \frac{mM_p t}{\sqrt{12\pi}} \quad \ldots\ldots\ldots (8.9)$$

In agreement with equation (8.8) for short time .We can further investigate the velocity of the slow-roll approximation by evaluating the potential as a function of time & checking that it is constant for short time. We do this by substituting our solution for $\varphi(t)$ back into the potential. We get,

$$V(\varphi) = \frac{1}{2} m^2 \phi^2$$

$$\Rightarrow V = \frac{1}{2} m^2 \varphi_0^2 e^{-\frac{2mM_p t}{\varphi_0 \sqrt{12\pi}}}$$

$$\Rightarrow V = \frac{1}{2} m^2 \varphi_0^2 e^{-\frac{mM_p t}{\varphi_0 \sqrt{3\pi}}}$$

$$\Rightarrow V \approx \frac{1}{2} m^2 \varphi_0^2 \left(1 - \frac{mM_p t}{\varphi_0 \sqrt{3\pi}} \right)$$

$$\therefore V \approx \frac{1}{2} m^2 \varphi_0 \left(\varphi_0 - \frac{mM_p t}{\sqrt{3\pi}} \right) \quad \dots \dots \dots (8.10)$$

Thus we see that for m << M_p or for short times the potential is indeed constant.

For short times (or for m << M_p) we have verified that φ & V are approximately constant.

This means that $\dot{\varphi} \approx 0$ & $\rho \approx V_0$ which give exponential inflation. (Also $\rho \approx -V_0$ .So that $P = -\rho$)

In order to solve the horizon, flatness & monopole problems .Most models requires a high degree of inflation typically amounting to about 60 e-folds.

Given $R(t) = R_0 e^{Ht}$ the number of e-folds after time t is ,

$$N = \ln(\frac{R(t)}{R_0}) = \ln(\frac{R_0 e^{Ht}}{R_0})$$

$$\therefore N = \ln(e^{Ht}) = Ht$$

After one lifetime τ the number of e-folds is,

$$\therefore N = H\tau = \sqrt{\frac{4\pi}{3}}.\frac{m}{M_p}.\varphi_0.\frac{\varphi_0\sqrt{12\pi}}{mM_p}$$

Where $H = \sqrt{\frac{4\pi}{3}}.\frac{m}{M_p}.\varphi_0$ and $\tau = \frac{\varphi_0\sqrt{12\pi}}{mM_p}$

$$\therefore N = \frac{4\pi r^2\varphi_0^2}{M_p^2} \quad \ldots\ldots\ldots (8.11)$$

Thus the requirement $N \geq 60$ yields

$$N \geq 60$$

$$\Rightarrow \frac{4\pi r^2\varphi_0^2}{M_p^2} \geq 60$$

$$\Rightarrow \frac{\varphi_0^2}{M_p^2} \geq 5$$

$$\therefore \varphi_0 \geq \sqrt{5}M_p \quad \ldots\ldots\ldots (8.12)$$

Notice how the flatness problem is solved in inflation .

We have ,

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G\rho}{3} - \frac{k}{R^2} \approx \frac{8\pi GV_0}{3} - \frac{k}{R^2}$$

Suring inflation (sloe-roll) V_0 stays constant but by the end of inflation ,

$$R = R_0 e^{60}$$

$$\text{Or, } \frac{1}{R^2} = \frac{1}{R_0^2}e^{-120}$$

The term $\frac{k}{R^2}$ has dropped by e^{-120} whereas V_0 has remained constant .Thus the term $\frac{k}{R^2}$ is entirely negligible .Inflation does not give K=0 ,but rather gives $\frac{k}{R^2} \approx 0$ which is equivalent to k=0 .This is an important destination .The universe can have k = 0 or k =1 or k= -1 .No matter what the value of k ,it gets diluted by inflation and it is equivalent to a universe with k=0 .Thus within our horizon the universe is flat .Quantum cosmology predict that a universe which arises via tunneling must have k=1 .This is perfectly okay with inflation which is simply dilutes the curvature .Quantum tunneling requires k=1. Inflation actually says *nothing* about the value of *k*. It simply predicts that $\frac{k}{R^2} \approx 0$ at the end of inflation. On Earth, the reason many people believe the Earth is flat is because we cannot see beyond the horizon. Up to the horizon it looks flat. If we could see beyond the horizon we would see the curvature. Similarly for our universe. According to inflation the size of the universe is much larger than the distance to the horizon (i.e. as far as we can see) the universe looks flat because $\frac{k}{R^2}$ is negligible. If we could see *beyond* the horizon we would see the curvature. And quantum tunneling predicts that what we would see would be a universe of positive curvature**[29]**.

8.5 Cosmological Constant associated with chaotic inflation :

Let us now calculate the density as a function of R .Now solving the equation

$$R(t) = R_0 e^{t\sqrt{\frac{4\pi}{3}}\frac{m}{M_p}\varphi_0} \qquad \text{For } t = t\,(R) \text{ as}$$

$$\Rightarrow e^{t\sqrt{\frac{4\pi}{3}}\frac{m}{M_p}\varphi_0} = \frac{R}{R_0}$$

$$\Rightarrow t\sqrt{\frac{4\pi}{3}}\frac{m}{M_p}\varphi_0 = \ln(\frac{R}{R_0})$$

$$\Rightarrow t = \frac{1}{\sqrt{\frac{4\pi}{3}}\frac{m}{M_p}\varphi_0}\ln(\frac{R}{R_0})$$

$$\Rightarrow t = \sqrt{\frac{3}{4\pi}}\frac{M_p}{m\varphi_0}\ln(\frac{R}{R_0}) \qquad \ldots\ldots\ldots (8.13)$$

So we get ,

$$\varphi(t) = \varphi_0 e^{\frac{-mM_p}{\varphi_0\sqrt{12\pi}}t}$$

$$\Rightarrow \varphi(R) = \varphi_0 e^{\frac{-mM_p}{\varphi_0\sqrt{12\pi}}.\sqrt{\frac{3}{4\pi}}.\frac{M_p}{m\varphi_0}.\ln(\frac{R}{R_0})}$$

$$\Rightarrow \varphi(R) = \varphi_0(\frac{R_0}{R})^{\frac{M_p^2}{4\pi\varphi_0^2}}$$

With

$$V(\varphi) = \frac{1}{2}m^2\varphi^2$$

$$\Rightarrow V(\varphi) = \frac{1}{2}m^2\varphi_0^2(\frac{R_0}{R})^{\frac{M_p^2}{2\pi\varphi_0^2}}$$

Also from equation (8.8) we get ,

$$\varphi(t) = \varphi_0 e^{\frac{-mM_p}{\varphi_0\sqrt{12\pi}}t}$$

$$\Rightarrow \dot{\varphi}(t) = \varphi_0(\frac{-mM_p}{\varphi_0\sqrt{12\pi}})e^{\frac{-mM_p}{\varphi_0\sqrt{12\pi}}t}$$

$$\Rightarrow \dot{\varphi}(t) = \frac{-mM_p}{\sqrt{12\pi}}.e^{\frac{-mM_p}{\varphi_0\sqrt{12\pi}}t}$$

$$\therefore \dot{\varphi}(R) = \frac{-mM_p}{\sqrt{12\pi}}\left(\frac{R_0}{R}\right)^{\frac{M_p^2}{4\pi\varphi_0^2}} \qquad \ldots\ldots\ldots (8.14)$$

Finally evaluating $\rho = \frac{1}{2}\dot{\varphi}^2 + V$ we have ,

$$\rho(R)=\frac{1}{2}m^2(\frac{M_p^2}{12\pi}+\varphi_0^2)(\frac{R_0}{R})^{\frac{M_p^2}{2\pi\varphi_0^2}}$$

$$\therefore \rho(R)\propto\frac{1}{R^m} \quad \left[\text{Here } m=\frac{M_p^2}{2\pi\varphi_0^2} \quad \& \quad \frac{1}{2}m^2\left(\frac{M_p^2}{12\pi}+\varphi_0^2\right)R^{\frac{M_p^2}{2\pi\varphi_0^2}}=Constnat\right]$$

Recall previously that an inflationary solution requires m < 2 yielding,

$$\varphi_0>\frac{M_p}{\sqrt{4\pi}}=0.3M_p$$

Now there arises a question that what is the equation of state?

Using, $p=\frac{1}{2}\dot{\varphi}^2-V$ we obtain

$$P=\frac{1}{2}(\frac{M_p^2}{12\pi}-\varphi_0^2)\left(\frac{R_0}{R}\right)^{\frac{M_p^2}{2\pi\varphi_0^2}}$$

We find that for inflation to occur we need $\varphi_0>\frac{M_p}{\sqrt{4\pi}}$ write this as, $\varphi_0=l\frac{M_p}{\sqrt{4\pi}}$ with $l>1$.

Then density becomes,

$$\rho(R)=\frac{m^2M_p^2}{8\pi}(\frac{1}{3}+l^2)(\frac{R_0}{R})^m \quad \ldots\ldots\ldots (8.15)$$

$$P(R)=\frac{m^2M_p^2}{8\pi}(\frac{1}{3}-l^2)(\frac{R_0}{R})^m \quad \ldots\ldots\ldots (8.16)$$

Defining, $\kappa\equiv\frac{m^2M_p^2}{8\pi}(\frac{R_0}{R})^m$ we write ,

$$\rho=(\frac{1}{3}+l^2)\kappa \quad \& \quad P=(\frac{1}{3}-l^2)\kappa$$

$$P=\frac{1-3l^2}{1+3l^2}\rho$$

The requirement $l > 1$ yields,

$$P < -\frac{1}{2}\rho \quad \dots\dots\dots (8.17)$$

Which means negative pressure writing $P = \frac{\gamma}{3}\rho$ gives, $\gamma = -\frac{3}{2}$

These results are in agreement with our previous constraints that in order to have positive $\ddot{R}$

We needed, $P = \frac{\gamma}{3}\rho$ or $\gamma < -1$

Our chaotic inflation model is the slow roll approximation gives negative pressure (but not P=-ρ) & corresponds to a weak decaying cosmological constant.

8.6 Inflation via modified gravity:

This requires one to go beyond standard Einstein gravity and consider modified versions, for example in the context of f(R)-theories.

In these theories the action is,

$$S = \int d^4x\sqrt{-g}\,\frac{M_P^2}{2}\,f(R) + \int d^4x\ L_M\,(g_{\mu\upsilon}\,,\psi_M) \quad \dots\dots\dots (8.18)$$

Where f (R) is an arbitrary function of the Ricci scalar R and L_M is a matter Lagrangian which is minimally coupled to gravity. This includes the Starobinsky model of inflation, which is one of the earliest models of inflation. The Starobinsky model features an R^2-term added to the Einstein-Hilbert action,

$$f(R) = R + \frac{R^2}{6M^2} \quad \ldots \ldots \ldots (8.19)$$

Where M is a new mass scale. We consider the Starobinsky model of Inflation in detail below. We begin our discussion by considering the field equations associated to the general action. These may be found by varying the action with respect to $g_{\mu\upsilon}$,

$$F(R)R_{\mu\vartheta} - \frac{1}{2} f(R)g_{\mu\vartheta} - \nabla_\mu \nabla_\vartheta F(R) + g_{\mu\vartheta} \Box F(R) = M_P^{-2} T_{\mu\vartheta}^M \quad \ldots \ldots \ldots (8.20)$$

Where $F(R) \equiv \frac{\delta f}{\delta R}$ & $T_{\mu\vartheta}^M$ is the energy-momentum tensor of the matter fields. We obtain the Starobinsky-Einstein equation

$G_{\mu\vartheta} \equiv R_{\mu\vartheta} - \frac{1}{2} R g_{\mu\vartheta} = -8\pi G T_{\mu\vartheta}$ by setting *f(R) =R* and *F(R)* = 1.

Now from equation (8.20) we have,

$$F(R)R_{\mu\vartheta} - \frac{1}{2} f(R)g_{\mu\vartheta} - \nabla_\mu \nabla_\vartheta F(R) + g_{\mu\vartheta} \Box F(R) = M_P^{-2} T_{\mu\vartheta}^M$$

$$\Rightarrow F(R)g^{\mu\vartheta}R_{\mu\vartheta} - \frac{1}{2} f(R)g^{\mu\vartheta}g_{\mu\vartheta} - \nabla_\mu \nabla_\vartheta F(R)g^{\mu\vartheta} + g_{\mu\vartheta}g^{\mu\vartheta} \Box F(R) = M_P^{-2} g^{\mu\vartheta}T_{\mu\vartheta}^M \quad \ldots\ldots.(8.21)$$

Here,

$$F(R)g^{\mu\vartheta}R_{\mu\vartheta} = F(R)[g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33}]$$

$$= F(R)\left[(-1)(-\frac{3\ddot{R}}{R}) + \frac{1}{R^2}(2\dot{R}^2 + R\ddot{R}) + \frac{1}{R^2r^2}r^2(2\dot{R}^2 + R\ddot{R}) + \frac{1}{R^2r^2\sin^2\theta}r^2\sin^2\theta(2\dot{R}^2 + R\ddot{R})\right]$$

$$= F(R)\left(\frac{3\ddot{R}}{R} + \frac{2\dot{R}^2}{R^2} + \frac{\ddot{R}}{R} + \frac{\ddot{R}}{R} + \frac{2\dot{R}^2}{R^2} + \frac{2\dot{R}^2}{R^2} + \frac{\ddot{R}}{R}\right)$$

$$= F(R)\left[6(\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2})\right]$$

$$= F(R)R$$

$$\frac{1}{2}f(R)g^{\mu\vartheta}g_{\mu\vartheta} = \frac{1}{2}f(R)[g^{00}g_{00} + g^{11}g_{11} + g^{22}g_{22} + g^{33}g_{33}]$$

$$= \frac{1}{2}f(R)\left[(-1)(-1) + \frac{1}{R^2}R^2 + \frac{1}{R^2r^2}R^2r^2 + \frac{1}{R^2r^2\sin^2\theta}R^2r^2\sin^2\theta\right]$$

$$= \frac{1}{2}f(R)(1+1+1+1) \quad = \frac{4}{2}f(R)$$

$$g^{\mu\vartheta}\nabla_\mu\nabla_\vartheta F(R) = g^{00}\nabla_0\nabla_0 F + g^{11}\nabla_1\nabla_1 F + g^{22}\nabla_2\nabla_2 F + g^{33}\nabla_3\nabla_3 F$$

$$= g^{00}(\delta_0\delta_0 F - \Gamma^0_{00}\dot{F}) + g^{11}(\delta_1\delta_1 F - \Gamma^0_{11}\dot{F}) + g^{22}(\delta_2\delta_2 F - \Gamma^0_{22}\dot{F}) + g^{33}(\delta_3\delta_3 F - \Gamma^0_{33}\dot{F})$$

$$= (-1)\left(\frac{d^2F}{dt^2} - 0\right) + \frac{1}{R^2}\left(\frac{d^2F}{dr^2} - R\dot{R}\frac{dF}{dt}\right) + \frac{1}{R^2r^2}\left(\frac{d^2F}{d\theta^2} - R\dot{R}r^2\frac{dF}{dt}\right) + \frac{1}{R^2r^2\sin^2\theta}\left(\frac{d^2F}{d\varphi^2} - R\dot{R}r^2\sin^2\theta\frac{dF}{dt}\right)$$

$$= -\frac{d^2F}{dt^2} + \frac{1}{R^2}\left(0 - R\dot{R}\frac{dF}{dt}\right) + \frac{1}{R^2r^2}\left(0 - R\dot{R}r^2\frac{dF}{dt}\right) + \frac{1}{R^2r^2\sin^2\theta}\left(0 - R\dot{R}r^2\sin^2\theta\frac{dF}{dt}\right)$$

$$= -\frac{d^2F}{dt^2} - \frac{\dot{R}}{R}\frac{dF}{dt} - \frac{\dot{R}}{R}\frac{dF}{dt} - \frac{\dot{R}}{R}\frac{dF}{dt} = -\frac{d^2F}{dt^2} - 3H\frac{dF}{dt}$$

$$= -(\frac{d^2}{dt^2} + 3H\frac{d}{dt})F \quad = \Box\ \mathrm{F} \qquad [\text{for } \Box = -(\frac{d^2}{dt^2} + 3H\frac{d}{dt})F\ ]$$

And,

$$g_{\mu\vartheta}g^{\mu\vartheta}\ \Box F(R) = \ \Box F(R)[g^{00}g_{00} + g^{11}g_{11} + g^{22}g_{22} + g^{33}g_{33}]$$

$$= \Box F(R)(1+1+1+1)$$

$$= 4\ \Box\ F(R)$$

Now from equation (8.21) we get,

$$F(R)R - \frac{4}{2}f(R) - \Box\ F(R) + 4\ \Box\ F(R) = M_P^{-2}g^{\mu\vartheta}T^M_{\mu\vartheta}$$

$$\Rightarrow F(R)R - 2f(R) + 3\Box\ F(R) = M_P^{-2}g^{\mu\vartheta}T^M_{\mu\vartheta} \quad \dots\dots\dots (8.22)$$

This reveals an extra propagating scalar degree of freedom ψ = *F(R)* as compared to standard Einstein gravity. We will soon see that this extra scalar degree of freedom may be used to drive inflation. In Einstein gravity the term F(R) vanishes and $R = -M_P^{-2} g^{\mu\vartheta} T_{\mu\vartheta}^{M}$ such that the Ricci scalar is determined by the matter content in the standard manner **[29]**.

In the following we consider vacuum solutions with $T_{\mu\vartheta}^{M} = 0$. In equation (8.20) we will consider the effects of integrating out matter fields. Also we consider flat FRW space-time (K=0) with metric,

$$ds^2 = -dt^2 + R^2(t)\delta_{ij} dx^i dx^j \quad \dots \dots \dots (8.23)$$

We have,

$$x^i, x^j = (x^0, x^1, x^2, x^3) = (t, r, \theta, \varphi) \quad \& \quad g_{ij} = \begin{cases} 1 & \text{if } \quad \mathrm{i = j} \\ 0 & \text{if } \quad \mathrm{i \neq j} \end{cases}$$

From the metric tensor,

$$ds^2 = g_{ij} dx^i dx^j$$

$$\Rightarrow ds^2 = g_{00} dx^0 dx^0 + g_{11} dx^1 dx^1 + g_{22} dx^2 dx^2 + g_{33} dx^3 dx^3$$

$$\Rightarrow ds^2 = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$$

$$\Rightarrow ds^2 = -dt^2 + R^2 dr^2 + R^2 r^2 d\theta^2 + R^2 r^2 \sin^2\theta d\varphi^2$$

$$\therefore ds^2 = -dt^2 + R^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)] \quad \& \quad R = \frac{6(R\ddot{R} + \dot{R}^2)}{R^2}$$

We know,

$$H = \frac{\dot{R}}{R}$$

$$\Rightarrow \frac{d}{dt}(H) = \frac{d}{dt}(\frac{\dot{R}}{R})$$

$$\Rightarrow \dot{H} = \frac{R\ddot{R} - \dot{R}^2}{R^2} = \frac{R\ddot{R}}{R^2} - \frac{\dot{R}^2}{R^2} = \frac{\ddot{R}}{R} - H^2$$

$$\therefore \frac{\ddot{R}}{R} = \dot{H} + H^2$$

$$\therefore R = \frac{6(R\ddot{R} + \dot{R}^2)}{R^2} = 6(\frac{R\ddot{R}}{R^2} + \frac{\dot{R}^2}{R^2}) = 6(\frac{\ddot{R}}{R} + H^2) = 6(\dot{H} + H^2 + H^2) = 6(\dot{H} + 2H^2)$$

The Ricci scalar is,

$$R = (\dot{H} + 2H^2) \quad \ldots \ldots \ldots (8.24)$$

With H the Hubble constant .Since we are studying inflation we are interested in (quasi) de Sitter solutions with H and R constant. In this case the term F(R) vanished from the trace equation which then reads,

$$0 + F(R)R - 2f(R) = 0$$

$$\Rightarrow F(R)R - 2f(R) = 0 \quad \ldots \ldots \ldots (8.25)$$

$$\Rightarrow F(R)R = 2f(R) = F(R)R = \frac{\delta f}{\delta R}R \qquad [\because F(R) = \frac{\delta f}{\delta R}R]$$

$$\Rightarrow 2\frac{\delta R}{R} = \frac{\delta f}{f}$$

$$\Rightarrow \ln f = 2\ln R + \ln R_0$$

$$\Rightarrow f(R) = R_0 R^2$$

$$\therefore f(R) \propto R^2$$

The model $f(R) \propto R^2$ solves this condition & gives rise to an exact de-sitter solution.

We may consider this is as a correction to Einstein gravity & write,

$$f(R) = R + \frac{R^2}{6M^2}$$

$$\Rightarrow \frac{R}{2}F(R) = R(1 + \frac{R}{6M^2})$$

$$\therefore F(R) = 1 + \frac{R}{3M^2} \qquad \ldots \ldots \ldots (8.26)$$

Where M is a mass scale .Then at high R-values where the R^2-term dominates we obtain quasi de Sitter expansion $F(R)R - 2f(R) \approx 0$. This is the famous Starobinsky model of inflation. During inflation R decreases such that Inflation ends when the quadratic term becomes smaller than the linear term R ~M^2.

Now we first insert the Starobinsky model and the FRW-metric in the field equations (8.20) then we can get the following calculation,

$$R_{\mu\vartheta} - \frac{1}{2}Rg_{\mu\vartheta} = -8\pi G T_{\mu\vartheta}$$

For μ = υ = 0, $$R_{00} - \frac{1}{2}Rg_{00} = -8\pi G T_{00}$$

$$\Rightarrow \frac{-3\ddot{R}}{R} - \frac{1}{2}(-1)\frac{6(R\ddot{R} + \dot{R}^2)}{R^2} = -8\pi G T_{00}$$

$$\Rightarrow \frac{-3\ddot{R}}{R} + \frac{3R\ddot{R}}{R^2} + \frac{3\dot{R}^2}{R^2} = -8\pi G T_{00}$$

$$\Rightarrow \frac{-3\ddot{R}}{R} + \frac{3\ddot{R}}{R} + \frac{3\dot{R}^2}{R^2} = -8\pi G T_{00}$$

$$\Rightarrow \frac{3\dot{R}^2}{R^2} = -M_P^{-2}T_{00} \qquad [\because M_P^2 = \frac{1}{\sqrt{8\pi G}}]$$

$$\therefore M_P^{-2}T_{00} = -\frac{3\dot{R}^2}{R^2}$$

Now,

$$F(R)R_{\mu\vartheta} - \frac{1}{2}f(R)g_{\mu\vartheta} - \nabla_\mu\nabla_\vartheta F(R) + g_{\mu\vartheta}\Box F(R) = M_P^{-2}T^M_{\mu\vartheta}$$

For μ = υ = 0,

$$F(R)R_{00} - \frac{1}{2} f(R) g_{00} - \nabla_0 \nabla_0 F(R) + g_{00} \, \Box F(R) = M_P^{-2} T_{00}^M$$

$$\Rightarrow F(R)(-\frac{3\ddot{R}}{R}) - \frac{R}{4} F(R)(-1) - \delta_0 \delta_0 F + \Gamma_{00}^0 \dot{F} + (-1)\{-(\frac{d^2}{dt^2} + 3H \frac{d}{dt})F\} = M_P^{-2} T_{00}^M$$

$$\Rightarrow -\frac{3\ddot{R}}{R} F(R) + \frac{R}{4} F(R) - \frac{d^2 F}{dt^2} + 0 + \frac{d^2 F}{dt^2} + 3H \frac{dF}{dt} = M_P^{-2} T_{00}^M$$

$$\Rightarrow (\frac{R}{4} - \frac{3\ddot{R}}{R}) F(R) + 3H \frac{dF}{dt} = M_P^{-2} T_{00}^M$$

$$\Rightarrow \left[\frac{6(2H^2 + \dot{H})}{4} - 3(\dot{H} + H^2) \right] F(R) + 3H \frac{d}{dt} \left(1 + \frac{R}{3M^2} \right) = M_P^{-2} T_{00}^M$$

$$\Rightarrow (3H^2 + \frac{3}{2}\dot{H} - 3\dot{H} - 3H^2) \left(1 + \frac{R}{3M^2} \right) + 3H \frac{d}{dt} \left(1 + \frac{R}{3M^2} \right) = M_P^{-2} T_{00}^M$$

$$\Rightarrow -\frac{3}{2}\dot{H}(1 + \frac{R}{3M^2}) + 3H \frac{d}{dt} \left(1 + \frac{R}{3M^2} \right) = M_P^{-2} T_{00}^M$$

$$\Rightarrow -\frac{3}{2}\dot{H} \left(1 + \frac{6(2H^2 + \dot{H})}{3M^2} \right) + 3H \frac{d}{dt} [1 + \frac{6(2H^2 + \dot{H})}{3M^2}] = -\frac{3\dot{R}^2}{R^2}$$

$$\Rightarrow -\frac{3}{2}\dot{H} - \frac{6H^2\dot{H}}{M^2} - \frac{3\dot{H}^2}{M^2} + \frac{3H}{M^2} \frac{d}{dt}(M^2 + 4H^2 + 2\dot{H}) = -\frac{3\dot{R}^2}{R^2}$$

$$\Rightarrow -\frac{3}{2}\dot{H} - \frac{6H^2\dot{H}}{M^2} - \frac{3\dot{H}^2}{M^2} + \frac{3H}{M^2} \times 8H\dot{H} + \frac{3H}{M^2} \times 2\ddot{H} = -3H^2$$

$$\Rightarrow -\frac{3}{2}\dot{H} - \frac{6H^2\dot{H}}{M^2} - \frac{3\dot{H}^2}{M^2} + \frac{24\dot{H}H^2}{M^2} + \frac{6\ddot{H}H}{M^2} = -3H^2$$

$$\Rightarrow \left(-\frac{3}{2}\dot{H} - \frac{3\dot{H}^2}{M^2} + \frac{18\dot{H}H^2}{M^2} + \frac{6\ddot{H}H}{M^2} \right) \frac{M^2}{6H} = -3H^2 \times \frac{M^2}{6H}$$

$$\Rightarrow -\frac{\dot{H}M^2}{4H} - \frac{1}{2}\frac{\dot{H}^2}{H} + 3H\dot{H} + \ddot{H} = -\frac{1}{2}M^2 H$$

$$\therefore \ddot{H} - \frac{\dot{H}^2}{2H} + \frac{1}{2}M^2 H + 3H\dot{H} = \frac{\dot{H}M^2}{4H}$$

Again, for μ = υ = 1, $R_{11} - \frac{1}{2} R g_{11} = -8\pi G T_{11}$

$$\Rightarrow R\ddot{R} + 2\dot{R}^2 - \frac{1}{2} R^2 \frac{6(R\ddot{R} + \dot{R}^2)}{R^2} = -8\pi G T_{11}$$

$$\Rightarrow R\ddot{R} + 2\dot{R}^2 - 3R\ddot{R} - 3\dot{R}^2 = -8\pi G T_{11}$$

$$\Rightarrow -2R\ddot{R} - \dot{R}^2 = -8\pi G T_{11}$$

$$\therefore 2R\ddot{R} + R^2 = M_P^{-2} T_{11} \qquad [\because M_P^2 = \frac{1}{\sqrt{8\pi G}}]$$

Now, for μ = υ = 1,

$$F(R)R_{11} - \frac{1}{2} f(R) g_{11} - \nabla_1 \nabla_1 F(R) + g_{11} \Box F(R) = M_P^{-2} T_{11}^M$$

$$\Rightarrow (2\dot{R}^2 + \ddot{R}R)F(R) - \frac{R}{4} R^2 F(R) - \delta_1 \delta_1 F + \Gamma_{11}^0 \dot{F} + R^2 \{-(\frac{d^2}{dt^2} + 3H \frac{d}{dt})F\} = M_P^{-2} T_{11}^M$$

$$\Rightarrow (2\dot{R}^2 + \ddot{R}R - \frac{R^3}{4})F(R) - \frac{d^2 F}{dr^2} + R\dot{R}\dot{F} - R^2 \frac{d^2 F}{dt^2} - 3HR^2 \frac{dF}{dt} = M_P^{-2} T_{11}^M$$

$$\Rightarrow (2\dot{R}^2 + \ddot{R}R - \frac{R^3}{4})F(R) - 0 - 2R\dot{R}\frac{dF}{dt} - R^2 \frac{d^2 F}{dt^2} = M_P^{-2} T_{11}^M$$

$$\Rightarrow (2\dot{R}^2 + \ddot{R}R - \frac{R^3}{4})(1 + \frac{R}{3M^2}) - 2R\dot{R}\frac{d}{dt}(1 + \frac{R}{3M^2}) - R^2 \frac{d^2}{dt^2}(1 + \frac{R}{3M^2}) = M_P^{-2} T_{11}^M$$

$$\Rightarrow 2\dot{R}^2 + \ddot{R}R - \frac{R^3}{4} + \frac{2\dot{R}^2 R}{3M^2} + \frac{R^2 \ddot{R}}{3M^2} - \frac{R^4}{12M^2} - \frac{2\dot{R}^2 R}{3M^2} - \frac{R^2 \ddot{R}}{3M^2} = M_P^{-2} T_{11}^M$$

$$\Rightarrow 2\dot{R}^2 + R\ddot{R} - \frac{R^3}{4} - \frac{R^4}{12M^2} = \dot{R}^2 + 2R\ddot{R}$$

$$\Rightarrow \dot{R}^2 - R\ddot{R} - \frac{R^3}{4} - \frac{R^4}{12M^2} = 0$$

$$\Rightarrow \ddot{R} - \frac{\dot{R}^2}{R} + \frac{R^2}{4} + \frac{R^3}{12M^2} = 0$$

$$\Rightarrow \ddot{R} - \frac{\dot{R}}{R}\dot{R} + \frac{R^2}{4}\left(1 + \frac{R}{3M^2}\right) = 0$$

$$\therefore \ddot{R} - H\dot{R} + \frac{R^2}{4}F = 0$$

So finally we get,

$$\left.\begin{aligned} &\ddot{H} - \frac{\dot{H}^2}{2H} + \frac{1}{2}M^2H + 3H\dot{H} = \frac{\dot{H}M^2}{4H} \\ &\ddot{R} - H\dot{R} + \frac{R^2}{4}F = 0 \end{aligned}\right\} \quad \dots\dots\dots \ (8.27)$$

The first equation is the (0,0)-component which have been inserted in the (i, i)-component to obtain the second equation. When deriving these equations it is useful to know that the FRW-metric yields,

$$\Box F = \frac{1}{\sqrt{-g}}\delta_\mu(\sqrt{-g}\,g^{\mu\vartheta}\delta_\mu F) = -(\frac{d^2}{dt^2} + 3H\frac{d}{dt})F$$

$$\nabla_\mu\nabla_\vartheta F = \delta_\mu\delta_\vartheta F - \Gamma^0_{\mu\vartheta}\dot{F} \ , \ \Gamma^0_{00} = 0 \quad \& \quad \Gamma^0_{ij} = R\dot{R}\delta_{ij}$$

As we did earlier, we quantify slow-roll by smallness of the Hubble slow-roll parameters,

$$\varepsilon_H = \left|\frac{\dot{H}}{H^2}\right| << 1 \qquad , \qquad \eta_H = \left|\frac{\ddot{H}}{H\dot{H}}\right| << 1$$

The first two terms in equation (8.27) may then be neglected. From equation (8.26) we find that $R \approx 12H^2$, hence $\ddot{R}$ can also be neglected. The slow-roll approximation then becomes,

$$0 + 0 + \frac{1}{2}M^2H = -3H\dot{H} + 0$$

$$\therefore \dot{H} \approx -\frac{1}{6}M^2$$

The term may readily be integrated to obtain the slow-roll solution,

$$\frac{dH}{dt} \approx -\frac{1}{6}M^2$$

$$\Rightarrow dH \approx -\frac{1}{6}M^2 dt$$

Integrating,

$$H \approx H_i - \frac{1}{6}M^2 \int_{t_i}^{t} dt$$

$$\therefore H \approx H_i - \frac{1}{6}M^2 (t - t_i)$$

$$R \approx R_i \exp[H_i (t - t_i) - \frac{1}{12}M^2 (t - t_i)^2]$$

$$R \approx 12H^2 - M^2 \qquad \ldots \ldots \ldots (8.28)$$

Where i denotes the initial conditions .It can be shown that the slow-roll trajectory is an attractor in phase space and hence the further evolution is largely independent on the Initial conditions, as we discussed in section before. Accelerated expansion occurs as long as the slow-roll parameter ε_H is smaller than unity.

$$\varepsilon_H = -\frac{\dot{H}}{H^2} \cong \frac{M^2}{6H^2} \qquad \ldots \ldots \ldots (8.29)$$

Hence inflation occurs for $H^2 > M^2$. Inflation ends when $\varepsilon_H = 1$ *i.e.* $H_{end} \cong \frac{M}{\sqrt{6}}$. It follows that this corresponds to the time at which the Ricci scalar decreases to $R \sim M^2$.

8.7 Starobinsky Inflation in the Einstein frame:

The *f*(R) theory,

$$F(R)R - 2f(R) + 3\Box F(R) = M_P^{-2} g^{\mu\vartheta} T^M_{\mu\vartheta} \quad \dots\dots\dots (8.30)$$

This equation may be cast in a form that features a potential for the extra scalar degree of freedom which appeared above. This can be done by considering the following linear representation in terms of a new field y.

$$S = \int d^4x \sqrt{-g}\, \frac{M_P^2}{2} [f(y) + f'(y)(R-y)] \quad \dots\dots\dots (8.31)$$

We set $T^M_{\mu\vartheta} = 0$ since we will insert the Starobinsky model shortly .The equation of motion for y is,

$$f''(y)(R-y) = 0$$

If $f''(y) \neq 0$ it follows that y = R & we recover the original action equation (8.30).

By inserting the scalar degree of freedom,

$$\Psi \equiv f'(y) = F(y)$$

$$\Rightarrow f'(y) = \Psi \quad \dots\dots\dots (8.32)$$

$$\Rightarrow f(y) = \Psi y \quad \text{[Integrating w.r.to y]}$$

$$\Rightarrow f(R) = \Psi R \quad \text{[for y = R]}$$

$$\Rightarrow f(y) = \Psi R \quad \dots\dots\dots (8.33)$$

Again,

$$f'(y)(R-y) = (R-y)\Psi = R\Psi - y\Psi = f(y) - y\Psi$$

$$\Rightarrow f'(y)(R-y) = f(y(\Psi)) - y(\Psi)\Psi$$

$$\Rightarrow f'(y)(R-y) = -[y(\Psi)\Psi - f(y(\Psi))]$$

Putting these values in equation (8.31) we get,

$$S = \int d^4x \sqrt{-g}\, \frac{M_P^2}{2} [\Psi R - \{y(\Psi)\Psi - f(y(\Psi))\}]$$

$$\Rightarrow S = \int d^4x\sqrt{-g}\left[\frac{1}{2}M_P^2\Psi R - \frac{1}{2}M_P^2\{y(\Psi)\Psi - f(y(\Psi))\}\right]$$

$$\Rightarrow S = \int d^4x\sqrt{-g}\left[\frac{1}{2}M_P^2\Psi R - V(\Psi)\right] \quad \dots\dots\dots (8.34)$$

Where,

$$V(\Psi) = \frac{1}{2}M_P^2[y(\Psi)\Psi - f(y(\Psi))]$$

Hence we have obtained and action for the scalar degree of freedom ψ with potential V (ψ) which is equivalent to the f (R)-theory. It appears to have the same form as the non-minimally coupled models we considered earlier

$$S_j = \int d^4x\sqrt{-g}\left[\frac{M_P^2 + \xi\varphi^2}{2}R - \frac{1}{2}g^{\mu\vartheta}\delta_\mu\varphi\delta_\vartheta\varphi - V(\varphi)\right]$$

except that there is no kinetic term. We will discuss similarities and differences within the framework of Starobinsky inflation shortly. First we proceed by performing a conformal transformation. To do this it is convenient to reinsert F(R) and write the action in the form,

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}FR - V\right] \quad \dots\dots\dots (8.35)$$

Let us briefly repeat the steps of the conformal transformation. The metric and Ricci scalar transform as

$$g_{\mu\vartheta} \rightarrow \hat{g}_{\mu\vartheta} = \Omega^2 g_{\mu\vartheta}$$

$$R = \Omega^2[\hat{R} + 6\Box\ln\Omega - 6\hat{g}^{\mu\vartheta}(\delta_\mu\ln\Omega)(\delta_\vartheta\ln\Omega)] \quad \dots\dots\dots (8.36)$$

The transformed action then reads,

$$S = \int d^4x\sqrt{-\hat{g}}[\frac{M_P^2}{2}F\Omega^2[\hat{R} + 6\Box\ \ln\Omega - 6\hat{g}^{\mu\vartheta}(\delta_\mu\ln\Omega)(\delta_\vartheta\ln\Omega) - V]$$

$$\Rightarrow S = \int d^4x\sqrt{-\hat{g}}[\frac{M_P^2}{2}F\Omega^{-2}[\hat{R} + 6\Box\ln\Omega - 6\hat{g}^{\mu\vartheta}(\delta_\mu\ln\Omega)(\delta_\vartheta\ln\Omega) - \Omega^{-4}V]$$

$$\dots\dots\dots (8.37)$$

We land in the Einstein frame where the action is linear in $\hat{R}$ if we choose

$$\Omega^2 = F$$

We also see that the action may be canonically normalized by the field redefinition,

$$\chi = \sqrt{\frac{3}{2}} M_P \ln F$$

Defining the Einstein frame potential $U(\chi)$ as,

$$U(\chi) = \Omega^{-4} V = \frac{V}{F^2} = \frac{1}{F^2} \cdot \frac{1}{2} M_P^2 [y(\psi)\psi - f(y(\psi))] = \frac{1}{2F^2} M_P^2 (y\psi - f)$$

$$\therefore U(\chi) = \frac{1}{2F^2} M_P^2 (RF - f) \quad [\because \Psi(y) = F(y) \quad \& \quad \text{y = R}] \qquad \ldots\ldots\ldots (8.38)$$

The action finally takes the form from (8.37) we get,

$$S_E = \int d^4x \sqrt{-\hat{g}} [\frac{M_P^2}{2} F\Omega^{-2} R - \frac{M_P^2}{2} F\Omega^{-2} \{6 \Box \ln\Omega - 6\hat{g}^{\mu\vartheta} (\delta_\mu \ln\Omega)(\delta_\vartheta \ln\Omega)\} - \Omega^{-4} V]$$

$$= \int d^4x \sqrt{-\hat{g}} [\frac{M_P^2}{2} \Omega^2 \Omega^{-2} R - \frac{M_P^2}{2} \Omega^2 \Omega^{-2} \{6 \Box \ln\Omega - 6\hat{g}^{\mu\vartheta} (\delta_\mu \ln\Omega)(\delta_\vartheta \ln\Omega)\} - \Omega^{-4} V]$$

$$\therefore S_E = \int d^4x \sqrt{-\hat{g}} [\frac{M_P^2}{2} R - \frac{1}{2} \hat{g}^{\mu\vartheta} \delta_\mu \chi \delta_\vartheta \chi - U(\chi)] \qquad \ldots\ldots\ldots (8.39)$$

We may now follow the same steps as earlier, and analyze inflation using the Einstein frame potential within the standard slow-roll paradigm. We proceed by inserting the Starobinsky model,

$$f(R) = R + \frac{R^2}{6M^2} \qquad \Rightarrow F(R) = 1 + \frac{R}{3M^2} \qquad \ldots\ldots\ldots (8.40)$$

The field redefinition then reads

$$\chi = \sqrt{\frac{3}{2}} M_P \ln(1 + \frac{R}{3M^2}) \qquad \text{[Using (8.40)]}$$

Using this relation, the Einstein frame potential *i.e.* equation (8.38) becomes

$$U(\chi) = \frac{M_P^2(FR - f)}{2F^2} = \frac{M_P^2}{2}\left(\frac{R(1+\frac{R}{3M^2}) - (R + \frac{R^2}{6M^2})}{(1+\frac{R}{3M^2})^2}\right)$$

$$\Rightarrow U(\chi) = \frac{M_P^2}{2}\left(\frac{R + \frac{R^2}{3M^2} - R - \frac{R^2}{6M^2}}{(1+\frac{R}{3M^2})^2}\right) = \frac{M_P^2}{2} \times \frac{R^2}{6M^2} \times \frac{1}{(1+\frac{R}{3M^2})^2}$$

$$\Rightarrow U(\chi) = \frac{M_P^2}{2} \times \frac{1}{6M^2} \times 9M^4(F-1)^2 \frac{1}{F^2} \qquad \text{[by using (8.40)]}$$

$$\Rightarrow U(\chi) = \frac{3}{4} M_P^2 M^2 (1 - \frac{1}{F})^2$$

$$\Rightarrow U(\chi) = \frac{3}{4} M_P^2 M^2 \left(1 - \frac{1}{1+\frac{R}{3M^2}}\right)^2$$

$$\Rightarrow U(\chi) = \frac{3}{4} M_P^2 M^2 \left[1 - \left(1 + \frac{R}{3M^2}\right)^{-1}\right]^2$$

$$\Rightarrow U(\chi) = \frac{3}{4} M_P^2 M^2 \left[1 - \exp\left\{\ln(\left(1 + \frac{R}{3M^2}\right)^{-1}\right\}\right]^2$$

$$\Rightarrow U(\chi) = \frac{3}{4} M_P^2 M^2 \left[1 - \exp\left\{-\ln\left(1 + \frac{R}{3M^2}\right)\right\}\right]^2$$

$$\Rightarrow U(\chi) = \frac{3}{4} M_P^2 M^2 \left[1 - \exp\left\{\frac{-2}{\sqrt{6}M_P} \times \sqrt{\frac{2}{3}} M_P \ln\left(1 + \frac{R}{3M^2}\right)\right\}\right]^2$$

$$\therefore U(\chi) = \frac{3}{4} M_P^2 M^2 \left(1 - \exp\left[\frac{-2\chi}{\sqrt{6}M_P}\right]\right)^2$$

Except for the overall coefficient, this is the same as the large field limit of the quartic potential with non-minimal coupling.

The two potentials coincide if we make the identification

$$U(\chi(\varphi)) \;=\Omega^{-4}\;\; V(\varphi) \;\cong \frac{\lambda M_P^4}{4\xi^2}\left(1-\frac{M_P^2}{\xi\varphi^2}\right)^2$$

$$U(\chi) \cong \frac{\lambda M_P^4}{4\xi^2}\left(1-\exp\left[\frac{-2\chi}{\sqrt{6}M_P}\right]\right)^2$$

The two potentials coincide if we make the identification,

$$M^2 = \frac{\lambda}{3\xi^2} M_P^2$$

Hence, by using our earlier results we find that the Planck constraint on the amplitude of scalar perturbations

$$\frac{U_*}{\varepsilon_{V_*}} = (0.0269 M_P)^4 \qquad \Rightarrow \xi \sim 48000\sqrt{\lambda}$$

constrains the mass parameter M to be $M \sim 10^{-5}\, M_P$.

The slow-roll parameters are the same as for the quartic potential since the overall coefficient of the potential drop out in the derivation,

$$\varepsilon_V = \frac{4}{3}\left(e^{\frac{2\chi}{\sqrt{6}M_P}}-1\right)^{-2} \cong \frac{4}{3}e^{\frac{-4\chi}{\sqrt{6}M_P}} \quad \& \quad \eta_V \cong -\frac{4}{3}e^{\frac{-2\chi}{\sqrt{6}M_P}}$$

Note that the similarity only holds in the large field approximation of the quartic potential.

Setting $N_* = 60$, the Starobinsky model then gives the same values of r and η_S which we obtained,

$$r \sim 0.0033 \quad , \quad \eta_S = 0.966$$

This is in excellent agreement with results from Planck.

8.8 Comparison with the quartic potential:

Let us briefly touch upon the similarities of the Starobinsky model and the quartic potential with a non-minimal coupling. We follow which provides a nice comparison between Higgs inflation and the Starobinsky model**[59]**. We begin by noting that in the linear representation which we considered earlier, the action may explicitly be written as,

$$S = \int d^4x\sqrt{-g}\,\frac{M_P^2}{2}\left[R + \frac{2R\psi}{M_P M} - \frac{6\psi^2}{M_P^2}\right] \quad \ldots\ldots\ldots (8.41)$$

$$\rightarrow \int d^4x\sqrt{-g}\,\frac{M_P^2}{2}\left[R + \frac{R^2}{6M^2}\right] \quad \ldots\ldots\ldots (8.42)$$

Note that ψ now has mass dimension 2. The arrow indicates that we may obtain the Starobinsky action in the pure f (R) form by integrating out using its equation of motion. Consider now the action for the non-minimally coupled quartic potential**[29]**,

$$S_J = \int d^4x\sqrt{-g}\left[\frac{M_P^2 + \xi\varphi^2}{2}R - \frac{1}{2}g^{\mu\vartheta}\delta_\mu\varphi\delta_\vartheta\varphi - \frac{\lambda}{4}\varphi^4\right] \quad \ldots\ldots\ldots (8.43)$$

During slow-roll inflation the kinetic term is by definition negligible. The action then reads

$$S_J = \int d^4x\sqrt{-g}\left[\frac{M_P^2 + \xi\varphi^2}{2}R - \frac{\lambda}{4}\varphi^4\right]$$

$$\Rightarrow S_J = \int d^4x\sqrt{-g}\,\frac{M_P^2}{2}\left[1 + \frac{\xi\varphi^2}{M_P^2}R - \frac{\lambda}{2M_P^2}\varphi^4\right] \ldots\ldots\ldots (8.44)$$

Hence the inflation is an auxiliary field in this regime, and may be integrated out by means of its equation of motion

$$\varphi^2 = \frac{\xi R}{\lambda} \quad \ldots\ldots\ldots (8.45)$$

Inserting this in the Jordan frame action we obtain,

$$S_J = \int d^4x\sqrt{-g}\,\frac{M_P^2}{2}[R + \frac{\xi R}{M_P^2}.\frac{\xi R}{\lambda} - \frac{\lambda}{2M_P^2}\frac{\xi^2 R^2}{\lambda^2}]$$

$$\Rightarrow S_J = \int d^4x\sqrt{-g}\,\frac{M_P^2}{2}[R + \frac{\xi^2 R^2}{\lambda M_P^2} - \frac{\xi^2 R^2}{2M_P^2\lambda}]$$

$$\Rightarrow S_J = \int d^4x\sqrt{-g}\,\frac{M_P^2}{2}[R + \frac{\xi^2 R^2}{2\lambda M_P^2}] \quad \ldots\ldots\ldots (8.46)$$

Therefore, the non-minimally coupled quartic potential is equivalent to the Starobinsky model during inflation. If we make the identification

$$M^2 = \frac{\lambda}{3\xi^2}M_P^2$$

it exactly coincides with the f (R)-representation in (8.42).
i.e.

$$S_J = \int d^4x\sqrt{-g}\,\frac{M_P^2}{2}\left[R + \frac{\xi^2 R^2}{2\lambda}.\frac{1}{\frac{3M^2\xi^2}{\lambda}}\right]$$

$$\Rightarrow S_J = \int d^4x\sqrt{-g}\,\frac{M_P^2}{2}\left(R + \frac{\xi^2 R^2}{2\lambda}.\frac{\lambda}{3M^2\xi^2}\right)$$

$$\Rightarrow S_J = \int d^4x\sqrt{-g}\,\frac{M_P^2}{2}\left(R + \frac{R^2}{6M^2}\right)$$

Of course, this is the same conclusion as the one we drew earlier using the Einstein frame actions. However, this representation clarifies that the equivalence arises since the kinetic term in the model with the quartic potential is negligible during inflation. We implicitly made the same approximation when we derived the Einstein frame potential for the quartic potential.

The approximation was made by going to the large field regime $\varphi >> M_P\sqrt{\xi}$ in the field redefinition,

$$\frac{d\chi}{d\varphi} = \sqrt{\Omega^{-2} + \frac{3}{2}M_P^2\left(\frac{d}{d\varphi}\ln\Omega^2\right)^2} \rightarrow \chi \approx \sqrt{\frac{3}{2}}M_P \ln\Omega^2(\varphi) \quad \text{for} \quad \varphi >> \frac{M_P}{3}$$

It is also important to note that the non-minimal coupling ξ naturally appears in the model with quartic potential equation (8.43), whereas it can be absorbed in the auxiliary field in the linear representation of the Starobinsky model equation (8.44) & (8.42). Hence ξ is redundant in the Starobinsky model. This difference arises since there is no kinetic term for the auxiliary field in the linear representation of the Starobinsky model **[59]**.

Let us now consider the observables of the two models and at what level they differ.

In particular we consider differences in the (r, η_s)-plane by comparing the slow-roll parameter ε_V of the two models. To do so, we compare the action of the quartic potential with non-minimal coupling and kinetic term equation (8.43) with the linear representation equation (8.44) of the Starobinsky model. We compute the slow-roll parameter ε_V for both models in the Einstein frame.

$$\varepsilon_V = \frac{1}{2}M_P\left(\frac{1}{U}\frac{dU}{d\varphi}\right)^2\left(\frac{d\varphi}{d\chi}\right)^2$$

The Einstein frame potential U is the same for the two models whereas the field redefinition $\chi(\varphi)$, which is related to the kinetic term, differ. Recall from

$$\frac{d\chi}{d\varphi} = \sqrt{\Omega^{-2} + \frac{3}{2}M_P^2\left(\frac{d}{d\varphi}\ln\Omega^2\right)^2} = \sqrt{\frac{1+(\xi+6\xi^2)\varphi^2/M_P^2}{(1+\xi\varphi^2/M_P^2)^2}}$$

That the field redefinition for the model with quartic potential and kinetic term is,

$$\frac{d\chi}{d\varphi} = \sqrt{\Omega^{-2} + \frac{3}{2}M_P^2\left(\frac{d}{d\varphi}\ln\Omega^2\right)^2}$$

In the Starobinsky model the term Ω^{-2} vanishes. This is exactly equivalent to the large field approximation for the model with quartic potential and kinetic term. Of course, we do not perform the large field approximation here, since the slow-roll parameters would then coincide. Using

$$N_* = \frac{1}{M_p^2}\int_{\varphi_{end}}^{\varphi_*} \frac{U}{dU/d\varphi}\left(\frac{d\chi}{d\varphi}\right)^2 d\varphi$$

$$\Rightarrow N_* \approx \frac{3\xi}{4M_p^2}(\varphi_*^2 - \varphi_{end}^2)$$

$$\Rightarrow \varphi_* \approx \frac{9M_P}{\sqrt{\xi'}} \quad \text{For } N_* = 60$$

as the number of e-folding one may find the following relation between the slow-roll parameters

$$\frac{\varepsilon_{V,\varphi^4-inlfation}}{\varepsilon_{V,Starobinsky}} = \frac{8N\xi}{1+\frac{4}{3}N+8N\xi} \approx \frac{1-10^{-5}}{6\lambda}$$

The difference is extremely small and we do not expect observable differences in the (r, η_s)-plane unless there is a strong dependence on model dependent post inflationary physics.

8.9 Marginally deformed Starobinsky Gravity:

In the previous section we have seen that gravity itself may be responsible for inflation. This requires one to go beyond standard Einstein gravity, for example by modifying the gravitational action via f(R)-theories. In particular we have seen that Inflation occurs in the Starobinsky model. We now consider quantum-induced marginal deformations of the Starobinsky action. We parameterize the deformations by $R^{2(1-\alpha)}$ where α is a positive parameter smaller than one half. The Starobinsky model is recovered for α = 0**[59]**.

As we shall see, deformations of the Starobinsky action may lead to sizeable amplitude of primordial tensor modes, even for small α. originally we compared the deformed model with the BICEP2 results which indicated the presence of primordial tensor modes. In this section we compare only with the Planck results and note that independently on the validity of the BICEP2 results, it is interesting to know how deformations of the Starobinsky model alter the inflationary observables. In particular we argue that deformations may arise if a matter theory of particle physics is embedded in the gravitational theory.

8.10 Motivation:

According to cosmology can be used qualitatively to establish the quantization of gravity. In fact, by combining cosmological observations with an effective field theory (EFT) treatment of gravity one can start estimating the parameters entering gravity's effective action. An actual discovery of primordial tensor modes can therefore be used to determine these parameters at the inflationary scale, which may turn out to be close to the grand unification energy scale.

To lowest order, the effective action for gravity can be parameterized as,

$$S = \int d^4x\sqrt{-g}[-\frac{M_P^2}{2}R + R_0R^2 + R_1R^3 + c_0C^2 + e_0E + ...]$$

Beyond an expansion in the Ricci scalar R, we formally included the Weyl conformal tensor C^2 and the Euler four dimensional topological term E. However we can drop E since it is a total derivative.

Furthermore when gravity is quantized around the Friedmann Lemaitre Robertson Walker metric the Weyl terms are sub–leading since the geometry is conformally flat. We are left with an f (R) form of the EFT. In particular the first two terms reproduce the Starobinsky model. Higher powers of R, C^2 and E are naturally suppressed by the Planck mass scale. If inflation occurs at energy scales much below the Planck scale the EFT is accurate. We must, however, take into account also marginal deformations including, for example, logarithmic corrections to the action above. Because of the similarity between the EFT description of gravity and the chiral Lagrangian for Quantum Chromo Dynamics we expect the quantum-induced logarithmic corrections to play a fundamental role for a coherent understanding of low energy gravitational dynamics at the inflationary scale. This is exactly what happens in hadronic processes involving pions at low energies.

8.11 Inflation In the modified Starobinsky Model:

We encode these ideas as deformations of the Starobinsky action

$$S = \int d^4x\sqrt{-g}[\frac{M_P^2}{2}R + aM_P^{4\alpha}R^{2(1-\alpha)}] \quad \ldots\ldots\ldots (8.47)$$

Where a is now a dimensionless parameter. Now replacing a with the dimension full parameter

$$a \to \frac{1}{12M^2M_P^2} \quad \& \quad \alpha = 0$$

Then we get,

$$S = \int d^4x\sqrt{-g}[\frac{M_P^2}{2}R + \frac{1}{12M_P^2M^2}M_P^{4\times 0}R^{2(1-0)}]$$

$$\Rightarrow S = \int d^4x\sqrt{-g}[\frac{M_P^2}{2}R + \frac{1}{12M_P^2M^2}M_P^0R^2]$$

$$\Rightarrow S = \int d^4x\sqrt{-g}\,\frac{M_P^2}{2}[R + \frac{R^2}{6M_P^4M^2}]$$

The equivalence between the Starobinsky model and non-minimally coupled large field φ^4 – inflation allows us to map the deformed Starobinsky action into the model with potential $\lambda(\frac{\varphi}{\Lambda})^{4\gamma}$ which we considered,

$$S_j = \int d^4x\sqrt{-g}\left[\frac{M_P^2 + \xi\varphi^2}{2}R - \frac{1}{2}g^{\mu\vartheta}\delta_\mu\varphi\delta_\vartheta\varphi - \lambda\varphi^4(\frac{\varphi}{\Lambda})^{4\gamma}\right] \quad \ldots\ldots\ldots (8.48)$$

During Inflation the kinetic term (i.e. $g^{\mu\vartheta}\delta_\mu\varphi\delta_\vartheta\varphi$) is negligible, which as we have seen, corresponds to the large field regime $\varphi >> M_P\sqrt{\xi}$ with large non-minimal coupling ξ**[29]**. The action then reads,

$$S_J = \int d^4x\sqrt{-g}\left[\frac{M_P^2 + \xi\varphi^2}{2}R - \lambda\varphi^4(\frac{\varphi}{\Lambda})^{4\gamma}\right] \quad \ldots\ldots\ldots (8.49)$$

This is equivalent to the linear representation of the deformed Starobinsky action (8.47) if we make the following identifications,

$$\alpha = \frac{\gamma}{1+2\gamma} \quad , \quad a^{1+2\gamma} = \left(\frac{\xi}{4}\frac{(1+2\gamma)}{(1+\gamma)}\right)^{2(1+\gamma)}\frac{1}{\lambda(1+2\gamma)} \quad \ldots\ldots\ldots (8.50)$$

These results are obtained straightforwardly by following the steps outlined in comparison with the quartic potential. As we have seen ξ is redundant in the linear representation of the Starobinsky model, however we will retain the explicit dependence on ξ to ease the comparison between the two models. The slow-roll analysis is the same as,

$$U(\chi) = \Omega^{-4}V(\varphi(x)) = \frac{M_P^4}{(M_P^2 + \xi\varphi^2)^2}\lambda\varphi^4(\frac{\varphi}{\Lambda})^{4\gamma}$$

$$= \underbrace{\frac{\lambda M_P^2}{\xi^2}(1-\exp[\frac{-2\chi}{\sqrt{6}M_P}])^2}_{\Phi^4\text{–inflation}} . \underbrace{(\frac{M_P}{\sqrt{\xi}\Lambda})^{4\gamma}\exp[\frac{4\gamma\chi}{\sqrt{6}M_P}]}_{\text{correction from }\gamma}$$

It leads to the Einstein frame potential,

$$U(\chi) = \underbrace{\frac{\lambda M_P^4}{\xi^2}(1+\exp[\frac{-2\chi}{\sqrt{6}M_P}])^{-2}}_{\Phi^4\text{–inflation}} \underbrace{\xi^{-2\gamma}\exp[\frac{4\gamma\chi}{\sqrt{6}M_P}]}_{\text{correction from }\gamma}$$

As well as the tensor-to-scalar ratio and scalar spectral index

$$r = 0.00\underbrace{33+0.2}_{\Phi^4\text{–inflation}}7\gamma + 8.73\gamma^2 + O(\gamma^3) \quad , N_* = 60 \quad \ldots\ldots\ldots (8.51)$$

$$\eta_S = 0.96\underbrace{7+1.2}_{\Phi^4\text{–inflation}}3\gamma - 15.8\gamma^2 + O(\gamma^3) \quad , N_* = 60 \quad \ldots\ldots\ldots (8.52)$$

The under braced φ^4-terms refers to the potential one would obtain by setting $\gamma = 0$ and the Starobinsky model. The expansions show that the (r,η_s)-values of the Starobinsky model are sensitive to even small corrections in γ (or equivalently α). In particular we find that deformations of the Starobinsky action may lead to primordial tensor modes.

We argued in before that the Starobinsky model and the non-minimally coupled quartic potential with kinetic term are probably indistinguishable in the (r,η_s)- plane. Note that the same argument holds for the deformed Starobinsky model and the deformed quartic potential with a kinetic term, since the models have the same Einstein frame potential in the large field limit**[29]**.

In the next Figure generic modifications of the Starobinsky model are confronted with Planck data. We observe that cosmology may constrain the deformation parameter α, and as we will show shortly, α holds information regarding the generic particle content embedded in this gravity model of inflation.

8.12 Field theoretical approach to quantum gravity:

We now argue that these marginal deformations, expected from a purely phenomenological standpoint, arise naturally within a field-theoretical approach to quantum gravity. To gain insight we start by expanding equation (8.47) in powers of α and write

$$S_J \cong \int d^4x\sqrt{-g}\left\{\frac{M_P^2}{2}R + aR^2\left[1-2\alpha\log\left(\frac{R}{M_P^2}\right)\right]+O(\alpha^2)\right\} \quad \ldots\ldots\ldots (8.53)$$

The logarithmic term is reminiscent of what one would obtain via trace-log evaluations of quantum corrections. There are several possible sources for these corrections. They may arise for example by integrating out matter fields, or they can arise directly from gravity loops. To sum-up the entire series of logarithmic corrections, and hence recover the $R^{2(1-\alpha)}$, we expect that a renormalization group improved computation is needed**[29]**.

This suggests that we would be able to determine α if a more fundamental theory was at our disposal. In the absence of a full theory of quantum gravity we start here by comparing different predictions for the coefficient of the logarithmic term in equation (8.53) stemming out from.

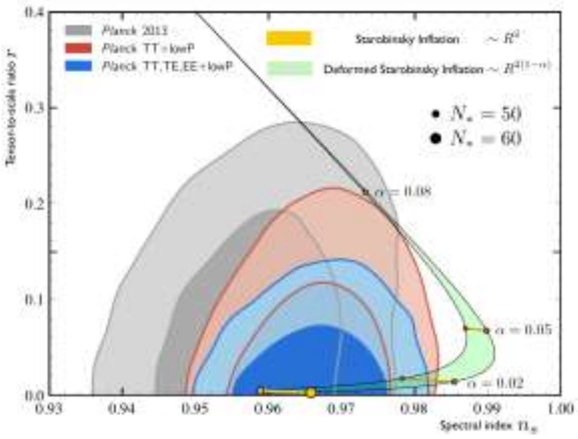


Figure: *Comparison with Planck results in the (r, η_S)-plane. The marginally deformed Starobinsky model gives the light green region. This region is obtained by letting N_* and α span the intervals N_* = [50,60] and α = [0, 0.15]. We find that the r and η_S depends sensitively on the value of α, which is related to the microscopic theory dictating the trace-log quantum corrections.*

1. Integrating out minimally coupled non-interacting NS real scalar fields (only non–conformal invariant matter contributes).

2. Gravity corrections via the effective field theory (EFT) approach.

3. Gravity corrections within higher derivative gravity (HDG).

For dimensional reasons these corrections can be parameterized by an $a(R/\mu^2)R^2$ term, where *a* is now a function of R/μ^2, with μ the renormalization scale. Explicit computations via heat kernel methods show that leading order quantum fluctuations will induce a logarithmic form for a as in (8.53). This fact alone immediately shows the link between the exponent α and the coefficient of the beta function related to the coupling of the R^2 term, as a scale derivative with respect to the mass scale in (8.53) shows. But we can give a better argument noticing that, because a depends on the ratio R/μ^2 we have $2R\delta_R a = -\mu\delta_\mu a$ a and one can determine the R dependence once the beta function, with respect to μ, of a is known. Non–local R^2 *log*(□/ μ^2) quantum corrections can also be derived in a similar way. To the lowest order the beta function is

$\mu\delta_\mu a = \frac{C}{(4\pi)^2}$ with C a constant depending on the source of quantum corrections considered. After an RG improvement, the equation for a reads,

$$R\delta_R a = \frac{-C}{2(4\pi)^2} a \qquad \dots \dots \dots (8.54)$$

The improvement is related to the appearance of a factor $a(R/\mu^2)$ on the right hand-side of the equation above. If one sets $a(R/\mu^2) = 1$ on the right-hand side, we only obtain the first logarithmic correction of (8.53).

Using (8.54) we construct the log-resumed solution

$$aR = a(R_0)\left(\frac{R}{R_0}\right)^{\frac{-C}{2(4\pi)^2}}$$

Here $R_0 = \mu_0^2$ is a given renormalization scale. We therefore have $\alpha = \frac{C}{4(4\pi)^2}$ and the constant *a* in equation (8.47) is *a(R₀)*. If C > 0 this would naturally lead to a positive α.

An explicit evaluation of C gives,

$$\left.\begin{aligned} C &= \frac{N_S}{72} && \text{Minimally coupled scalars} \\ C &= \frac{1}{4} && \text{EFT gravity} \\ C &= \frac{5}{36} && \text{HDG} \end{aligned}\right\} \quad \dots \dots (8.55)$$

Remarkably we deduce a positive exponent regardless of the underlying theory used to determine the associated quantum corrections to the gravitational action. Massive particles (we consider scalars of mass m for simplicity) lead to the beta function $\mu\delta_\mu a = \frac{C}{(4\pi)^2}(1+\frac{m^2}{\mu^2})$. When the renormalization scale is taken to be the Planck mass the effect of the mass term is negligible. Smaller renormalization scales generally tend to reduce the value of C and thus of α, but in particular they do not affect its sign.

From (8.55) we deduce that quantum gravitational contributions can account, at most, for a 3% increase in *r* as compared to the original Starobinsky model. Therefore any larger value of *r* can only be generated by adding matter corrections. This in turn can be used to constrain particle physics models minimally coupled to f (R) gravity.

Furthermore, as it is evident form Figure, for small r the spectral index (η_S) depends strongly on the particular value of α. For example we find that if Ns~ 90 or higher, the contour cross outside the one sigma confidence level provided by Planck. This corresponds to α ~ 0.02. To exemplify our results further, we may compare this with popular models of grand unification (GUT) such as minimal SU that features 34 scalars and (non) minimal SO featuring (297) 109 scalars. It is clear that only models with a low content of scalars are preferred by current experiments. Values of r around and above 0.2 can be achieved only by allowing for the presence of thousands of scalars.

This corresponds to the upper part of Figure.

To conclude, we have found that if inflation is driven by an *f* (R) theory of gravity, a natural form for this function is the marginally deformed Starobinsky action provided in (8.47) with a positive α. The size of α is related to the microscopic theory dictating the trace-log quantum corrections. This form can be tested by current and future experimental results and constitutes a natural generalization of the original Starobinsky model**[29]**.

References

References

(1) **Alpher R. A., Bethe H.** and **Gamow G.,** Phys. Rev. 73, Issue 7 (1948) 803.

(2) **Bergmann P. G.** (Professor of Physics, Syracuse University), "*Introduction to the theory of Relativity*", with a foreword by *Albert Einstein*.

(3) **Biswas D.**, *article in "An exact scalar field inflationary cosmological model and other problems of inflationary cosmology."*- Page Updated: Tuesday, 05-25-2010.*Netaji Park, Chakdaha, Nadia, 741222, India.*

(4) **Bondi H.**, "*Cosmology*", Cambridge University Press 1960.

(5) **Chow T. L.**, "*General Relativity and Cosmology*" (wuerz-Publishing Ltd. , Winnipeg. Canada,1994).

(6) **Coles P.**, "*The Origin and Evolution of Cosmic Structure*", Second Edition (Wiley, 2002).

(7) **Collins P.D.B.** , **Martin A. D.**, **Squires E. J.**, "*Particle Physics and Cosmology*", (Wiley, 1989); University of Durham, Durham, England.

(8) **Cresser J. D.**, "*Lecture notes on The Special Theory of Relativity",* Macquarie University 2003.

(9) **de Sitter, W.** 1917, Proc. Akad. Weteusch Amsterdam 19, 1217.

(10) **Dirac P. A. M.**, "*General Theory of Relativity*", John Wiley & Sons 1975.

(11) **Einstein A.**, "*Cosmological considerations in the General Theory of Relativity*," Sitzungsber. Preuss. Akad. Wiss. Phys.-math. Klasse VI (1917) 142.

(12) **Einstein A., Preuss S., Wiss A.. d..** Phys.-Math 142 (1917).

(13) **Foster J.** and **Nightiangle J . D.**, "*A Short Course in General Relativty*", 2nd edition (Springer-verlag, 1995) QC 173.6.F67.

(14) **Freedman W. L.** et al. [HST Collaboration], Astrophysics. J. 553, 47 (2001).

(15) **Friedman, A.** 1922, Z. Phys.,10,377.

(16) **Friedman, A.** 1924, Z. Phys.,21,326.

(17) **Faraoni V.,** *article in "General relativity and gravitation" · September 2012,* Bishop's University.

(18) **Garcia J. –**Bellido, "The Origin of Matter and Strtucture in the Universe", Phil. Trans. R. Soc. Lond. 357,3237 (1999).

(19) **Guth H.**, "*The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems*," Phys.Rev. D23 (1981) 347–356.

(20) **Guth H.**, "*An Inflationary Universe*", Perseus Books ,Reading 1997

(21) **Guth A.** and **Steinhardt P.**, "*The Inflationary Universe*", in the Newq Physics, edited by P. Davies, (Cambridge University Press 1989).

(22) **Guth A.** and **Y. Pi S.,** Phys**.** Rev. Lett 49 (1982) 1110.

(23) **Hawking S. W.,**Phys. Lett. B. 115 (1982) 295.

(24) **Hawking S. W. and Penrose R.,** "*The Singularity of Gravitational collapse and Cosmology*" proc. Soc. Lon. A 314, P.529.

(25) **Hawkings S.**, "*The Brief History of Time*".

(26) **Hooft G.'t**, "*Introduction to General Relativity*" (Caput College Lectures, Web Draft, 1998), Institute for Theoretical Physics, Utrecht University, Princetonplein 5,3584 CC Utrecht, the Netherlands

(27) **Islam J N "***An Introduction to Mathematical Cosmology*" (Second Edition); Research Centre of Mathematical and Physical Science. University of Chittagong.

(28) **Joshi P. S.,** "*Global Aspects in Gravitation and Cosmology*", Oxford University Press Inc., New York.

(29) **Jørgensen J. J.,** *a thesis on "Models of Inflation with non-minimal coupling"*, April 15, 2015,University of Southern Denmark .

(30) **Kolb W** and **Turner M. S.**, "*The Early Universe*".(Addition Wesley,1990).

(31) **Lawden D . F.**, "*An Introduction to Tensor Calculus*", Relativity and Cosmology. (Third edition)

(32) **Layer B. R. , Mukanda N.** and **Vishveshwara** Gravitation**,** "*Gauge Theories and the Early Universe***".**

(33) **Lemaitre, G. 1927 ,** Ann. De la Societe Scientific de Bruxelles, 47,49.

(34) **Liddle Andrew R.**, "*Inflationary Cosmology: Status and Prospects*", astro-ph/011155v1.

(35) **Liddle A.** (University of Sussex) ,David H. Lyth (University of Lancaster), "*Cosmological Inflation and Large Scale Structure*" (CUP,2000).

(36) **Liddle A.**, "*An Introduction to Modern Cosmology*" (2nd edition), University of Sussex, UK.

(37) **Liddle A. R.** and **Lyth D. H., "***Cosmological Inflation and Large Scale structure".*Cambridge University Press, Cambridge, UK, 2000.

(38) **Linde A.**, "*Particle Physics and Inflationary Cosmology*", (Web Version 2004).

(39) **Linde D.,** *"The self-producing Inflationary Universe",* Scientific Am. November, 32, 1994.

(40) **Linde D.**, "*Chaotic Inflation*," Phys. Lett. B129 (1983) 177–181.

(41) **Maneekhum P.,** *"The tracker solution for cosmological scalar field",* July 2008, Naresuan University.

(42) **Mavromatos N. E.**, *"General Relativity and Cosmology",* King's Colge London 2008.

(43) **Narlikar J. V.,** "*An Introduction to Cosmology*", (3rd edition); Inter-University Centre for Astronomy and Astrophysics, Pune ,India. Cambridge University Press 2002

(44) **Norbury J. W.**, "*General Relativity and Cosmology*" for Undergraduates. Physics Department, University of Wisconsin-Milwaukee.

(45) **Nekoogar F.**, Multiversal Journeys, March 2011, http://www.springer.com/978-1-4419-8293-3

(46) **Peebles P. J. E.** and **Yu J. T.,** Astrophys. J 162 (1970) 815.

(47) **Penzias A. A.** and **Wilson R.,** Astrophys. J 142 (1965) 419.

(48) **Prakash S.** , "*Relativistic Mechanics*" , Vardman (P. G) College, Binjor-246701 . Rohilkhand University.

(49) **Raine D. and Thomas T.** "*An Introduction to the Science of Cosmology",* IOP Publishing Ltd 2001.

(50) **Rindler W.** , "*Relativity: Special , General and Cosmologcal*", Second Edition , Oxford University Press 2006.

(51) **Rindler W.** , "*Essential Relativity*", Second Edition. Springer-verlag; New York-Heidelberg-Berlin.

(52) **Robertson, H.P.** 1935, Ap. J. 82,248.

(53) **Robertson H. P.,** “Relativistic Cosmology”, Rev.Mod.Phys.5.P.62

(54) **Robertson, H.P.** 1928, Phil. Mag .5,p.835.

(55) **Schutz F.** , “*A First Course in General Relativity*”, Cambridge University Press 1990.

(56) **Schutz B.F.**, “*A first Course in General Relativity*”. Cambridge University press, New York, 1990.

(57) **Shafieloo A.,** *“Confronting Cosmological Models with Observations”* July 2008, University of Pune.

(58) **Sheykhi A.,** *article in* “*Physical review d: particles and fields” · November 2011,* Shiraz University.

(59) **Starobinsky A. A.,** “*Stochastic De Sitter (Inflationary) Stage In The Early Universe,"* in: Current Topics in Field Theory, Quantum Gravity and Strings.

(60) **Walker, A.**G., 1936, Proc. London Math. Soc. 42,90.

(61) **Weinberg S.**, “*Cosmology*”, Oxford University Press Inc.,New York 2008.

(62) **Weinberg S.**, “*Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*”, John Wiley & Sons Inc 1972.

(63) **Weinberg A. S.**, “*Gravitation and Cosmology: Principles and applications of the General Theory of Relativity”*. Massachusetts Institute of Technology.

(64) **WMAP** Collaboration, C. L. Bennett et al., "*First year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Preliminary maps and basic results*".- Page Updated: Tuesday, 05-25-2010

(65) **WMAP-**Webmaster: Britt Griswold, NASA Official: Dr. Edward J. Wollack. http://wmap.gsfc.nasa.gov.- Page Updated: Tuesday, 05-25-2010