STUDIES ON SPHERICALLY SYMMETRIC STATIC AND NONSTATIC SINGULAR RELATIVISTIC COSMOLOGIES

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Dedicated

To

My respected

Father

And beloved

Mother

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CONTENTS

CONTENTS

ABSTRACT

In this thesis, we study on "**SPHERICALLY SYMMETRIC STATIC AND NON-STATIC SINGULAR RELATIVISTIC COSMOLOGIES**". Soon after the Einstein's equations were discovered, Karl Schwarzschild (1916) found out an exact solution by considering a spherically symmetric static metric, then Friedman (1922) showed that the universe (non-static) must have originated a finite time ago from an epoch of infinite density and curvatures if the evolving matter obeys the dynamical equations of general relativity theory, together with the assumptions of homogeneity and isotropy, which is the non-static singular relativistic cosmology.

In recent years, these have been some interest in studying the mathematical and physical interpretations of different models and theories of relativistic cosmology. The project of this research is firstly to study the physical and mathematical properties of the known solutions and secondly to attempt to find out new physically interesting solutions, with particular references.

GENERAL INTRODUCTION

This thesis is mainly expository, but the last chapter carries some original calculations, which we treat as the new results.

The title of this thesis is "**STUDIES ON SPHERICALLY SYMMETRIC STATIC AND NON-STATIC SINGULAR RELATIVISTIC COSMOLOGIES**".

Spherical symmetry has played an important role in the development of general relativity. The exact solutions of Einstein's field equation which provided the decisive experimental verification of the theory, namely the Schwarzschild external solution and the Robertson-Walker-Friedman cosmological solutions, were found under the assumption that space-time was spherically symmetric.

A space-time is said to be spherically symmetric if colloquially expressed, it is possible to rotate it leaving its metric (and any other non-metric fields) unchanged. In more precise terms for every rotation R (a 3×3 rotation matrix) in the rotation group SO(3), there is an isometry of the space-time $\phi(R)$and the isometries constitute what is called an action of the group, meaning that the consumption of the isometries corresponds to the composition of corresponding rotations:

$$\phi(R)\phi(S) = \phi(RS)$$

In analytic mechanics, one knows the symmetries of a Lagrangian or Hamiltonian result in conservation laws. That is, there is a conserved quantity, whenever, a symmetry exists. These general principles also exist in the general theory of relativity and are used to deduce, from the symmetries of Schwarzschild space-time, constants of motion for the trajectories (geodesics) of freely falling particle in the gravitational field outside a star. The same constants of motion are obtained in a different language in differential geometry, where a killing vector in the standard tool for the description of symmetry.

In general, the spherical symmetry of a space-time can be defined vigorously in terms of the killing vectors; there must be three linearly independent space-like killing vector fields X^1, X^2, X^3in the space-time which satisfy the commentator equations:

$$[X^1, X^2] = X^3, \qquad [X^2, X^3] = X^1, \qquad [X^3, X^1] = X^2$$

And there orbits must be closed. Using the properties one could then again derive the line element for a spherically symmetric space-time.

NON-STATIC SINGULAR RELATIVISTIC COSMOLOGIES:

Further attempts were made by Friedman, Lemaitre and Robertson to investigate the most general quadratic line element which would describe a non-static but isotropic and homogeneous universe. The most satisfactory non-static cosmological model was given by Robertson. It is kinematical in nature and ignores the usual dynamical equations of general relativity. The assumptions of Robertson model are:

1. There exists a cosmic time which is orthogonal to the spatial geometry

$$ds^2 = dt^2 + g_{ij} dx^2 dy^2$$

2. The three-dimensional spatial surfaces belonging to $t = constant$ are locally isotropic and homogeneous.

The entire thesis contains nine chapters except the general introduction. The introductory chapter does not contain any mathematical work. It is almost ornamented. Every chapter has got an introduction.

The work presented here has been largely derived from the books: Introduction to Mathematical Cosmology by **Jamal Nazrul Islam** [1], Cambridge University Press, 2002; Global Aspect in Gravitation and Cosmology by **Pankaj Joshi** [2], Oxford University Press, Inc. New York (1993); Gauge theories and the Early Universe by **B. R. Iyer, N. Mukunda** and **CV. Vishveshwara** [3]; Principles and Applications of the General theory of Relativity by **Steven-Wienberg** [4]; An Introduction to Cosmology by **Jayant Vishnu Narlikar** [5], Cambridge University Press; Black Hole Astrophysics, in

General Relativity by **Blandford R.D.** and **Thorne K.** (1979)[6]; An Einstein Centenary Survey (ed. **S.W.Hawking** and **W.Israel**)[7], Cambridge University Press, Cambridge; Spherically symmetric Models in General Relativity, Mon. Nof. Astron. Soc.107 P.400. Mach's Principle and Relativistic Theory of Gravitation by **Brans C**. and **Dicke R. H**.(1968) [8],phy. Rev. 124, P 925 and more, along with other books and papers, all of which have been mentioned in the references.

However, the books and papers cited here the calculations are not given in detail many of the steps are omitted. We have carried out most calculations in detail and checked the relevant equations in the references mentioned here.

The various chapters are organized as follows:

CHAPTER 1 contains "AN OVERVIEW OF VECTOR AND TENSOR ANALYSIS". A vector is a quantity having both magnitude and direction, such as displacement, velocity, force and acceleration. The tensors are invented as an exclusion of vectors to formalize the manipulation of geometric entities. Tensors are geometric objects defined on a manifold, which remain invariant under the change of coordinates. There are various matter fields, defined on a space-time such as the electromagnetic fields or dust and so on, which are represented by the stress energy tensors of space-time. On the other hand, the global geometry and curvature of the manifold are described by fields such as the metric tensor and the curvature tensor. In the general theory of relativity it is required that the form of physical laws must remain unchanged under a general transformation of coordinates.

CHAPTER 2 deals with "THE SPECIAL THEORY OF RELATIVITY". The special theory of relativity leads with systems known as inertial systems, that is, the systems which more in uniform rectilinear motion relative to one another. According to this "All systems of coordinates are equally suitable for description of physical phenomena". If we extend this principle to accelerate systems, i.e. the systems moving with accelerated velocity relative to one another, the theory of relativity is called "General Theory of Relativity". This special theory of relativity has **two** postulates:

(1) The fundamental laws of physics have the same form for all inertial systems, i.e. for all reference systems at rest or moving with constant linear velocity relative to one another.

(2) The velocity of light in vacuum is independent of the relative motion of the two fundamental postulates in the special theory of relativity. The first postulate is the exclusion of the conclusion drawn from Newtonian mechanics, since velocity is not absolute, but relative which is a fact drawn from the failure of the experiments to determine the velocity of earth relative to ether.

We know that the speed of light is not constant under Galilean transformations and the first postulate is the conclusion from Newtonian mechanics; thus second postulate is not true according to Galilean transformations. Actually this is true since the velocity of light calculated by any mea is a constant. Thus the second postulate is very important and only this postulate is responsible to differentiate the classical theory and Einstein's theory of relativity. According to Einstein, the theory of relativity is applicable to laws of optics. Thus for the constancy of velocity of light we have to introduce the new transformation equations which fulfill the requirements:

1. The speed of light C must have the same value in every inertial frame.

2. The transformation must be linear and for low speeds i.e. $v \ll c$ they should approach the Galilean transformations.

3. They should not be based on "**Absolute Time and Absolute Space**".

CHAPTER 3 deals with "THE GENERAL THEORY OF RELATIVITY". The two fundamental theories in physics which have been of great importance in studying the behavior of matter are:

(i) Newtonian theory of gravitation which describes the behavior of one mass point on the other and

(ii) The electrodynamics which describes the behavior of charged matter in the presence of electromagnetic fields.

The special theory of relativity had its only accounts for inertial systems, in the region of free space, where gravitational effect, can be neglected. In this system the law of inertia holds good and the physical laws retain the same form. The special theory of relativity does not account for non-inertial (i.e. accelerated systems). For example the "**clock paradox**" and universal phenomenon of gravitation could not be counted by special theory of relativity. Thus naturally we wish to extend the principle of relativity in such a way that it may hold even for non-inertial systems and consequently the extended theory may explain the non-inertial phenomenon like clock paradox and particularly the phenomenon of gravitation. The extended theory is known as the general theory of relativity. In developing the general theory of relativity it is helpful to the phenomenon of gravitation.

On the generalization the theoretical predictions led to the small deviations from the observed phenomenon in the following cases:

(a) ***Advanced of Perihelion of planets***:

The special theory of relativity of gravitation leads to the precession of the perihelion of planets; but the precession of a planet observed and that accounted by special theory of relativity are not in the same amounts. For example, in the case of planet mercury the special theory of relativity accounts a retardation of perihelion at the rate 7.2 seconds of arc per century; while observations show that the perihelion of mercury advanced at the rate of 43 seconds of arc per century, thus the effect observed is six times greater in magnitude as accounted by special theory of relativity.

(b) ***Shift in Spectral Lines***:

The special theory of relativity of gravitation predicts no shift of special lines emitted by atoms even in powerful gravitational fields; while observations indicate a shift of spectral lines towards red.

(c) ***Deflection of Light Rays Due To Gravitational Field***:

The special theory of relativity predicts that when the light rays pass close to the sun, they are deflected by an amount of 0.88 second of an arc; but observations show that the deflection is 1.75 seconds of arc; which is twice the result of special theory of relativity of gravitation.

The special theory of relativity, due to these durations from observations, every not be considered wrong since for low velocities it reduces to Newtonian theory which is valid to a high degree of accuracy. However the special theory of relativity of gravitation seems be slightly wrong for the phenomenon in which the velocity approaches the velocity of light, because it could not account the correct amount of bending of light rays towards the sun. The special theory of relativity of gravitation seems to fail for fixed particles in the gravitational fields; because it could not predict the red shift of spectral lines emitted by atoms even in powerful gravitational fields. The special theory of relativity also seems to be wrong for the phenomenon when velocity and gravitational field both are present since it could not predict correctly the procession of perihelion of planets which is considered to be due to their velocity and the strength of gravitational field.

Thus it is obvious that Minkowskian space-time continuum does not account the natural phenomenon of gravitation in non-inertial frames. This is due to the fact that in general theory of relativity we consider that in general theory of relativity we consider that the motion in gravitational field is some type of inertial motion as accelerating the passengers relative to a suddenly stopped carriage. The acceleration is due to Newton's law of inertia, "A body at rest remains at rest and a body in motion moves with the same velocity till no external force is applied." This also contains why the acceleration due to action of gravity is independent of the mass of the body. As the force of gravitation is zero at infinite distance from the mass or attracting bodies; therefore it is assumed the space close to the attracting bodies or the mass does not follow the Minkowskian space-time continuous. Hence for accounting the observed facts of gravitation we have to modify the Minkowskian space-time continuum. The observations of gravitational red shift on earth-support the modification in Minkowskian space-time structure.

In Minkowski space-time continuum the line element is given by:

$$\left.\begin{array}{c} ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \\ or, ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \end{array}\right\} \quad (1)$$

The modified form of above equation in terms or form is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \quad (2)$$

Obviously equation (2) may reduce to (1) in special case. Thus in regions far from gravitational field the general theory of relativity reduces to special one and the Minkowski space-time continuum holds in regions far from gravitational field.

The element in (2) represents the curved geometry. Thus according to general theory of relativity the space is curved in a gravitational field. Since the space is curved in gravitational field, therefore the geometry of space in a gravitational field in Riemannian, for the idea was originally developed by Riemann. Thus free motion in a gravitational field for in a non-Euclidean space is not straight, but curvilinear.

Thus if we consider a very small element of this curve, it will be nearly a straight line therefore it is very difficult to distinguish the gravitational and inertial forces in small regions of space.

In such regions, a non-inertial co-ordinate system is equivalent to inertial system is equivalent to inertial system in which an additional gravitational field operates with the same acceleration of falling bodies which is due to inertial and non-inertial systems is merely that in the latter the inertial forces come into play.

Hence the theory of gravitation also deals with non-inertial systems unlike special theory which deals only inertial systems, due to this reason the theory of gravitation is called "General Theory Of Relativity".

CHAPTER 4 deals with the "SYMMETRIC SPACE". As we know that Einstein's exterior equations $R_{\mu\nu} = 0$ by setting $T_{\mu\nu} = 0$ are a set of coupled non-linear partial differential equations for the ten unknown functions $g_{\mu\nu}$. The interior equations $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$ may involve other unknown functions such as the mass-energy density and the pressure. Because of the freedom to carry out general coordinate transformations one can in general impose four condition on the ten functions $g_{\mu\nu}$. Later we have shown in this chapter explicitly how this is done in a case involving symmetries.

In most situations of physical interest one has space-time symmetries which reduce further the number of unknown functions. To determine the simplest form of the metric (that is, the form $g_{\mu\nu}$) when one has a given space-time symmetry is a non-inertial problem. For example, Newtonian theory spherical symmetry is usually defined by a centre and the property that all points at any given distance from the centre are equivalent. This definition cannot be taken over directly to general relativity. In the latter, centre may not be accessible to physical measurement, as is indeed the case in the Schwarzschild geometry. One therefore has to find some coordinate independent and covariant manner of defining space-time symmetries such as axial symmetry and stationary. This is done with the help of killing vectors.

CHAPTER 5 deals with "SPHERICALLY SYMMETRIC SCHWARZSCHILD SOLUTION AND ITS PROPERTIES". The Schwarzschild solution represents the geometry exterior to a spherically symmetric (which is a function of r only, the radius vector) massive body such as a star and has been used extremely to verify the predictions of the general theory of relativity experimentally. This is the empty exterior solution where the Ricci tensor vanishes and which is matched at the boundary to the interior solution inside the body.

In spite of such predominant global features evident in the structure of gravitational theory, most of the calculations were done, until the early 1960s, using a local coordinate system defined in the effort was devoted them to solving Einstein equations using various simplifying assumptions, which form a rather complicated set of non-linear partial differential equations. The situations and approach changed considerably when the so called "Schwarzschild Singularity" problem came up. The Schwarzschild exterior solutions of Einstein's field equation describe the gravitational field outside a spherically symmetric star where there is no matter present and the space-time is empty. The space-time distance(t, r, θ, ϕ) coordinates, between the two infinitesimally separated events is given by the metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\theta^2), \qquad (3)$$

Here 'm' represents the mass of the star and the boundary of the star lies at $r = r_b$. The range of 't' and 'r' coordinates is given by $-\infty < t < \infty$, and $r_b < r < \infty$, and $\theta \; and \; \phi$ are the usual coordinates on the sphere. It is clear now that if $r_b < 2m$ or if equation (3) represents the geometry outside that of a point particle of mass m placed at $r = 0$, then the above space-time has an apparent singularity at $r = 2m$ as seen by the divergence of metric component in equation(3) and this value. It was through initially that the above represents a singularity in the space-time itself and that physics goes seriously wrong at $r = 2m$. It was realized only after considerable effort that this is not a genuine space-time singularity but merely a coordinate defect, and what was really needed was an exclusion of the finiteness of curvature components at $r = 2m$. The point is, the coordinate system used above breaks down at this value of r and it describes only a patch of the space-time defined by the above coordinate range. Such an exclusion covering the rest of the space-time was obtained by Kruskal (1960) and Szekeres (1960) and this may be regarded as an important insight involving a global approach.

Similar such developments which could be mentioned here are Alexanderov's (1950, 1967) work on space-time topology and the analysis of the Cauchy's problem in general relativity [20].

CHAPTER 6 deals with the "STELLAR EQUILIBRIUM AND COLLAPSE". The theory of stellar evolution states that stars whose masses are of the order of the sun's mass can reach a final equilibrium state as a white dwarf, or a neutron star. But for much larger masses, no such equilibrium is possible, and in such a case the star will contract to such an extent that the gravitational effects will overcome the internal pressure and stresses which will not be able to halt further contraction. General relativity predicts that a spherically symmetric star will necessarily contract until all matter contained in the star (collapses) arrives at space-time singularity at the center of symmetry.

The fundamental motivation for the concept of a black hole comes from an examination of the spherically symmetric homogeneous collapse which has two outstanding features. **Firstly,** for a star undergoing a complete gravitational collapse, a region of trapped surfaces from below $r = 2M$, from which no light rays escape to an observer at infinity. Thus, a black hole forms in the space-time. **Secondly,** the ultimate fate of the star undergoing the collapse is an infinite curvature singularity $r = 0$, which is completely hidden with the trapped surface region and the black hole. Here no emission of light rays from the singularity could go out any observer at infinity and the singularity is causally disconnected from the outside space-time.

CHAPTER 7 deals with "SINGULARITIES IN COSMOLOGY". The singularity is not really a tangible object either. According to the General theory of Relativity the singularity is a point of infinite space-time curvature. This means that the force of gravity has become infinitely strong at the centre of a black hole. Everything that falls into a black hole by passing the event horizon, including light, will eventually reach the singularity of a black hole. Before something reaches the singularity it is torn apart by intense gravitational forces. Even the atoms themselves are torn apart by the gravitational forces. The singularity is the region of space-time which cannot be explained anyway either by theoretically or mathematically; it can be understood only by hypothetically. It is the region from which nothing can be communicated with the external world. So, singularity can be regarded as nobody knows phenomena.

After Einstein proposed the general theory of relativity describing the gravitational force in terms of space-time curvature and proposed the field equations relating the geometry and smaller content of the space-time manifold, the earliest solutions found for the field equations were the Schwarzschild metric representing the gravitational field around an isolated body such as a spherically symmetric star, and the Friedman cosmological models.

Each of these solutions contained a space-time singularity where the curvatures and the densities were infinite and the physical description would break down. In the Schwarzschild solution such a singularity was present at $r = 0$ where as in the Friedman models it was found at the epoch $t = 0$ which is beginning of the universe and origin of time when the scale factor R(t) also vanishes and also objects are crushed to Zero volume due to infinite gravitational tidal forces.

In the Friedman-Robertson-Walker models, the Einstein equations imply that if $\rho + 3p > 0$ at all times, where ρ is the total energy density and p is the pressure, there is a singularity at $t = 0$ which could be identified as the origin of the universe. If $p + \rho > 0$ at all times then it is seen that along all the past directed trajectories meeting this singularity, $\rho \to \infty$ and also the curvature scalars $R = R_{ij}R^{ij} \to \infty$. Again, all the past directed non-space-like geodesics are incomplete in the above sense.

Thus, there is an essential curvature singularity at $t = 0$ which cannot be transformed away by any coordinate transformations. In fact, similar behaviour has been generalized to the class of spatially homogeneous cosmological models as shown by Ellis and King (1974) which satisfy the positivity of energy condition $\rho \geq 0,\ p \geq 3p \geq 0\ and\ \ 1 \geq 3\, dp/d\rho \geq 0$.

The existences of such singularities where the curvature scalars and densities diverge imply a genuine space-time pathology where the usual laws of physics must break down. The existences of the geodesics in completeness in these space-times imply that, for example, a time-like observer suddenly disappears from the space-time after a finite amount of proper time.

Even though the physical problem posed by the existence of such a strong curvature singularity was realized immediately in the solutions which turned out to have several important implications towards the experimental verification of the general theory of relativity, initially this phenomenon was not taken seriously. It was generally thought that the existence of such a singularity must be a consequence of the very high degree of symmetry imposed on the space-time while deriving these solutions subsequently, the distinction between a genuine singularity and a mere coordinate singularity became clear and it was realized that the singularity at $r = 2m$ in the Schwarzschild space-time was a coordinate singularity which could be removed by a suitable coordinate transformation. It was clear, however, that the genuine curvature singularity at $r = 0$ cannot be removed by any such coordinate transformation. The hope was then that when more general solutions are considered with a less degree of symmetry requirements, such singularity will be avoided. This issue was sorted out when a detailed study of the structure of a general space-time and the associated problem of space-time singularity was taken up by Hawking, Penrose, and Geroch.

It was shown by this work that a space-time will admit singularities within a very general frame work provided it satisfies certain reasonable assumptions such as the positivity of energy, a suitable causality assumption and a condition such as the existence of trapped surfaces. It thus follows that the space-time singularities form a general feature of the relativity theory. In fact, these considerations ensure the existence of singularities in other theories of gravity, which are based on a space-time manifold framework and satisfy the general conditions stated above.

CHAPTER 8 deals with "THE NATURE OF SINGULARITIES IN SYMMETRIC SCALAR FIELD COSMOLOGIES". The nature of singularities in general solution of the Einstein equations is a subject about which much remains to be learned. Various classes of singularities have been defined which represent possible models for general behaviour. Examples are curvature singularities, crushing singularities, velocity-dominated singularities and isotropic singularities are discussed here.

CHAPTER 9 deals with, "SCHWARZSCHILD METRIC WITH COSMOLOGICAL CONSTANT". In this chapter we have introduced the cosmological constant and applying to the Einstein Field Equation to have an exact solution which gives us the limiting conditions of the event horizon.

“Matter is represented by curvature, but not every curvature does represent matter there may be curvature in vacuuM.”

G.LEMATIRE

CHAPTER 1

AN OVERVIEW OF VECTOR AND TENSOR ANALYSIS

1.1 INTRODUCTION

As we know vector is an entity, which has both magnitude and direction. Tensors are invented as an extension of vectors to formalize the manipulation of geometric entities. Tensors are geometric objects defined on a manifold, which remain invariant under the change of coordinates. There is various matter fields defined on a space-time such as the electromagnetic fields or dust and so on, which are represented by the stress-energy tensor of the space-time. On the other hand, the global geometry and curvature of the manifold are described by fields such as the metric tensor, and the curvature tensor. In the general theory of relativity it is required that the form of physical laws must remain unchanged under a general transformation of coordinates (principle of general covariance). Thus, physical fields are represented by various tensor fields on the space-time and the laws governing them are expressed as tensor equations which remain valid under arbitrary coordinate transformations. When one specializes to an inertial coordinate system, these laws reduce to the equations of special relativity.

1.2 VECTOR

A vector is a quantity having both magnitude and direction, such as displacement, velocity, force and acceleration.

Graphically a vector is represented by an arrow OP[**Fig: 1.1**] defining the direction, the magnitude of the vector being indicated by the length of the arrow. The tall end O of the arrow is called the **terminal point** or **terminus**.

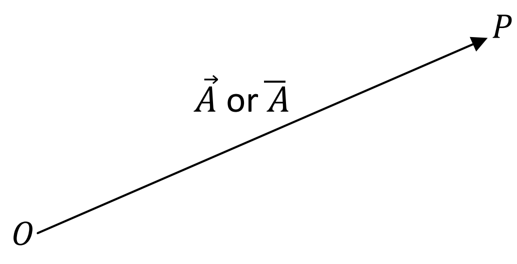

Fig: 1.1

Analytically a vector is represented by a letter with an arrow over it as $\vec{A}$ or $\bar{A}$ [**Fig: 1.1**] and its magnitude is denoted by $|\bar{A}|$ or A. In printed works, bold face type, such **A**, is used to indicate as $\overline{op}$. In such case we shall denote its **magnitude** by op, $|\overline{op}|$, $|op|$.

1.3 TENSORS

In general, a tensor T of (r, s)type at $p \in M$ is a multi-linear real-valued function on the Cartesian product

$$T_p^* \times \dots \times T_p^* \times T_P \dots \times T_P \rightarrow \mathbb{R}, \qquad (1.3.1)$$

Where there are r-factors of T_p^* and s-factors of T_P. Thus, T acts on one-forms and vectors in general to produce a real number.

If T is a tensor of (r, s) type at $p \in M$,

$$T(\omega_1, \dots, \omega_r, V_1, \dots, V_s) = T\left(\omega_{i_1} e^{i_1}, \dots, \omega_{i_r} e^{i_r}, V^{j_1} e_{j_1}, \dots, V^{j_s} e_{j_s}\right). \qquad (1.3.2)$$

Using the multi-linear property of T, the above can be written as

$$T^{i_1 \dots i_r}_{j_1 \dots j_s} \omega_{i_1} \dots \omega_{i_r} V^{j_1} \dots V^{j_s},$$

Where we have defined

$$T^{i_1 \dots i_r}_{j_1 \dots j_s} \equiv T\left(e^{i_1}, \dots, e^{i_r}, e_{j_1}, \dots, e_{j_s}\right), \qquad (1.3.3)$$

And $\{e_i\}$ and $\{e^i\}$ are basis vectors at p for T_p and T_p^* respectively.

The space of all the tensors of type (r, s) at p is called the tensor product $T_s^r(p)$ and is denoted by

$$T_s^r(p) = T_P \otimes \dots \otimes T_P \otimes T_p^* \otimes \dots \otimes T_p^*, \qquad (1.3.4)$$

Where there are r-factors of $T_P$and s-factors of T_p^*. The dimension of T_s^r is n^{r+s}, where n is the dimension of the manifold. This is a vector space of all (r, s) tensors over real numbers with the addition of tensors and scalar multiplication defined in a natural manner. In particular, a vector is a tensor of type (1,0) and a one-form is a tensor of type (0,1). In terms of the basis vectors $\{e_i\}$ and $\{e^i\}$for the tangent space and cotangent space at p, the set

$$\left\{e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s}\right\}.$$

forms a basis for the tensor product $T_s^r(p)$ with all the indices running from 1 to n. Then, any tensor $T \in T_s^r$can be expressed as

$$T = T^{i_1 \dots i_r}_{j_1 \dots j_s} e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s},$$

Where, the tensor components $T^{i_1 \dots i_r}_{j_1 \dots j_s}$ are defined as above. Consider now a change of coordinates, which causes a change of basis $\{e_i\}$ and $\left\{e^{i'}\right\}$ and similarly,$\{e_j\}$ going to$\{e_{j'}\}$. In particular, let us choose a coordinate basis $\{\partial/\partial x^i\}$ for T_P and corresponding basis $\{dx^i\}$ for the cotangent space T_p^*. Then, under a change of coordinates, the components of T in the new coordinates $\{x^{i'}\}$ can be written as

$$T^{i_1'\dots i_r'}_{j_1'\dots j_s'} = T\left(dx^{i_1'},\dots,dx^{i_r'},\frac{\partial}{\partial x^{j_1'}},\dots,\frac{\partial}{\partial x^{j_s'}}\right)$$

Since $x^{i'}$can be treated as functions of x^i, substituting in the above for $\partial/\partial x^{i'}$ and $dx^{i'}$ gives for the transformed components of the tensor T,

$$T^{i_1'\dots i_r'}_{j_1'\dots j_s'} = T^{i_1\dots i_r}_{j_1\dots j_s}\frac{\partial x^{i_1'}}{\partial x^{i_1}}\dots\frac{\partial x^{i_r'}}{\partial x^{i_r}}\frac{\partial x^{j_1}}{\partial x^{j_1'}}\dots\frac{\partial x^{j_s}}{\partial x^{j_s'}} \qquad (1.3.5)$$

Thus, for the transformation of the components of a vector V and a one-form ω we get

$$V^{i'} = \frac{\partial x^{i'}}{\partial x^i}V^i, \qquad \omega_{i'} = \frac{\partial x^i}{\partial x^{i'}}\omega_i \qquad (1.3.6)$$

Even when one chooses a general set of basis vectors rather than a choice of a coordinate basis, the formula for transformation of the components of a tensor can be written in a similar manner.

If T is a tensor of type (r,s), the contraction of T over a contravariant index and a covariant index is defined to be a tensor $C(T)$ of type $(r-1,s-1)$. For example, if we contract over the first contravariant and covariant indices, this gives

$$C_1^1(T) = T^{ij\dots l}_{im\dots n}e_j \otimes \dots \otimes e_l \otimes e^m \otimes \dots \otimes e^n. \qquad (1.3.7)$$

Using the relationships given above for the transformation of components of a tensor under the change of basis vectors, it is again possible to see that the contraction C_1^1 is independent of the basis used, that is, it is invariant under change of coordinates. Similarly, one could contract T over any pair of a contravariant and covariant indices.

In the space of all tensors of type (r,s)at p, the addition of two tensors T and T' is defined as

$$(T+T')(\omega^1,\dots,\omega^r,X_1,\dots,X_s) = T(\omega^1,\dots,\omega^r,X_1,\dots,X_s) + T'(\omega^1,\dots,\omega^r,X_1,\dots,X_s)$$

And the multiplication by a real number α is defined by

$$(\alpha T)(\omega^1, \dots, \omega^r, X_1, \dots, X_s) = \alpha T(\omega^1, \dots, \omega^r, X_1, \dots, X_s).$$

The operation of outer product on two tensors T and T' of type $(r, s)$and (r', s') could now be defined in terms of their components to give a new tensor $T \otimes T'$ given by,

$$(T \otimes T')^{i_1 \dots i_{r+r'}}_{j_1 \dots j_{s+s'}} = T^{i_1 \dots i_r}_{j_1 \dots j_s} T'^{i_{r+1} \dots i_{r+r'}}_{j_{s+1} \dots j_{s+s'}}.$$

This offers a way of constructing new tensors out of the vectors and dual vectors.

A tensor field of (r, s) type on M is an assignment of a tensor of the same type at all p in M. Such a tensor field is called C^k differentiable if all the components of T are having the same differentiability as functions of coordinates.

Finally, we discuss the symmetry properties of tensors. Suppose T is a (0,2) type tensor. Then it acts on pairs of vectors $V.W$ to produce a real number $T(V, W) = T_{ij} V^i V^j$. Then T is called symmetric if the result is the same even when we change the slots for V and W, that is,

$$T(V, W) = T(W, V).$$

If $\{e^i\}$is a basis for the tangent space, this amounts to the condition $T(e_i, e_j) = T(e_j, e_i)$, which is same as saying that

$$T_{ij} = T_{ji}$$

Similarly, T is called anti-symmetric if

$$T_{ij} = -T_{ji}$$

It is convenient to formulate this in terms of symmetric part is written as,

$$T_{(ij)} = \frac{1}{2!}(T_{ij} + T_{ji}),$$

And its anti-symmetric part is written as

$$T_{[ij]} = \frac{1}{2!}(T_{ij} - T_{ji}).$$

Then T is called **symmetric** if $T_{(ij)} = T_{ij}$ and it is called **antisymmetric** if $T_{[ij]} = T_{ij}$. In general, for a tensor $T_{i_1,\dots,i_r}$ of type $(0, r)$, $T_{(i_1,\dots,i_r)}$ is defined as the sum over all permutations of indices $i_1, \dots, i_r$ divided by $r!$. Thus, for example,

$$T^i_{[jkl]} = \frac{1}{3!}\left[T^i_{jkl} + T^i_{klj} + T^i_{ljk} - T^i_{kjl} - T^i_{lkj} - T^i_{jlk}\right].$$

In general, a tensor of type (r, s) is called symmetric over a collection of indices if it equals its symmetric part over these indices, and antisymmetric tensors are defined in a similar manner.

1.4 CO-ORDINATE TRANSFORMATIONS

Let $(x^1, x^2, \dots, x^N)$and $(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^N)$ be co-ordinates of a point in two different frames of reference. Suppose there exists N independent relations between the co-ordinates of the two systems having the form,

$$\left.\begin{aligned} \bar{x}^1 &= \bar{x}^1(x^1, x^2, \dots, x^N) \\ \bar{x}^2 &= \bar{x}^2(x^1, x^2, \dots, x^N) \\ \vdots \quad & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \bar{x}^N &= \bar{x}^N(x^1, x^2, \dots, x^N) \end{aligned}\right\} \tag{1.4.1}$$

Where we can indicate equation (1.4.1) briefly by,

$$\bar{x}^k = \bar{x}^k(x^1, x^2, x^3, \dots, x^N) \tag{1.4.2}$$

Where, it is supposed that, the functions involved are single-valued, continuous and have continuous derivatives. Then conversely to each set of $(x^1, x^2, \dots, x^N)$, given by,

$$x^k = x^k(\bar{x}^1, \bar{x}^2, \bar{x}^3, \dots, \bar{x}^N) \qquad ; k = 1, 2, \dots, N \tag{1.4.3}$$

Here, the relations (1.4.2) or (1.4.3) define a transformation of co-ordinates from one frame of reference to another.

1.5 THE SUMMATION CONVENTION

In writing an expression, such as,$a_1x^1 + a_2x^2 + \cdots + a_nx^n$, we can use the short notation

$$\sum_{i=1}^{3} a_i x^i$$

An even shorter notation in simply top write it as,$a_i x^i$, where we adopt the convention that whenever an index (subscript or superscript) is repeated in a given term, we are to sum over that index from 1 to N unless otherwise specified. This is called the **summation convention**.

Clearly, instead of using the index I we could have used another letter, say p, and the sum could be write $a_p x^p$. Any index which repeated in a given term, so that the summation convention applies, is called a **dummy index** or **umbrall index**.

An index occurring only one N in a given term is called a free index and can stand for any of the numbers in equation. $\bar{x}^k = \bar{x}^k(x^1, x^2, x^3, \dots, x^N)$ or, $x^k = x^k(\bar{x}^1, \bar{x}^2, \bar{x}^3, \dots, \bar{x}^N)$, each of which represent N quantities.

1.6 THE LINE ELEMENT

We know that the geometry on the surface of a sphere is Riemannian. Consider a point P on the surface of a sphere of unit. Its position is given by two angular co-ordinates (θ, φ), θ being the co-latitude $(\theta = \angle NOP)$ and φ be the longitude $(\varphi = \angle XUP)$. If θ is constant and φ varies from 0 to 2π, the point P moves along the small circle of radius $\sin\theta$. If φ is constant and θ varies from 0 to π, the point P moves along the meridian large circle NPS (Fig:1.2a). If $Q(\theta + d\theta, \varphi + d\varphi)$ is a neighboring point on the sphere at an infinitesimal distance from P (Fig: 1.2b) then we have,

$$ds^2 = d\theta^2 + sin^2\theta d\varphi^2 \qquad (1.6.I)$$

is customary to regard relations like equation (1.6.1) as defining a line element. Equation (1.6.1) is known as the **line element** on the surface of a sphere or a line element of a two-dimensional Riemannian space.

If however, we were considering two neighbouring points on a plane with co-ordinates $(x, y), (x + dx, y + dy)$, the line element would be,

$$ds^2 = dx^2 + dy^2 \qquad (1.6.2)$$

As the geometry on the plane is Euclidean, we can say that equation (1.6.2) is a line element of a **two-dimensional Euclidean space**.

On comparison of equation (1.6.2), we see that there are many points of similarity between them, e.g. both are quadratic in differentials of the co-ordinates. However, there is one striking difference. The one feature that distinguishes a line element of Riemannian space from that of a Euclidean space, is that the co-efficient of the quadratic terms (some of them at least) are functions (not constant functions) of the co-ordinates in a Riemannian space. We begin our study of space-time with this distinction in mind.

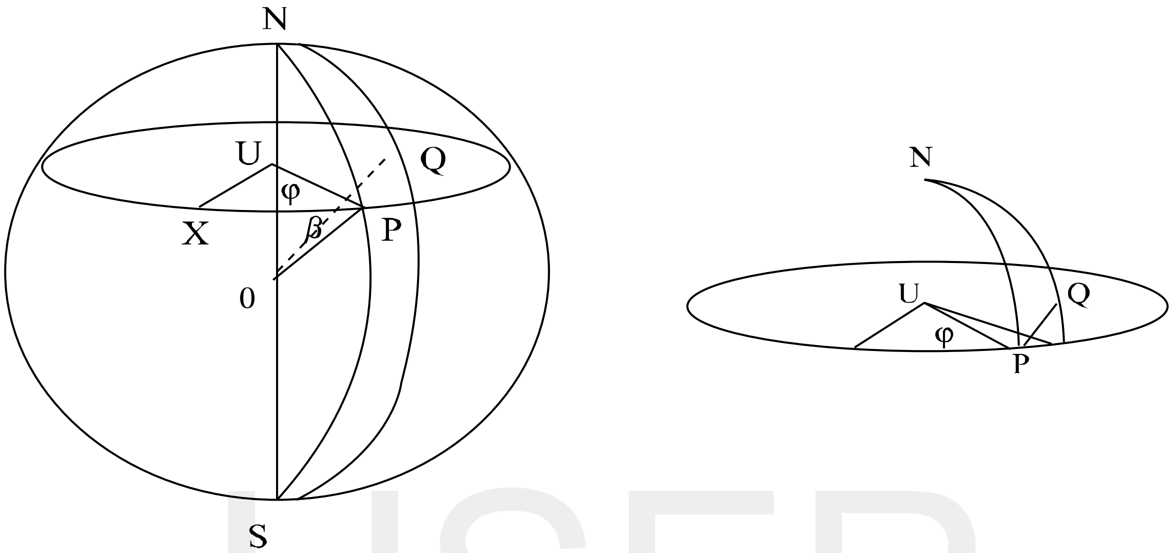

Fig: 1.2

In a four-dimensional space-time, let (x^1, x^2, x^3, x^4)be the co-ordinates of an event. A neighbouring event will have the co-ordinates $(x^1 + dx^1, x^2 +$ $dx2,x3+dx3,x4+dx4$. We take the line element as a quadratic in dx^1, dx^2, dx^3, dx^4.

Therefore,

$$\begin{aligned} ds^2 = g_{11}(dx^1)^2 + 2g_{12}(dx^1)(dx^2) + 2g_{13}(dx^1)(dx^3) + 2g_{14}(dx^1)(dx^4) \\ + \ g_{22}(dx^2)^2 + 2g_{23}(dx^2)(dx^3) + 2g_{24}(dx^2)(dx^4) \\ + 2g_{34}(dx^3)(dx^4) + g_{44}(dx^4)^2 \end{aligned}$$

Or, we may write it as,

$$ds^2 = \sum_{i=1}^{4} \sum_{k=1}^{4} g_{ik} dx^i dx^k \qquad (1.6.3)$$

Where g_{ik} are 10 functions of "**position**" out of 16 terms and 6 terms are symmetric, i.e. of the four variables (x^1, x^2, x^3, x^4) symmetric in i & k so that $g_{ik} = g_{ki}$.

1.7 THE FUNDAMENTAL TENSOR (THE METRIC TENSOR)

We now turn to the basic line element $ds^2 = g_{ik}dx^i dx^k$. We have seen that dx^i is a vector and so $dx^i dx^k$ is a contravariant tensor of rank two. Add to this the basic geometric requirement that ds^2 is an invariant and we have all the ingredients of the quotient law to show that g_{ik} must be a second rank covariant tensor. This tensor determines the nature of the corresponding Riemannian geometry and so it is often called the **fundamental tensor** or the **metric tensor**.

Consider the 16 functions g_{ik} arranged as a 4×4 matrix $\|g_{ik}\|$. If the determinant of this matrix is not zero, it will have an inverse matrix. Call the inverse matrix $\|g_{ik}\|$. If follows from the rules of inversion of metrics that $g_{ik}g^{kl} = \delta_i^l$; where $\boldsymbol{\delta_i^l}$ is the **kronecker delta**. We first show that δ_i^l is a tensor.

To show that δ_i^l is a tensor, we use the quotient law. Let B_{ik} is an arbitrary tensor. Then the product,

$B_{lk}\delta_i^l = B_{ik}$

Which is a tensor, and so by the quotient law δ_i^l becomes a tensor.

1.8 CONTRAVARIANT VECTOR

Consider a set of n quantities $A^1, A^2, \dots \dots \dots \dots, A^n$ in a co-ordinate system of variables $x^i(x^1, x^2, \dots \dots \dots \dots, x^n)$ and let these quantities have n other quantities $\bar{A}^1, \bar{A}^2, \dots \dots \dots \dots, \bar{A}^n$ in other co-ordinate system of variable $\bar{x}^\mu(\bar{x}^1, \bar{x}^2, \dots \dots \dots \dots, \bar{x}^n)$. If these quantities obey the transformation relation, then we have,

$$\bar{A}^\mu = \sum_{i=1}^{n} \frac{\partial \bar{x}^\mu}{\partial x^i} A^i \qquad ; \mu = 1,2, \dots \dots \dots \dots n$$

Which by the conventions adopted can simply be written as,

$$\bar{A}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^i} A^i \qquad (1.8.1)$$

Then the quantities A^i are said to be the components of a **contravariant vector** or a **contravariant tensor of first rank**.

Any n functions can be chosen as the components of a contravariant vector in a system of variables $\bar{x}^\mu$.

Then, multiplying equation (1.8.1) by $\frac{\partial x^j}{\partial \bar{x}^\mu}$ and taking the sum over the index μ from 1 to n, we get,

$$\frac{\partial x^j}{\partial \bar{x}^\mu}\bar{A}^\mu = \frac{\partial x^j}{\partial \bar{x}^\mu}\frac{\partial \bar{x}^\mu}{\partial x^i}A^i$$

$$= \frac{\partial x^j}{\partial x^i}A^i$$

$$= \delta_i^j A^i$$

$$= A^j$$

$$\therefore A^j = \frac{\partial x^j}{\partial \bar{x}^\mu}\bar{A}^\mu \qquad (1.8.2)$$

Here equation (1.8.2) represents the solution of equation (1.8.1).

Also, the transformation of differentials dx^i and $d\bar{x}^\mu$ in the co-ordinate system of variables x^i and $\bar{x}^\mu$ respectively then we get,

$$d\bar{x}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^i}dx^i \qquad (1.8.3)$$

As equation (1.8.1) and equation (1.8.3) are similar transformation equations. We can say that the differentials dx^i from the components of contravariant vector whose components in any other system are the differentials $d\bar{x}^\mu$ of that system. Also we conclude that the components of a contravariant vector are actually the components of a contravariant tensor of rank one.

Let us now consider a further change of variables from $\bar{x}^\mu$ to x'^p then the components A'^p must be given by,

$$A'^p = \frac{\partial x'^p}{\partial \bar{x}^\mu}\bar{A}^\mu$$

$$= \frac{\partial x'^p}{\partial \bar{x}^\mu}\frac{\partial \bar{x}^\mu}{\partial x^i}A^i \qquad [By\ using\ (1.8.1)]$$

$$= \frac{\partial x'^p}{\partial x^i}A^i \qquad (1.8.4)$$

The equation (1.8.4) has the same form as equation (1.8.1). This indicates that the transformations of contravariant vector form a **group**.

1.9 COVARIANT VECTOR

Consider a system of n quantities $A_1, A_2, \dots \dots \dots \dots, A_n$ in a co-ordinate system of variables x^i i.e. $(x^1, x^2, \dots \dots \dots \dots, x^n)$ and are related to n other quantities $\bar{A}_1, \bar{A}_2, \dots \dots \dots \dots, \bar{A}_n$ in another co-ordinate system of variables $\bar{x}^\mu$ i.e. $(\bar{x}^1, \bar{x}^2, \dots \dots \dots \dots, \bar{x}^n)$. If these quantities obey the transformation equations, then we have,

$$\bar{A}_\mu = \sum_{i=1}^{n} \frac{\partial x^i}{\partial \bar{x}^\mu} A_i \, ; \qquad i = 1,2, \dots \dots \dots, n$$

Which by the conventions adopted can simply be written as,

$$\bar{A}_\mu = \frac{\partial x^i}{\partial \bar{x}^\mu} A_i \tag{1.9.1}$$

Then the quantities A_i are said to be the components of a **covariant vector** or a **covariant tensor of rank one.**

Any n functions can be chosen as the components of a covariant vector in co-ordinate system of variables x^i and equation (1.9.1) determine the n-components in the new system of variables $\bar{x}^\mu$.

Multiplying equation (1.9.1) by $\frac{\partial \bar{x}^\mu}{\partial x^j}$ and taking the sum over the index μ from 1 to n then we get,

$$\frac{\partial \bar{x}^\mu}{\partial x^j} \bar{A}_\mu = \frac{\partial x^i}{\partial \bar{x}^\mu} \frac{\partial \bar{x}^\mu}{\partial x^j} A_i$$

$$= \frac{\partial x^i}{\partial x^j} A_i$$

$$= \delta^i_j A_i$$

$$= A_j$$

$$\therefore \; A_j = \frac{\partial \bar{x}^\mu}{\partial x^j} \bar{A}_\mu \tag{1.9.2}$$

Here equation (1.9.2) is the solution of equation (1.9.1).

Let us now consider a further change of variables from $\bar{x}^{\mu}$ to x'^{p}. Then the new components $\bar{A}_p$ must be given by,

$$A'_p = \frac{\partial \bar{x}^{\mu}}{\partial x'^{p}} \bar{A}_{\mu}$$

$$= \frac{\partial \bar{x}^{\mu}}{\partial x'^{p}} \frac{\partial x^{i}}{\partial \bar{x}^{\mu}} A_i$$

$$= \frac{\partial x^{i}}{\partial x'^{p}} A_i \qquad (1.9.3)$$

The equation (1.9.3) has the same form of equation (1.9.1). This indicates that the transformation of co-variant vectors form a group. As,

$$\frac{\partial \Psi}{\partial \bar{x}^{\mu}} = \frac{\partial \Psi}{\partial x^{i}} \frac{\partial x^{i}}{\partial \bar{x}^{\mu}}$$

$$= \frac{\partial x^{i}}{\partial \bar{x}^{\mu}} \frac{\partial \Psi}{\partial x^{i}}$$

It follows from equation (1.9.1) that $\frac{\partial \Psi}{\partial x^{i}}$ form the components of a covariant vector whose components in any other system are the corresponding partial derivatives $\frac{\partial \Psi}{\partial \bar{x}^{\mu}}$. This covariant vector is called **grad Ψ**.

1.10 SCALARS OR INVARIANTS

Suppose ϕ is a function of the co-ordinates x^k i.e. $\phi(x^k)$ and let $\bar{\phi}$ denote the functional value under a transformation to a new set of co-ordinates $\bar{x}^k$ i.e. $\bar{\phi}(\bar{x}^k)$. Then ϕ is called a **scalar** or **invariant** with respect to the co-ordinate transformation if

$$\phi(x^k) = \phi[x^k(\bar{x}^k)]$$

$$= \bar{\phi}(\bar{x}^k)$$

$$i.e. \;\; \phi = \bar{\phi}$$

Clearly, a scalar or invariant does not change under any change of co-ordinates. It is also called a **tensor of rank zero**.

1.11 COVARIANT DIFFERENTIATION

Let a scalar field derivative $S_{,v}$ is a covariant vector. Now A_p is a vector, its derivative with respect to x^r is denoted by$A_{p,r}$. Is it a tensor?

Now we examine how $A_{p,r}$ transform under a change of co-ordinates system. Hence,

$$A'_j = \frac{\partial x^p}{\partial x'^j} A_p$$

$$\therefore \quad A'_j = A_p x^p_{,j'} \qquad (1.11.1)$$

Here differentiating equation (1.11.1) with respect to x'^q, then, we get,

$$\frac{\partial A'^j}{\partial x'^q} = \frac{\partial}{\partial x'^q}\left(A_p x^p_{,j'}\right)$$

$$= A_{p,r} x^r_{,q'} + A_p x^p_{,j'q'} \qquad (1.11.2)$$

Here the second term on the right is not transform as a tensor. Thus $A_{p,r}$ is not a tensor. Now we can however modify the process of differentiation to get a tensor. Here the second derivative of equation (1.11.2) is denoted by $x^p_{,j'q'}$.

In general non zero and indicator that the transformation co-efficient in equation (1.11.1) vary with position in space time. Thus when we seek to construct to derivative $\frac{\partial A_p}{\partial x^r}$, we are forced to define it as a limit,

$$\frac{\partial A_p}{\partial x^r} = \lim_{\delta x^r \to 0}\left[\frac{A_p\left(x^k + \delta x^k - A_p(x^k)\right)}{\delta x^r}\right] \qquad (1.11.3)$$

However the two terms in the numerator do not transform as vector at the same point because of the variation of the transformation co-efficient with position in space time.

Therefore, their difference is not expected to be a vector. This situation is illustrated by the following figure

The vector field has the component A_p at P and $A_p + dA_p$ at Q. If A_p were transported parallel to itself along an infinitesimal curve connecting P to Q, it components at Q would be $A_p + \delta A_p$.

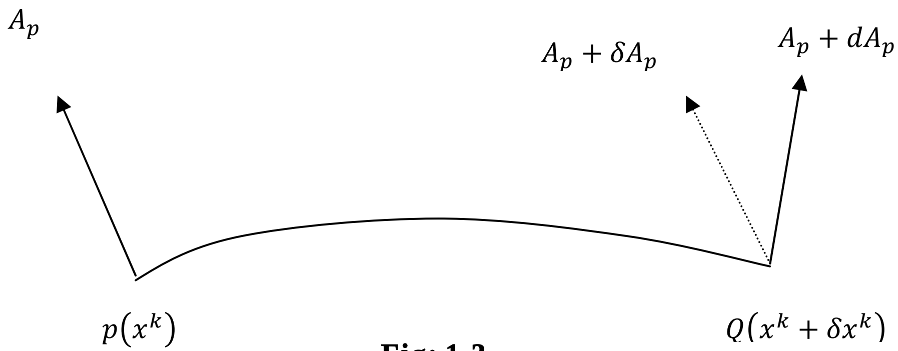

Fig: 1.3

In the above figure, we see that $\mathrm{P}(x^q)$ and $\mathrm{Q}(x^k + \delta x^k)$ be two neighboring points. With the vectors A^p at P is moved to Q as its magnitude and the direction did not be changed. In the above figure, this is shown by a dotted vector Q. The difference between the vector $A^p(x + x^k)$ and this dotted vector is another vector at Q. So, we may after all be able to define a process of differentiation of vectors , provided we know what happens to A_p during a parallel transport from P to Q and then a simple calculation we get as,

$$\delta A_p = \Gamma^{\ell}_{\mathrm{pk}} \mathrm{A}_{\ell} \delta \mathrm{x}^{\mathrm{k}} \qquad (1.11.4)$$

Where the co-efficient Γ^{ℓ}_{pk} are function of space and time, these quantities are called the **three index symbol** or the **Christoffel symbols**. Now we take the difference between continuous and the dotted vector Q is given by

$$A_p(x^k + \delta x^k) - \left[A_p(x^k) + \delta A_p\right]$$

$$= A_p(x^k + \delta x^k) - A_p(x^k) - \delta A_p$$

$$= \frac{A_p(x^k + \delta x^k) - A_p(x^k)}{\delta x^k} . \delta x^k - \Gamma^{\ell}_{\mathrm{pk}} \mathrm{A}_{\ell} \delta \mathrm{x}^{\mathrm{k}}$$

[By using equation (1.11.4)]

$$= \left[\frac{A_p(x^k + \delta x^k) - A_p(x^k)}{\delta x^k} - \Gamma^{\ell}_{\mathrm{pk}} \mathrm{A}_{\ell}\right] \delta x^k$$

$[By\ using$ (1.11.3)]

Now we redefine the derivative of vector,

$$\lim_{\delta x^k \to 0} \frac{\delta A_p}{\delta x^k} = \lim_{\delta x^k \to 0} \frac{A_p(x^k + \delta x^k) - A_p(x^k)}{\delta x^k} - \Gamma^{\ell}_{\mathrm{pk}} \mathrm{A}_{\ell}$$

Therefore,

$$A_{p,k} = \frac{\partial A_p}{\partial x^k} - \Gamma^{\ell}_{pk} \tag{1.11.5}$$

In the above manner we can state the above equation in the following form,

$$A_{p,k} = \frac{\partial A_p}{\partial x^k} - \Gamma^{\ell}_{pk} \tag{1.11.6}$$

In the above manner we can state the above equation in the following form,

$$A^{p}_{,k} = \frac{\partial A^p}{\partial x^k} + \Gamma^{p}_{k\ell} A^{\ell} \tag{1.11.7}$$

This derivative, by definition must transform a tensor. It is called a **co-variant derivative** and will be a **semicolon**, as against the **ordinary derivative** which is denoted by **comma**.

Similarly, the rule can be combined and extend the above result as,

$$A^{p_1 \ldots\ldots p_m}_{r_1 \ldots\ldots r_n, k} = \frac{\partial A^{p_1 \ldots\ldots p_m}_{r_1 \ldots\ldots r_n}}{\partial x^k} - \Gamma^{s}_{r_1 k} A^{p_1 \ldots\ldots p_m}_{s r_2 \ldots\ldots r_n} - \Gamma^{s}_{r_2 k} A^{p_1 \ldots\ldots p_m}_{r_1 s r_3 \ldots\ldots r_m} - \Gamma^{s}_{r_n k} A^{p_1 \ldots\ldots p_m}_{r_1 \ldots\ldots r_{n-1}, s}$$
$$+ \Gamma^{p_1}_{ks} A^{s p_2 \ldots\ldots p_m}_{r_1 \ldots\ldots r_n} + \Gamma^{p_2}_{ks} A^{p_1 s p_3 \ldots\ldots p_m}_{r_1 \ldots\ldots r_n} + \Gamma^{p_m}_{ks} A^{p_1 \ldots\ldots p_{m-1} s}_{r_1 \ldots\ldots r_n}$$

1.12 SYMMETRIC AND ANTISYMMETRIC TENSOR

SYMMETRIC TENSOR

A tensor is called symmetric with respect to two contravariant or two covariant indices if its components remain unaltered upon interchanged of indices. This if $A^{mpr}_{qs} = A^{pmr}_{qs}$ the tensor is symmetric in m and p. If a tensor is symmetric with respect to any two covariant and any two covariant indices it is called **symmetric**.

ANTISYMMETRIC TENSOR

A tensor is called anti-symmetric with respect to two contravariant of two covariant indices if its components change sign upon interchange of the indices. Thus $A^{mpr}_{qs} = -A^{pmr}_{qs}$ the tensor is **anti-symmetric** or **skew-symmetric** in m and p. If a tensor is skew-symmetric with respect to any two contravariant and any two covariant indices, it is called anti-symmetric or skew-symmetric.

1.13 CONTRACTION

The operation of contraction consists of identifying a lower index with an upper index in a mixed tensor. This procedure **reduces** the **rank** of tensor by 2.

Thus $A^i B_k$ is a tensor of rank 2 if A^i and B_k are vector. The identification $i = k$ gives a scalar. Hence,

$$A^i B_i = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 \qquad (1.13.1)$$

As in special relativity, we define a vector A^i to be space-like, time-like or null-like, then according to,

$$g_{ik} A^i A^k < 0, \;\; g_{ik} A^i A^k > 0, \;\; g_{ik} A^i A^k = 0 \qquad (1.13.2)$$

It is convenient to define associated tensors by the relations,

$$A_i = g_{ik} A^k, A^k = g^{ik} A_i \qquad (1.13.3)$$

Thus, $g_{ik} A^i A^k = A_k A^k$. The operations embodied in equation (1.13.3) are called lowering and raising the indices. We may frequently refer to A^i and A_i as the same object.

From the above manipulations of tensors it is clear the product of two tensors is a tensor. A reverse result is sometimes useful I deducing that a certain quantity is a tensor. This result is known as the **quotient law**. It states that, if a relation, such as,

$$PQ = R$$

holds in all co-ordinate frames, where p is an arbitrary tensor of rank m and R is a tensor of rank $(m + n)$ then Q is a tensor of rank n.

1.14 THE CHRISTOFFEL SYMBOLS

The Γ^i_{kl} introduced above is called the affine connection used for connecting a vector at p to a vector at neighbouring point Q. We now specialize this connection by assuming,

(i) $\Gamma^i_{kl} = \Gamma^i_{lk}$ $\qquad$ (ii) $g_{ik;l} = 0$

When connections follow these assumptions, the geometry of space time becomes Riemannian geometry. We simplify assumption (ii) to get Γ^i_{kl} in terms of g_{ik} and its derivatives,

$$g_{ik;l} = g_{ik,l} - \Gamma^n_{il} g_{nk} - \Gamma^n_{kl} g_{in} = 0$$

Define, $\Gamma^n_{il} g_{nk} = \Gamma_{k/il}$. Then we have,

$$g_{ik,l} = \Gamma_{k/il} + \Gamma_{i/kl} \qquad (1.14.1)$$

$$g_{kl,i} = \Gamma_{l/ik} + \Gamma_{k/il} \qquad (1.14.2)$$

$$g_{li,k} = \Gamma_{i/lk} + \Gamma_{l/ik} \qquad (1.14.3)$$

Take {(1.14.1) + (1.14.2) − (1.14.3)} then we have,

$$g_{ik,l} + g_{kl,i} - g_{li,k} = 2\Gamma_{k/il}$$

$$or, \qquad \Gamma_{k/il} = \frac{1}{2}\left[g_{ik,l} + g_{kl,i} - g_{li,k}\right]$$

$$or, \qquad g^{kn}\Gamma_{k/il} = \frac{1}{2} g^{kn}\left[g_{ik,l} + g_{kl,i} - g_{li,k}\right]$$

$$or, \qquad \Gamma^n_{il} = \frac{1}{2} g^{nk}\left[g_{ik,l} + g_{lk,i} - g_{il,k}\right] \qquad (1.14.4)$$

Here, Γ^n_{il} are often written as $\begin{Bmatrix} n \\ il \end{Bmatrix}$ and are called the three-index symbols. The symmetry in i & l reduces their number from **64 to 40**.

1.15 THE RIEMANNIAN-CHRISTOFFEL TENSOR

Let us consider a point with co-ordinate x^μ and a small closed curve C. Here U and V are two neighbouring points on C with co-ordinates $(x^\mu + \xi^\mu)$ and $(x^\mu + \xi^\mu + d\xi^\mu)$; where ξ^μ small.

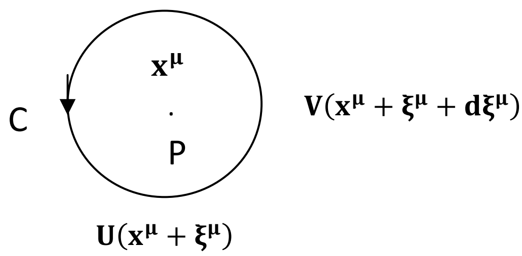

Fig: 1.4

Consider a contravariant vector $A^{\mu}$is displaced from U toV parallel to itself. The change of its components is given by,

$$\delta A^{\mu} = -\Gamma^{\mu}_{\nu\lambda} A^{\nu} d\xi^{\lambda} \tag{1.15.1}$$

When $\Gamma^{\mu}_{\nu\lambda}$ and A^{ν} are computed at U.

Now the small displacement from P to U is ξ.

Therefore,

$$\left(\Gamma^{\mu}_{\nu\lambda}\right)_{U=p+\xi} = \left(\Gamma^{\mu}_{\nu\lambda}\right)_{p} + \frac{\partial \Gamma^{\mu}_{\nu\lambda}}{\partial x^{\sigma}} \xi^{\sigma} \quad \text{[By Taylor's expansion]} \tag{1.15.2}$$

Also,

The vectorA^{μ}in equation (1.15.1) represents the vector after its parallel displacement from P to U.

Therefore,

$$(A^{\nu})_{U=p+\xi} = (A^{\mu})_{p} - \Gamma^{\nu}_{\varepsilon\sigma} A^{\tau} \xi^{\sigma} \tag{1.15.3}$$

Where, $A^{\nu}, A^{\tau}, \Gamma^{\nu}_{\tau\sigma}$ are all to be evaluated at P.

Now, substituting equation (1.15.2) and equation (1.15.3) in equation (1.15.1), we find,

$$\delta A^{\mu} = -\left(\Gamma^{\mu}_{\nu\lambda} + \frac{\partial \Gamma^{\mu}_{\nu\lambda}}{\partial x^{\sigma}} \xi^{\sigma}\right) \left(A^{\nu} - \Gamma^{\nu}_{\tau\sigma} A^{\tau} \xi^{\sigma}\right) d\xi^{\lambda}$$

$$= -\left[\Gamma^{\mu}_{\nu\lambda} A^{\nu} + \frac{\partial \Gamma^{\mu}_{\nu\lambda}}{\partial x^{\sigma}} A^{\nu} \xi^{\sigma} - \Gamma^{\mu}_{\nu\lambda} \Gamma^{\nu}_{\tau\sigma} A^{\tau} \xi^{\sigma} + \Gamma^{\nu}_{\tau\sigma} \frac{\partial \Gamma^{\mu}_{\nu\lambda}}{\partial x^{\sigma}} A^{\tau} (\xi^{\sigma})^{2}\right] d\xi^{\lambda}$$

$$= -\left[\Gamma^{\mu}_{\nu\lambda} A^{\nu} + \frac{\partial \Gamma^{\mu}_{\nu\lambda}}{\partial x^{\sigma}} A^{\nu} \xi^{\sigma} - \Gamma^{\mu}_{\nu\lambda} \Gamma^{\nu}_{\tau\sigma} A^{\tau} \xi^{\sigma}\right] d\xi^{\lambda} \quad (since\ to\ first\ order\ in\ \xi)$$

$$= -\left[\Gamma^{\mu}_{\nu\lambda} A^{\nu} + \left(\frac{\partial \Gamma^{\mu}_{\nu\lambda}}{\partial x^{\sigma}} A^{\nu} - \Gamma^{\mu}_{\nu\lambda} \Gamma^{\nu}_{\tau\sigma} A^{\tau}\right) \xi^{\sigma}\right] d\xi^{\lambda}$$

$$= -\left[\Gamma^{\mu}_{\nu\lambda} A^{\nu} + \left(\frac{\partial \Gamma^{\mu}_{\nu\lambda}}{\partial x^{\sigma}} A^{\nu} - \Gamma^{\mu}_{\tau\lambda} \Gamma^{\tau}_{\nu\sigma} A^{\nu}\right) \xi^{\sigma}\right] d\xi^{\lambda} \tag{1.15.4}$$

Now integrating on both sides of equation (1.15.4) around C we find that,

$$\Delta A^{\mu} = \oint \delta A^{\mu} = -\left[\Gamma^{\mu}_{\nu\lambda} A^{\nu} \oint_{c} d\xi^{\lambda} + \frac{\partial \Gamma^{\mu}_{\nu\lambda}}{\partial x^{\sigma}} A^{\nu} \oint_{c} \xi^{\sigma} d\xi^{\lambda} - \Gamma^{\mu}_{\tau\lambda} \Gamma^{\tau}_{\nu\sigma} A^{\nu} \oint_{c} \xi^{\sigma} d\xi^{\lambda}\right] \tag{1.15.5}$$

But we know that,

$$\oint_{c} d\xi^{\lambda} = 0 \tag{1.15.6}$$

And,

$$\oint_{c} \xi^{\sigma} d\xi^{\lambda} = \oint [d(\xi^{\sigma}\xi^{\lambda}) - \xi^{\lambda} d\xi^{\sigma}]$$

$$= \oint d(\xi^{\sigma}\xi^{\lambda}) - \oint \xi^{\lambda} d\xi^{\sigma}$$

$$= -\oint \xi^{\lambda} d\xi^{\sigma} \quad ; [\oint d(\xi^{\sigma}\xi^{\lambda}) = 0] \tag{1.15.7}$$

Therefore,

$$\oint_{c} \xi^{\sigma} d\xi^{\lambda} = \frac{1}{2}\oint_{c} \left(\xi^{\sigma} d\xi^{\lambda} + \xi^{\sigma} d\xi^{\lambda}\right)$$

$$= \frac{1}{2}\oint_{c} \left(\xi^{\sigma} d\xi^{\lambda} - \xi^{\lambda} d\xi^{\sigma}\right) \quad [By\, using (1.14.7)]$$

$$= \alpha^{\sigma\lambda} \tag{1.15.8}$$

$$= -\alpha^{\lambda\sigma} \tag{1.15.9}$$

Now using equation (1.15.6) & equation (1.15.8) we get,

$$\Delta A^{\mu} = \left[\Gamma^{\mu}_{\tau\lambda} \Gamma^{\tau}_{\nu\sigma} A^{\nu} - \frac{\partial \Gamma^{\mu}_{\nu\lambda}}{\partial x^{\sigma}} A^{\nu}\right] \alpha^{\sigma\lambda}$$

$$= \left[\Gamma^{\mu}_{\tau\lambda} \Gamma^{\tau}_{\nu\sigma} - \frac{\partial \Gamma^{\mu}_{\nu\lambda}}{\partial x^{\sigma}}\right] A^{\nu} \alpha^{\sigma\lambda}$$

Since $A^{\mu}$is an arbitrary vector, implies that ΔA^{μ} is also an arbitrary vector. Thus,

$$Y_{\nu}^{\mu} = \left(\Gamma_{\tau\lambda}^{\mu}\Gamma_{\upsilon\sigma}^{\tau} - \frac{\partial\Gamma_{\nu\lambda}^{\mu}}{\partial x^{\sigma}}\right)\left(-\alpha^{\lambda\sigma}\right) \; [By\ using\ (1.15.9)]$$

$$= \left(\frac{\partial\Gamma_{\tau\lambda}^{\mu}}{\partial x^{\sigma}} - \Gamma_{\tau\lambda}^{\mu}\Gamma_{\nu\sigma}^{\tau}\right)\alpha^{\lambda\sigma}$$

Now, interchanging the indices λ & σ in the above equation we get,

$$Y_{\nu}^{\mu} = \frac{\partial\Gamma_{\nu\sigma}^{\mu}}{\partial x^{\lambda}} - \Gamma_{\tau\sigma}^{\mu}\Gamma_{\nu\lambda}^{\tau}\alpha^{\sigma\lambda} \qquad (1.15.11)$$

Therefore, adding equation (1.15.10) & equation (1.15.11) we get,

$$Y_{\nu}^{\mu} = \frac{1}{2}\left(\frac{\partial\Gamma_{\nu\sigma}^{\mu}}{\partial x^{\lambda}} - \frac{\partial\Gamma_{\nu\lambda}^{\mu}}{\partial x^{\sigma}} + \Gamma_{\tau\lambda}^{\mu}\Gamma_{\nu\sigma}^{\mu} - \Gamma_{\tau\sigma}^{\mu}\Gamma_{\nu\lambda}^{\tau}\right)\alpha^{\sigma\lambda} \qquad (1.15.12)$$

But here the bracket expression is **anti-symmetric** in σ and λ. Therefore equation (1.15.12) can be written as,

$$Y_{\nu}^{\mu} = \frac{1}{2}R_{\nu\lambda\sigma}^{\mu}\alpha^{\sigma\lambda}$$

Where,

$$R_{\nu\lambda\sigma}^{\mu} = \frac{\partial\Gamma_{\nu\sigma}^{\mu}}{\partial x^{\lambda}} - \frac{\partial\Gamma_{\nu\lambda}^{\mu}}{\partial x^{\sigma}} + \Gamma_{\lambda\tau}^{\mu}\Gamma_{\nu\sigma}^{\tau} - \Gamma_{\sigma\tau}^{\mu}\Gamma_{\nu\lambda}^{\tau} \qquad (1.15.13)$$

is called **Riemann-Christoffel tensor** or simply the **curvature tensor**, by using a closed path.

Now, using cyclic permutations in the indices v, λ, σ we get,

$$R_{\lambda\sigma\nu}^{\mu} = \frac{\partial\Gamma_{\lambda\nu}^{\mu}}{\partial x^{\sigma}} - \frac{\partial\Gamma_{\lambda\sigma}^{\mu}}{\partial x^{\nu}} + \Gamma_{\sigma\tau}^{\mu}\Gamma_{\lambda\nu}^{\tau} - \Gamma_{\nu\tau}^{\mu}\Gamma_{\lambda\sigma}^{\tau} \qquad (1.15.14)$$

Also using cyclic permutations in the indices λ, σ, v we get,

$$R_{\sigma\nu\lambda}^{\mu} = \frac{\partial\Gamma_{\sigma\lambda}^{\mu}}{\partial x^{\nu}} - \frac{\partial\Gamma_{\sigma\nu}^{\mu}}{\partial x^{\lambda}} + \Gamma_{\nu\tau}^{\mu}\Gamma_{\sigma\lambda}^{\tau} - \Gamma_{\lambda\tau}^{\mu}\Gamma_{\sigma\nu}^{\tau} \qquad (1.15.15)$$

Here, adding equations [(1.15.13), (1.15.14) & (1.15.15)] we get,

$$R_{\nu\lambda\sigma}^{\mu} + R_{\lambda\sigma\nu}^{\mu} + R_{\sigma\nu\lambda}^{\mu} = 0 \qquad (1.15.16)$$

Here equation (1.15.16) is called **cyclic property** of the **Riemann-Christoffel tensor** or the **curvature tensor** i.e. for anti-symmetric affine connection.

For the contraction of μ & σ the equation (1.15.16) becomes,

$$R^{\mu}_{\nu\lambda\mu} + R^{\mu}_{\lambda\mu\nu} + R^{\mu}_{\mu\nu\lambda} = 0$$

$$\Rightarrow \quad R^{\mu}_{\nu\lambda\mu} + R^{\mu}_{\lambda\mu\nu} = 0$$

$$\Rightarrow \quad R^{\mu}_{\nu\lambda\mu} - R^{\mu}_{\lambda\nu\mu} = 0$$

$$\Rightarrow \quad R_{\upsilon\lambda} - R_{\lambda\nu} = 0$$

$$\Rightarrow \quad R_{\upsilon\lambda} = R_{\lambda\nu}$$

Therefore Ricci tensor $R_{\upsilon\lambda}$ is symmetric for a metric affinity.

1.16 THE RICCI AND EINSTEIN TENSORS

THE RICCI TENSOR

The contraction of Riemannian-Christoffel tensor with respect to σ gives the second rank covariant tensor, called the Ricci tensor and denoted by $R_{\mu\upsilon}$.

Here Christoffel curvature tensor is,

$$R^{\lambda}_{\mu\nu\sigma} = \frac{\partial}{\partial x^{\nu}}\Gamma^{\lambda}_{\mu\sigma} - \frac{\partial}{\partial x^{\sigma}}\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\lambda}_{\nu\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\alpha\sigma} \qquad (1.16.1)$$

Where,

$$\Gamma^{\lambda}_{\mu\lambda} = \frac{\partial\left(log\sqrt{g}\right)}{\partial x^{\mu}}$$

So, equation (1.15.1) may be written as,

$$R_{\mu\nu} = \frac{\partial^2\left(log\sqrt{g}\right)}{\partial x^{\nu}\partial x^{\mu}} - \frac{\partial}{\partial x^{\lambda}}\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\alpha\lambda}$$

Also, by the process of contraction we can construct lower-rank tensors from R_{iklm} .

The tensor,

$$R_{kl} = g^{im}R_{iklm} = R^{m}_{klm}$$

is called the **Ricci tensor**. If we use the locally inertial co-ordinate system, we immediately see that

$$R_{kl} = R_{lk}$$

i.e. Ricci tensor is symmetric tensor.

EINSTEIN TENSOR

Here the Bianchi identities are given by,

$$R^{\lambda}_{\mu\nu\sigma;\rho} + R^{\lambda}_{\mu\sigma\rho;\nu} + R^{\lambda}_{\mu\rho\nu;\sigma} = 0 \qquad (1.16.2)$$

Apply anti-symmetric property in the second term of equation (1.16.2) we get,

$$R^{\lambda}_{\mu\nu\sigma;\rho} - R^{\lambda}_{\mu\rho\sigma;\nu} + R^{\lambda}_{\mu\rho\nu;\lambda} = 0 \qquad (1.16.3)$$

Contracting this with respect to λ & σ we have,

$$R^{\lambda}_{\mu\nu\lambda;\rho} - R^{\lambda}_{\mu\rho\lambda;\nu} + R^{\lambda}_{\mu\rho\nu;\lambda} = 0 \qquad (1.16.4)$$

By definition of Ricci tensor we have,

$$R_{\mu\nu;\rho} - R_{\mu\rho;\nu} + R^{\lambda}_{\mu\rho\nu;\lambda} = 0 \qquad (1.16.5)$$

Since the derivatives of fundamental tensors are zero, then multiplying the above equation by $g^{\mu\rho}$ on both sides, we may write,

$$\left(g^{\mu\rho}R_{\mu\nu}\right)_{;\rho} - \left(g^{\mu\rho}R_{\mu\rho}\right)_{;\nu} + \left(g^{\mu\rho}R^{\lambda}_{\mu\rho\nu}\right)_{;\lambda} = 0$$

$$\Rightarrow \quad R^{\rho}_{\nu;\rho} - R_{;\nu} + R^{\lambda}_{\nu;\lambda} = 0 \qquad (1.16.6)$$

Now, changing the dummy indices ρ & λ to μ we get,

$$R^{\mu}_{\nu;\mu} - R_{;\nu} + R^{\mu}_{\nu;\mu} = 0$$

$$\Rightarrow \quad 2R^{\mu}_{\nu;\mu} - R_{;\nu} = 0 \qquad (1.16.7)$$

But,

$$R_{;\nu} = \frac{\partial R}{\partial x^{\nu}} = \frac{\partial}{\partial x^{\mu}}(\delta^{\mu}_{\nu}R) = (\delta^{\mu}_{\nu}R)_{;\mu}$$

So, the equation (1.15.7) becomes,

$$2R^{\mu}_{\nu;\mu} - (\delta^{\mu}_{\nu}R)_{;\mu} = 0$$

$$\Rightarrow\ R^{\mu}_{\nu;\mu} - \frac{1}{2}(\delta^{\mu}_{\nu}R)_{;\mu} = 0$$

$$\Rightarrow\ \left(R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R\right)_{;\mu} = 0$$

$$\Rightarrow\quad G^{\mu}_{\nu;\mu} = 0 \qquad (1.16.8)$$

Where, the tensor $G^{\mu}_{\nu} = R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R$ is called the **Einstein tensor**. The Einstein tensor G^{μ}_{ν} plays a very fundamental role in the general theory of relativity.

Again, the covariant form of Einstein's tensor is given by,

$$G_{\lambda\mu} = g_{\lambda\mu}G^{\mu}_{\nu}$$

$$= g_{\lambda\mu}\left(R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R\right)$$

$$= g_{\lambda\mu}R^{\mu}_{\nu} - \frac{1}{2}g_{\lambda\mu}\delta^{\mu}_{\nu}R$$

$$= R_{\lambda\nu} - \frac{1}{2}g_{\lambda\nu}R$$

$$\therefore\ G_{\lambda\mu} = R_{\lambda\nu} - \frac{1}{2}g_{\lambda\nu}R \qquad (1.16.9)$$

This is the required co-variant form of Einstein's tensor.

Also, according to definition of divergence of a tensor we get,

$$div(G^{\mu}_{\nu}) = G^{\mu}_{\nu;\mu}$$

Now, using equation (1.15.8) we have,

$$G^{\mu}_{\nu;\mu} = 0$$

Thus, the divergence of Einstein's tensor is identically zero.

1.17 BIANCHI IDENTITIES

Here for the sake of simplicity we shall use geodesic co-ordinate system and establish an important identity known as the Bianchi identity.

We know at the pole of geodesic co-ordinate system, both kinds of Christoffel symbols vanish, but not necessarily their derivatives also.

i.e. $\Gamma^{\mu}_{\nu\lambda} = 0$ at the pole p, but $\frac{\partial \Gamma^{\mu}_{\nu\sigma}}{\partial x^{\sigma}} \neq 0$.

At the event of vanishing of the Christoffel symbols at a point the process of covariant differentiation reduces to ordinary partial differentiation.

Now, the mixed curvature tensor at the P_0 of geodesic co-ordinate system is given by,

$$R^{\lambda}_{\mu\nu\sigma} = \frac{\partial}{\partial x^{\nu}} \Gamma^{\gamma}_{\mu\sigma} - \frac{\partial}{\partial x^{\sigma}} \Gamma^{\lambda}_{\mu\nu} \quad (1.17.1)$$

Taking covariant derivative with respect to ρ of equation (1.17.1) at the pole of geodesic co-ordinate system we get,

$$R^{\lambda}_{\mu\nu\sigma;\rho} = \frac{\partial^2}{\partial x^{\rho} \partial x^{\nu}} \Gamma^{\lambda}_{\mu\sigma} - \frac{\partial^2}{\partial x^{\rho} \partial x^{\sigma}} \Gamma^{\lambda}_{\mu\nu} \quad \text{; at the pole of } P_0 \quad (1.17.2)$$

Permuting the indices ν, σ & ρ in a cyclic order we get two more equations from equation (1.17.1),

$$R^{\lambda}_{\mu\sigma\rho;\nu} = \frac{\partial^2}{\partial x^{\nu} \partial x^{\sigma}} \Gamma^{\lambda}_{\mu\rho} - \frac{\partial^2}{\partial x^{\nu} \partial x^{\sigma}} \Gamma^{\lambda}_{\mu\sigma} \quad (1.17.3)$$

$$R^{\lambda}_{\mu\rho\nu;\sigma} = \frac{\partial^2}{\partial x^{\sigma} \partial x^{\rho}} \Gamma^{\lambda}_{\mu\nu} - \frac{\partial^2}{\partial x^{\sigma} \partial x^{\nu}} \Gamma^{\lambda}_{\mu\rho} \quad (1.17.4)$$

So, adding equations (1.17.2), (1.17.3) & (1.17.4), then we get at the pole P_0,

$$R^{\lambda}_{\mu\nu\sigma;\rho} + R^{\lambda}_{\mu\sigma\rho;\nu} + R^{\lambda}_{\mu\rho\nu;\sigma} = 0 \quad (1.17.5)$$

We have proved the equation (1.17.5) at the pole P_0 of geodesic co-ordinate system; but since this is a tensor equation, it must hold good in every co-ordinates system. Further since P_0 is an arbitrary point of Riemannian space then equation (1.17.5) is true for all points of Riemannian space and for all co-ordinate systems. The relations expressed by equation (1.17.5) are called **Bianchi identities**.

The covariant form of Bianchi identities is obtained by taking inner product of equation (1.17.5) with $g_{\alpha\lambda}$; Viz.

$$\left(g_{\alpha\lambda}R^{\lambda}_{\mu\nu\sigma}\right)_{;\rho} + \left(g_{\alpha\lambda}R^{\lambda}_{\mu\sigma\rho}\right)_{;\nu} + \left(g_{\alpha\lambda}R^{\lambda}_{\mu\rho\nu}\right)_{;\sigma} = 0$$

$$\Rightarrow \quad R_{\alpha\mu\nu\sigma;\rho} + R_{\alpha\mu\sigma\rho;\nu} + R_{\alpha\mu\rho\nu;\sigma} = 0 \qquad (1.17.6)$$

1.18 MATTER TENSOR FOR A PERFECT FLUID

From any general co-ordinate system x^i, we transform to a locally inertial system x_0^i. In this system,

$$(g_{ik})_0 = diag(-1,-1,-1,1); \;\; (c = 1)$$

In the neighborhood of the point where this x_0^i systems holds, the perfect fluid is characterized by its density and equal pressure in all directions. Therefore, the matter tensor in this system has

$$T_0^{44} = \rho; \quad T_0^{11} = T_0^{22} = T_0^{33} = p; \quad T_0^{ab} = 0, \qquad a \neq b$$

Also, in this local frame the fluid is at rest so that

$$\frac{dx_0^1}{ds} = \frac{dx_0^2}{ds} = \frac{dx_0^3}{ds} = 0, \qquad \frac{dx_0^4}{ds} = 0$$

$$v_0^1 = v_0^2 = v_0^3 = 0, \qquad v_0^4 = 1$$

Now we are ready to find T^{ik} in the general co-ordinate system. We have the transformation $x_0^i \longrightarrow x^i$ giving,

$$T^{ik} = \frac{\partial x^i}{\partial x_0^a}\frac{\partial x^k}{\partial x_0^b}T_0^{ab}$$

$$= \frac{\partial x^i}{\partial x_0^4}\frac{\partial x^k}{\partial x_0^4}\rho + \left(\frac{\partial x^i}{\partial x_0^1}\frac{\partial x^k}{\partial x_0^1} + \frac{\partial x^i}{\partial x_0^2}\frac{\partial x^k}{\partial x_0^2} + \frac{\partial x^i}{\partial x_0^3}\frac{\partial x^k}{\partial x_0^3}\right)p$$

But the law of transformation of g^{ik} gives,

$$g^{ik} = \frac{\partial x^i}{\partial x_0^a}\frac{\partial x^k}{\partial x_0^b} g_0^{ab}$$

$$= \frac{\partial x^i}{\partial x_0^4}\frac{\partial x^k}{\partial x_0^4} - \left(\frac{\partial x^i}{\partial x_0^1}\frac{\partial x^k}{\partial x_0^1} + \frac{\partial x^i}{\partial x_0^2}\frac{\partial x^k}{\partial x_0^2} + \frac{\partial x^i}{\partial x_0^3}\frac{\partial x^k}{\partial x_0^3}\right)$$

Also, in the general co-ordinate system the velocity of the fluid is V^i, then

$$V^i = \frac{\partial x^i}{\partial x_0^a} V_0^a$$

$$= \frac{\partial x^i}{\partial x_0^4} V_0^4 + \frac{\partial x^i}{\partial x_0^1} V_0^1 + \frac{\partial x^i}{\partial x_0^2} V_0^2 + \frac{\partial x^i}{\partial x_0^3} V_0^3$$

$$= \frac{\partial x^i}{\partial x_0^4}.1 + \frac{\partial x^i}{\partial x_0^1}.0 + \frac{\partial x^i}{\partial x_0^2}.0 + \frac{\partial x^i}{\partial x_0^3}.0$$

$$= \frac{\partial x^i}{\partial x_0^4}$$

Here,

$$T^{ik} = V^i V^k \rho + p(V^i V^k - g^{ik})$$

$$= (p + \rho) V^i V^k - p g^{ik}$$

This is the required **matter tensor** for a perfect fluid.

"Rather than have one global frame with gravitational forces we have many local frames without gravitational forces."

STEPHEN SCHUTZ

CHAPTER 2

THE SPECIAL THEORY OF RELATIVITY

2.1 INTRODUCTION

The special theory of relativity had its origin in the development of **Electrodynamics.** The general theory of Relativity is the **relativistic theory of gravitation**.

The special theory of relativity makes only a restricted use of the idea that we can detect and measure the motion of a given body relative to other bodies, but cannot assign any meaning to its absolute motion, i.e. it merely considers the relativity of uniform translator motion, in the region of free space, where gravitational effects can be neglected. Consequently, by this assumption we can conclude that physical laws remain unchanged when subjected to the systems in which the law of inertia holds well. But in order to explain the "**clock paradox**" and universal law of gravitation in the special theory was extended to the non-inertial systems. On the generalization of the special theory of relativity for the gravitational forces, the theoretical predictions were not able to explain the observed phenomena. These deviations were due to the facts that:

1) The theory fails for fixed particles in the gravitational field which can be clearly observed in the red shift of spectral lines i.e., the atoms are fixed and spectral lines emitted by atoms are subjected to strong magnetic field.

2) The theory fails for the phenomena in which the velocity is comparable to the velocity of light, e.g., bending of light rays around the attracting body.

According to the special theory, the bending of light rays passing near the sun should be **0.88 seconds of arc** while in actual observation it is **1.75 seconds arc.**

3) The theory fails in the case when the velocity and gravitational field both are present as in the case of precession of perihelion of Mercury.

The advance of the perihelion of Mercury is predicted by the special theory to be **at the rate of 7.2 seconds of arc per century** while the **actual advance is 43 seconds of arc per century.**

The predictions given in special theory of relativity therefore, must be modified. As we see that the special theory of relativity deals with only the systems knows as inertial systems. In special theory, all the physical laws in nature are supposed to be invariant with respect to co-ordinate transformation. But this invariance is limited to inertial systems only, if we extend the above statement by saying that all the physical laws in nature are invariant relative to any co-ordinate transformation, i.e., for non-inertial systems also. Then this results the general theory of relativity. The study of gravitational phenomenon's with the help of the general theory of relativity gives small deviations from those obtained from the special theory and these deviations have been verified by experimental results.

2.2 POSTULATES OF SPECIAL THEORY OF RELATIVITY

The postulates of special theory of relativity are given below:

(i) **The nature laws must preserve their forms relative to all observers in a state of relative uniform motion.**
According to this postulate, velocity is not absolute but relative. It is a fact drawn from the failure of Michelson and Morley experiment which was performed to determine velocity of earth through either.

(ii) **The velocity of light in vacuum is independent of the velocity of observer or the velocity of source.**

According to Galilean transformations, this postulate is not true. In fact, it is confirmed experimentally that the velocity of light calculated by any method of constant. The second postulate is important in the sense that it gives a clear distinction between **classical theory** and **Einstein theory of relativity.**

2.3 LORENTZ TRANSFORMATIONS IN SPECIAL RELATIVITY

A frame of reference (or co-ordinate system) in which a freely moving body (or rest particle) proceeds uniformly with constant velocity along a straight line or remains at rest is called an **inertial frame**. In brief we may say, "an inertial frame is one in which Newton's first law is true." Newton says that,

"**If force does not act then the rest object which are at rest and the moving object which are at move relative to one another.**"

If two frames of reference move with uniform motion relative to each other and if one of them is an inertial frame then the other is also an inertial frame. Because in this system, two every free motion will be linear and uniform.

Let us consider two frames of reference be parallel to each other. Let S and $S^{'}$ be two inertial frames of reference, where $S^{'}$is moving with uniform velocity u along $x^{'}$-axis relative to$S$. Two observers situated at the origin $O$and $O^{'}$. Observe any point P from the systems S and $S^{'}$respectively. The event P determined by co-ordinates $(t, x, y, z)$and $(t^{'}, x^{'}, y^{'}, z')$ by observers O and $O^{'}$respectively. Each co-ordinate $(t^{'}, x^{'}, y^{'}, z')$ is a linear function of(t, x, y, z).

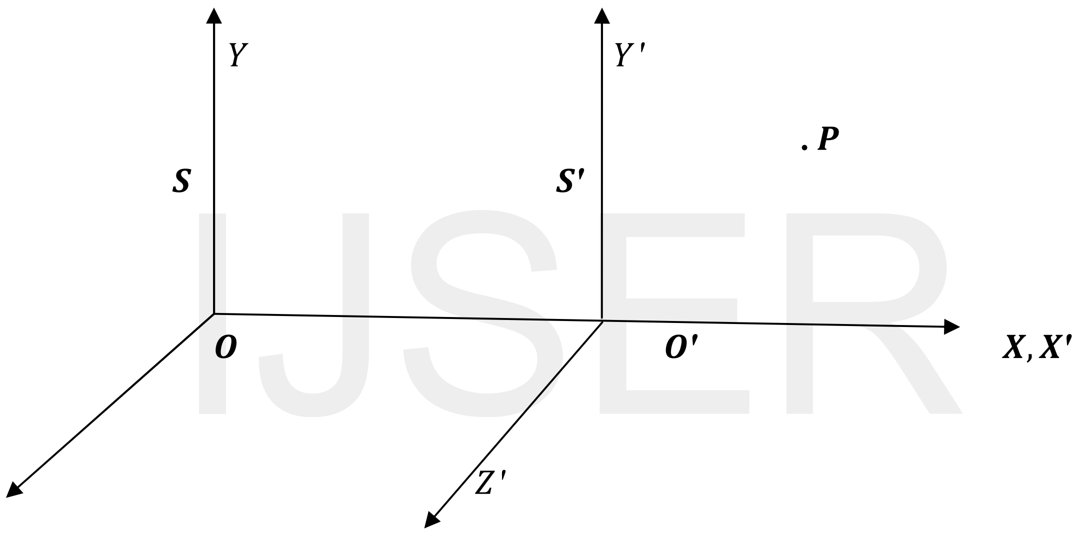

Fig: 2.1

Let us assume**,**

$$t^{'} = \alpha t + \beta x \quad (2.3.1)$$

$$x^{'} = \gamma t + \delta x \quad (2.3.2)$$

$$y^{'} = y \quad (2.3.3)$$

$$z^{'} = z \quad (2.3.4)$$

Suppose at the instant $t = 0$, a light source situated at the common origin O, O' in S, S' radiates a pulse of short durations. In time t, light will occupy a sphere whose centre is O in S and whose radius is ct.

$$\therefore c^2t^2 = x^2 + y^2 + z^2 \tag{2.3.5}$$

Similarly for the system S' we have,

$$c^2t'^2 = x'^2 + y'^2 + z'^2 \tag{2.3.6}$$

Thus the substitutions for (t', x', y', z') from equations (2.3.1), (2.3.2), (2.3.3), (2.3.4), in equation (2.3.6) we get equation (2.3.5). This will be true if

$$c^2t'^2 = x'^2 + y'^2 + z'^2 = k(c^2t^2 - x^2 - y^2 - z^2)$$

Where, k is the constant depending on u i.e., $k = k(u)$. It can be shown in fact $k = 1$. Therefore,

$$c^2t'^2 - x'^2 - y'^2 - z'^2 = (c^2t^2 - x^2 - y^2 - z^2)$$

$$or, \quad c^2t'^2 - x'^2 = c^2t^2 - x^2$$

$$or, \quad c^2(\alpha t + \beta x)^2 - (\gamma t + \delta x)^2 = c^2t^2 - x^2 \text{[By using (2.3.1) \& (2.3.2)]}$$

$$or, \quad c^2(\alpha^2t^2 + 2\alpha\beta tx + \beta^2x^2) - (\gamma^2t^2 + 2\gamma\delta tx + \delta^2x^2) = c^2t^2 - x^2$$

$$or, c^2(\alpha^2t^2 + 2\alpha\beta tx + \beta^2x^2) - (\gamma^2t^2 + 2\gamma\delta tx + \delta^2x^2) - c^2t^2 + x^2 = 0$$

Now equating the co-efficient of t^2, tx, x^2 in the above equation we get,

$$c^2\alpha^2 - \gamma^2 - c^2 = 0 \tag{2.3.7}$$

And

$$2\alpha\beta c^2 - 2\gamma\delta = 0$$
$$or, \ \alpha\beta c^2 - \gamma\delta = 0 \tag{2.3.8}$$

And

$$c^2\beta^2 - \delta^2 + 1 = 0 \tag{2.3.9}$$

Here the origin O' is given by,

$$x' = 0 = \gamma t + \delta x \qquad \text{[By using (2.3.2)]}$$

Therefore,

$$\gamma t + \delta x = 0$$

$$or, \ \delta x = -\gamma t$$

$$or, \quad \frac{x}{t} = -\frac{\gamma}{\delta}$$

$$or, u = -\frac{\gamma}{\delta} \qquad \left[\therefore \frac{x}{t} = u\right]$$

$$or, \gamma = -\delta u \qquad (2.3.10)$$

From equation (2.3.7) we get,

$$c^2\alpha^2 = \gamma^2 + c^2$$

$$or,\ \alpha^2 = \frac{\gamma^2}{c^2} + 1$$

$$or, \alpha = \pm\sqrt{\frac{\gamma^2}{c^2} + 1} = \pm\sqrt{1 + \frac{\gamma^2}{c^2}}$$

$$\therefore \alpha = \pm\sqrt{1 + \frac{\delta^2 u^2}{c^2}} \qquad \text{[By using (2.3.10)]}$$

Also from equation (2.3.9) we have,

$$\beta^2 = \delta^2$$

$$or,\ \ \beta^2 = \frac{1}{c^2}(\delta^2 - 1)$$

$$or, \quad \beta = \pm\tfrac{1}{c}\sqrt{\delta^2 - 1} \qquad (2.3.12)$$

Now, substituting the values of α, β, γ in equation (2.3.8) we get,

$$\left(\pm\sqrt{1 + \frac{\delta^2 u^2}{c^2}}\right)\left(\pm\frac{1}{c}\sqrt{\delta^2 - 1}\right)c^2 — \delta u \delta = 0$$

$$or, \left(\pm\sqrt{1 + \frac{\delta^2 u^2}{c^2}}\right)\left(\pm\frac{1}{c}\sqrt{\delta^2 - 1}\right)c^2 = -(-\delta u)\delta \qquad \text{[Squaring on both sides]}$$

$$or, \quad \left(1 + \frac{\delta^2 u^2}{c^2}\right)\frac{1}{c}(\delta^2 - 1)c^4 = \delta^4 u^2]$$

$$or, \quad \delta^2 - 1 + \frac{\delta^4 u^2}{c^2} - \frac{\delta^2 u^2}{c^2} = \frac{\delta^4 u^2}{c^2}$$

$$or,\ \ \delta^2 - \frac{\delta^2 u^2}{c^2} = 1$$

$$or,\ \ \delta^2\left(1 - \frac{u^2}{c^2}\right) = 1$$

$$or,\ \ \delta = \pm\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

When $u = 0; t = t^{'} = 0; x = x^{'} = 0, y = y^{'}, z = z'$ then from equation (2.3.2) we get $\delta = 1$, implies that δ is positive.

$$\therefore\ \ \ \delta = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Hence from equation (2.3.10) we get,

$$\gamma = -\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \times u$$

$$\therefore\ \ \ \gamma = -\frac{u}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Also, from equation (2.3.11) we get,

$$\alpha = \pm\sqrt{1 + \frac{u^2}{c^2}\left(\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\right)^2}$$

$$= \pm\sqrt{1 + \frac{u^2}{c^2}.\frac{1}{1 - \frac{u^2}{c^2}}}$$

$$= \pm\sqrt{1 + \frac{u^2}{c^2 - u^2}}$$

$$= \pm\sqrt{\frac{c^2 - u^2 + u^2}{c^2 - u^2}}$$

$$= \pm\sqrt{\frac{c^2}{c^2 - u^2}}$$

$$= \pm\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

When $u = 0; t = t' = 0; x = x' = 0, y = y', z = z'$ then from equation (2.3.1)

We get $\alpha = 1$, which implies that α is positive.

$$\therefore \quad \alpha = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Again, from equation (2.3.12) we get,

$$\beta = \pm\frac{1}{c}\sqrt{\left(\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\right)^2 - 1}$$

$$= \pm\frac{1}{c}\sqrt{\frac{1}{1 - \frac{u^2}{c^2}} - 1}$$

$$= \pm\frac{1}{c}\sqrt{\frac{c^2}{c^2 - u^2} - 1}$$

$$= \pm\frac{1}{c}\sqrt{\frac{c^2 - c^2 + u^2}{c^2 - u^2}}$$

$$= \pm\frac{1}{c}\sqrt{\frac{u^2}{c^2 - u^2}}$$

$$= \pm\frac{1}{c}\sqrt{\frac{u^2}{c^2\left(1 - \frac{u^2}{c^2}\right)}}$$

$$= \pm\frac{u}{c^2}.\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

When $u = o; t = t^{'} = 0; x = x^{'} = 0, y = y^{'}, z = z'$ then from equation (2.3.1)

We get $\beta = 0$. Therefore this does not tell us anything about origin.

Now, substituting all the values of $\alpha, \beta, \gamma, \delta$ in equation (2.3.8) we get,

$$\left(\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\right)\left(\pm\frac{u}{c^2\sqrt{1 - \frac{u^2}{c^2}}}\right)c^2 - \left(-\frac{u}{\sqrt{1 - \frac{u^2}{c^2}}}\right)\left(\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\right) = 0$$

$$or, \quad \left(\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\right)\left(\pm\frac{u}{c^2\sqrt{1 - \frac{u^2}{c^2}}}\right) = \left(-\frac{u}{\sqrt{1 - \frac{u^2}{c^2}}}\right)\left(\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\right)$$

Since the R.H.S is (–ve), so that in the L.H.S the value of β must be negative.

$$\therefore \quad \beta = -\frac{u}{c^2\sqrt{1 - \frac{u^2}{c^2}}}$$

Now, putting the values of $\alpha, \beta, \gamma, \delta$ in equation [(2.3.1), (2.3.2), (2.3.3), (2.3.4)]

we get,

$$t^{'} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}.t + \left(-\frac{u}{c^2\sqrt{1 - \frac{u^2}{c^2}}}\right)$$

$$= \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{\frac{ux}{c^2}}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$= \frac{t-\frac{ux}{c^2}}{\sqrt{1-\frac{u^2}{c^2}}} \qquad (2.3.13)$$

$$x' = \left(-\frac{u}{\sqrt{1-\frac{u^2}{c^2}}}\right).t + \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}.x$$

$$= \frac{x}{\sqrt{1-\frac{u^2}{c^2}}} - \frac{ut}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$= \frac{x-ut}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}} \qquad (2.3.14)$$

$$y' = y \qquad (2.3.15)$$

and $$z' = z \qquad (2.3.16)$$

Which are the required **Lorentz transformation** equations.

Also,

When $u \ll c$ then both $\frac{u}{c^2}$ and $\frac{u}{c}$ can be neglected. So the above Lorentz transformations becomes,

$$t' = t \qquad (2.3.17)$$

$$x' = x - ut \qquad (2.3.18)$$

$$y' = y \qquad (2.3.19)$$

$$z' = z \qquad (2.3.20)$$

Which are the required **Galilean transformation** equations.

"Only two things are infinite, the universe and human stupidity, and I am not sure about the former."

ALBERT EINSTEIN

CHAPTER 3

THE GENERAL THEORY OF RELATIVITY

3.1 INTRODUCTION

The special theory of relativity is applicable only to a special class of observers namely, to inertial observers. The Minkowskian space-time structure cannot be used to describe natural phenomena in a non-inertial frame. So there arises the necessity of extending the kinematical framework so that it applies generally. This extension should enable us to treat physics from any reference frame, inertial or non-inertial. Such a generalization of the theory is called general relativity.

General Relativity, a theory of gravity that describes the relationship between matter space-time, is the cornerstone of modern models of the universe. Cosmological models based on general relativity present up with two radically different alternatives for the nature of the universe and its fate.

3.2 THE PRINCIPLE OF EQUIVALANCE

In order to introduce the effects of gravitational field in the relativistic theory of gravitation, Einstein gave a principle known as the **Principle of Equivalence.**

A force which appears only due to acceleration of the non-inertial frame is called the inertial force. In order to introduce the effects of gravitational action Einstein pointed out that the inertial acceleration is similar to gravitational acceleration. By this analogy, **Einstein** gave the principle of equivalence which states, "***In the neighborhood of any given point it is absurd to distinguish between the gravitational field produced by the acceleration of masses and the field produced by accelerating an inertial frame of reference***". i.e. the gravitational mass and the initial mass both are same to each other.

In other words, "***A system which is stationary in a gravitational field of strength g is physically equivalent to a system which is in gravitational free space but accelerated in the opposite direction with an acceleration of g***" is known as the **Principle of Equivalence**.

According to the **Principle of Equivalence**, a uniform field can be replaced by a single system of reference. Hence all frames of reference become equally suitable for the description of physical laws. The principle of equivalence is true for electrical and optical phenomena as well. For example, if a ray of light follows a rectilinear path with respect an inertial frame of reference S', it is found that the same ray of light no longer follows the rectilinear path with respect to another system of reference S', which is an accelerated motion.

From this it follows that the rays of light in general, follow the curvilinear paths in gravitational fields. So, the **Principle Of Equivalence** is a very powerful tool in the general theory of relativity.

Mathematically,

If a_i is the inertial acceleration of the body due to application of inertial force F_i then by Newton's 2nd law of motion we have,

$$F_i = ma_i$$

Where m is the inertial mass of the body and may be written as $m_i$and then,

$$m_i = \frac{F_i}{a_i} \qquad (3.2.1)$$

Also, if g is the acceleration of a body in a gravitational field of attraction F_g, then

$$F_g = mg$$

Where m is the gravitational mass of the body and may be written as m_g and then,

$$m_g = \frac{F_g}{g} \qquad (3.2.2)$$

Since according to the principle of equivalence the inertial and gravitational forces are of the same nature and obey the same laws and by suitable choice of accelerated frame of reference a desired gravitational field can be produced, i.e. the **Principle Of Equivalence** implies,

$$\frac{F_i}{a_i} = \frac{F_g}{g}$$

$$i.e. \quad m_i = m_g \quad \text{[By using (3.2.1) \& (3.2.2)]}$$

Thus the **Principle Of Equivalence** necessarily implies the equality of the inertial and gravitational masses of the same body. One consequence of this equality of inertial and gravitational masses is that all bodies in the same gravitational field of force fall with acceleration. The equality of gravitational and inertial masses has been experimentally verified to a very high degree of accuracy by $E\ddot{o}tv\ddot{o}s$ in 1896, 1908 and recently by Dicke in 1962.

The preposition of equality of inertial and gravitational masses is sometimes itself known as the **Principle Of Equivalence**.

3.3 THE PRINCIPLE OF COVARIANCE

According to special theory of relativity the laws describing any phenomena in free space must be independent of the velocity of particular observer who makes measurements and must have the same form and contents; when referred to different sets of Cartesian axes which are in uniform relative translator motion. In the general theory, we make full use of general idea of relativity of all kinds of motion.

Here the laws must be expressible in a form which is independent of the particular space time co-ordinate chosen or in other words, ***laws of nature remain invariant with respect to any space time co-ordinate system***. This statement is called the **Principle Of General Covariance**.

So we must express all our laws by means of covariant equations that make no use of a particular co-ordinate system. This we do by use of tensor calculus, because the expression of a law by a tensor equation has an exactly the same form in all systems of co-ordinate. As we see the modified of the equation

$$ds^2 = -(dx^2 + dy^2 + dz^2) + c^2dt^2$$

In tensor form is:

$$ds^2 = g_{ij}dx^i dx^j \qquad (i, j = 1,2,3)$$

We see that the fundamental tensor g_{ij} is a covariant tensor of rank two obeying the transformation law,

$$g'_{ij} = \frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j}g_{ab}$$

Where the dashed quantities are used belong to the new co-ordinate system x'^i . Suppose the physical laws of nature in x^i co-ordinate system are represented by the equation,

$$A^i_j = B^i_j \qquad (3.3.1)$$

Then we can write the transformation law for this tensor as,

$$A'^i_j - B'^i_j = \frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^j}A^\alpha_\beta - \frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^j}B^\alpha_\beta$$

$$= \frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^j}\left(A^\alpha_\beta - B^\alpha_\beta\right)$$

$$= \frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^j}\left(A^\alpha_\beta - B^\alpha_\beta\right)\text{[By using (3.3.1)]}$$

$$= \frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^j}.0$$

$i.e.\quad A'^i_j - B'^i_j = 0$

$$\therefore \quad A'^i_j = B'^i_j \tag{3.3.2}$$

Thus we see that the tensorial quantities follow the general covariant laws.

3.4 GEODESIC CO-ORDINATES

The co-ordinate system x^i is called geodesic co-ordinate system with the pole P_0 if all the Christoffel symbols are zero at the point P_0, i.e. for geodesic co-ordinates,

$\Gamma_{l,jk} = \Gamma^l_{jk} = 0$, at the point P_0.

This indicates that at the pole of a geodesic co-ordinate system the first order covariant derivatives reduce to the corresponding ordinary derivatives.

For example, consider the covariant derivative of A^j_k with respect to x^l we get,

$$A^j_{k;l} = \frac{\partial A^j_k}{\partial x^l} + \Gamma^j_{ml}A^m_k - \Gamma^m_{kl}A^j_m$$

$$= \frac{\partial A^j_k}{\partial x^l} \text{ , at the pole } P_0 \text{ of geodesic coordinate System.}$$

Therefore, necessary and sufficient condition that a given co-ordinate system be geodesic with pole at P_0 .

We know,

$$\bar{\Gamma}^\lambda_{\mu\nu}\frac{\partial x^l}{\partial \bar{x}^\lambda} = \Gamma^l_{jk}\frac{\partial x^j}{\partial \bar{x}^\mu}\frac{\partial x^k}{\partial \bar{x}^\nu} + \frac{\partial^2 x^l}{\partial \bar{x}^\mu \partial \bar{x}^\nu} \tag{3.4.1}$$

Interchanging x and $\bar{x}$ co-ordinate systems, we get

$$\Gamma^{\lambda}_{\mu\nu}\frac{\partial \bar{x}^l}{\partial x^l} = \bar{\Gamma}^l_{jk}\frac{\partial x^j}{\partial \bar{x}^\mu}\frac{\partial x^k}{\partial \bar{x}^\nu} + \frac{\partial^2 x^l}{\partial \bar{x}^\mu \partial \bar{x}^\nu}$$

$$or, \qquad \frac{\partial^2 \bar{x}^l}{\partial x^\mu \partial x^\nu} - \Gamma^{\lambda}_{\mu\nu}\frac{\partial \bar{x}^l}{\partial x^\lambda} = -\bar{\Gamma}^l_{jk}\frac{\partial \bar{x}^j}{\partial x^\mu}\frac{\partial \bar{x}^k}{\partial x^\nu} \qquad (3.4.2)$$

For a given value of l, $\bar{x}^l$is a scalar function of x^μ and hence $\frac{\partial \bar{x}^l}{\partial x^\mu}$ is a covariant vector.

Let,

$$\frac{\partial \bar{x}^l}{\partial x^\mu} = \bar{x}^l_{;\mu} = A_\mu(say)$$

Hence equation (3.4.2) becomes,

$$\frac{\partial A_\mu}{\partial x^\nu} - \Gamma^{\lambda}_{\mu\nu}A_\lambda = -\bar{\Gamma}^l_{jk}\frac{\partial \bar{x}^j}{\partial x^\mu}\frac{\partial \bar{x}^k}{\partial x^\nu} \qquad (3.4.3)$$

But,

$$\frac{\partial A_\mu}{\partial x^\nu} - \Gamma^{\lambda}_{\mu\nu}A_\lambda = A_{\mu;\nu}$$

$$= \left(\bar{x}^l_{;\mu}\right)_{,\nu}$$

$$= \bar{x}^l_{;\mu\nu}$$

Hence equation (3.4.3) becomes,

$$\frac{\partial A_\mu}{\partial x^\nu} - \Gamma^{\lambda}_{\mu\nu}A_\lambda = -\bar{\Gamma}^l_{jk}\frac{\partial \bar{x}^j}{\partial x^\mu}\frac{\partial \bar{x}^k}{\partial x^\nu} \qquad (3.4.4)$$

But,

$$\frac{\partial A_\mu}{\partial x^\nu} - \Gamma^{\lambda}_{\mu\nu}A_\lambda = A_{\mu;\nu}$$

$$= \left(\bar{x}^l_{;\mu}\right)_{,\nu}$$

$$= \bar{x}^l_{;\mu\nu}$$

Hence equation (3.4.3) becomes,

$$\bar{x}^l_{;\mu\nu} = -\bar{\Gamma}^l_{jk}\frac{\partial \bar{x}^j}{\partial x^\mu}\frac{\partial \bar{x}^k}{\partial x^\nu} \qquad (3.4.5)$$

Now, let $\bar{x}^l$ be a geodesic co-ordinate system with the pole at P_0, then we get,

$$\bar{\Gamma}^l_{jk} = 0, \text{ at the pole } P_0,$$

Therefore equation (3.4.4) becomes,

$$\bar{x}^l_{;\mu\nu} = 0$$

This gives the necessary condition for a given co-ordinate system to be geodesic. Conversely suppose that,

$$\bar{x}^l_{;\mu\nu} = 0, \quad \text{at the pole } P_0.$$

Therefore equation (3.4.4) becomes,

$$\bar{\Gamma}^l_{jk} \frac{\partial \bar{x}^j}{\partial x^\mu} \frac{\partial \bar{x}^k}{\partial x^\nu} = 0, \qquad at\ the\ pole\ P_0 \tag{3.4.6}$$

As $\frac{\partial \bar{x}^j}{\partial x^\mu}$ and $\frac{\partial \bar{x}^k}{\partial x^\nu}$ are arbitrary then we have,

$$\bar{\Gamma}^l_{jk} = 0, at\ the\ pole\ P_0 \tag{3.4.7}$$

Which implies that $\bar{x}^l$ is a geodesic co-ordinate system with pole at P_0. Hence the necessary and sufficient condition that a given co-ordinate system be geodesic with the pole at P_0, is that their second covariant derivatives with respect to space co-ordinates must vanish at P_0 .

3.5 GEODESICS

Geodesics are curves in a manifold analogous to straight lines in Euclidean space. One way of characterizing a straight line is as the shortest curve between two points and this characterization could be used in a Riemannian manifold, where the length of a curve defined. It may be arises technical difficulties. To avoid this we adopt another characterization of a straight line, namely its straightness, as our guide to defining geodesics. We may characterize a curve in a manifold as being a geodesic, if there exist a parameterization of it such that the tangent vectors constitute a parallel field of vectors along the curve. Such a parameter is called **affine parameter**.

Let us consider, S be the length of the curve measured from the fixed point then the length ds of the element of arc joining the points x^p and $x^p + dx^p$ is given by,

$$ds^2 = g_{pq} dx^p dx^q \quad (3.5.1)$$

$$\therefore \quad ds = \left(g_{pq} dx^p dx^q\right)^{1/2} \quad (3.5.2)$$

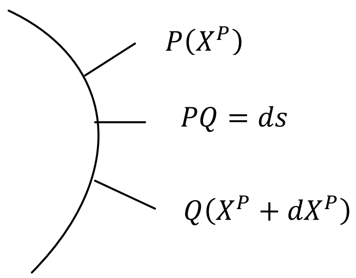

Fig: 3.5

Therefore,

$$\frac{ds}{dt} = \left(g_{pq} \frac{dx^p}{dt} \frac{dx^q}{dt}\right)^{1/2}$$

$$\therefore \; \dot{s} = \left(g_{pq} \frac{dx^p}{dt} \frac{dx^q}{dt}\right)^{1/2} \quad (3.5.3)$$

The length s of the arc joining the points which corresponds to the values t_0 and t_1 of the parameter is given by,

$$s = \int_{t_0}^{t} \left(g_{pq} \frac{dx^p}{dt} \frac{dx^q}{dt}\right)^{1/2} dt$$

$$= \int_{t_0}^{t} \varphi dt \quad (3.5.4)$$

Where

$$\varphi = \left(g_{pq} \frac{dx^p}{dt} \frac{dx^q}{dt}\right)^{1/2} \quad (3.5.5)$$

If the integral, $I = \int_{t_0}^{t} \varphi dt$ be the extremum then φ satisfies **Euler's Equation**.

i.e.

$$\frac{\partial\varphi}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial\varphi}{\partial\dot{x}^k}\right) = 0 \tag{3.5.6}$$

From equation (3.5.5) we have,

$$\varphi = \left(g_{pq}\frac{dx^p}{dt}\frac{dx^q}{dt}\right)^{1/2} = \dot{s}$$

$$or, \qquad \varphi = \left(g_{pq}\dot{x}^p\dot{x}^q\right)^{1/2} = \dot{s}$$

Therefore,

$$\frac{\partial\varphi}{\partial x^k} = \frac{1}{2}\left(g_{pq}\dot{x}^p\dot{x}^q\right)^{-1/2}\frac{\partial g_{pq}}{\partial x^q}\dot{x}^p\dot{x}^q$$

$$= \frac{1}{2\left(g_{pq}\dot{x}^p\dot{x}^q\right)^{1/2}}\frac{\partial g_{pq}}{\partial x^k}\dot{x}^p\dot{x}^q$$

$$= \frac{1}{2\dot{s}}\frac{\partial g_{pq}}{\partial x^q}\dot{x}^p\dot{x}^q \tag{3.5.7}$$

Again,

$$\frac{\partial\varphi}{\partial x^k} = \frac{1}{2}\left(g_{pq}\dot{x}^p\dot{x}^q\right)^{-1/2}\frac{\partial}{\partial\dot{x}^k}\left(g_{pq}\dot{x}^p\dot{x}^q\right)$$

$$= \frac{1}{2\left(g_{pq}\dot{x}^p\dot{x}^q\right)^{1/2}}\left[\left(g_{pq}\frac{\partial\dot{x}^p}{\partial\dot{x}^k}\right)\dot{x}^q + \left(g_{pq}\frac{\partial\dot{x}^q}{\partial\dot{x}^k}\right)\dot{x}^p\right]$$

$$= \frac{1}{2\dot{s}}\left[\left(g_{pq}\delta_k^p\right)\dot{x}^q + \left(g_{pq}\delta_k^q\right)\dot{x}^p\right]$$

$$= \frac{1}{2\dot{s}}\left[g_{kq}\dot{x}^p + g_{pk}\dot{x}^p\right]$$

$$= \frac{1}{2\dot{s}}\left[g_{pk}\dot{x}^p + g_{pk}\dot{x}^p\right][\text{since p \& } q \text{ } are \text{ } arbitrary]$$

$$= \frac{1}{\dot{s}}g_{pq}\dot{x}^p \tag{3.5.8}$$

Now, putting the above values of equation (3.5.7) & equation (3.5.8) in equation (3.5.6), we get,

$$\frac{1}{2\dot{s}}\frac{\partial g_{pq}}{\partial x^k}\dot{x}^p\dot{x}^q - \frac{d}{dt}\left(\frac{1}{\dot{s}}g_{pk}\dot{x}^p\right) = 0$$

$$or, \quad \frac{1}{2\dot{s}}\frac{\partial g_{pq}}{\partial x^k}\dot{x}^p\dot{x}^q + \frac{1}{\dot{s}^2}\frac{d\dot{s}}{dt}g_{pk}\dot{x}^p - \frac{1}{\dot{s}}\frac{\partial g_{pk}}{\partial x^q}\frac{dx^q}{dt}\dot{x}^p - \frac{1}{\dot{s}}g_{pk}\frac{d\dot{x}^q}{dt} = 0$$

$$or, \quad \frac{1}{2\dot{s}}\frac{\partial g_{pq}}{\partial x^k}\dot{x}^p\dot{x}^q + \frac{1}{\dot{s}^2}\ddot{s}g_{pk}\dot{x}^p - \frac{1}{\dot{s}}\frac{\partial g_{pk}}{\partial x^q}\dot{x}^q\dot{x}^p - \frac{1}{\dot{s}}g_{pk}\ddot{x}^p = 0$$

$$or, \quad \frac{1}{2}\frac{\partial g_{pk}}{\partial x^k}\dot{x}^p\dot{x}^q + \frac{\ddot{s}}{\dot{s}}g_{pk}\dot{x}^p - \frac{\partial g_{pk}}{\partial x^q}\dot{x}^q\dot{x}^p - g_{pk}\ddot{x}^p = 0$$

$$or, \quad g_{pk}\ddot{x}^p + \frac{\partial g_{pk}}{\partial x^q}\dot{x}^p\dot{x}^q - \frac{1}{2}\frac{\partial g_{pq}}{\partial x^k}\dot{x}^p\dot{x}^q = \frac{\ddot{s}}{\dot{s}}g_{pk}\dot{x}^p \qquad (3.5.9)$$

Now multiplying both sides of equation (3.5.9) by g^{rk} we have,

$$or, g^{rk}g_{pk}\ddot{x}^p + \frac{1}{2}g^{rk}\left(\frac{\partial g_{pk}}{\partial x^q} + \frac{\partial g_{qk}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^k}\right)\dot{x}^p\dot{x}^q = g^{rk}g_{pk}\dot{x}^p\frac{\ddot{s}}{\dot{s}}$$

$$or, \quad \delta^r_p\ddot{x}^p + g^{rk}[pq,k]\dot{x}^p\dot{x}^q = \delta^r_p\dot{x}^p\frac{\ddot{s}}{\dot{s}}$$

$$or, \quad \ddot{x}^p + \Gamma^r_{pq}\dot{x}^p\dot{x}^q = \dot{x}^r\frac{\ddot{s}}{\dot{s}} \qquad (3.5.10)$$

Now, choosing $s = t$ so that $\dot{s} = t$ and $\ddot{s} = 0$, then equation (3.5.10) becomes,

$$\ddot{x}^p + \Gamma^r_{pq}\dot{x}^p\dot{x}^q = 0$$

$$or, \quad \frac{d^2x^r}{ds^2} + \Gamma^r_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds} = 0 \qquad (3.5.11)$$

***This is the equation for a geodesic*.**

Thus a geodesic can be defined in an affine space and the parameter S is affine parameter.

Now, multiplying the above equation (3.5.11) by $g_{rm}\frac{dx^m}{ds}$ we have,

$$g_{rm}\frac{dx^m}{ds}\frac{d^2x^r}{ds} + \Gamma^r_{pq}g_{rm}\frac{dx^m}{ds}\frac{dx^p}{ds}\frac{dx^q}{ds} = 0 \qquad (3.5.12)$$

But from the metric tensor,

$$g_{rm}\Gamma^r_{pq} = g_{rm}\frac{1}{2}g^{rn}\left(g_{pn,q} + g_{qn,p} - g_{pq,n}\right)$$

$$= \frac{1}{2}\delta_m^n\left(g_{pn,q} + g_{qn,p} - g_{pq,n}\right)$$

$$= \frac{1}{2}\left(g_{pm,q} + g_{qm,p} - g_{pq,m}\right)$$

$$= \frac{1}{2}\left(g_{mp,q} + g_{mq,p} - g_{pq,m}\right) \qquad (3.5.13)$$

So, the equation (3.5.12) becomes, by using (3.5.13),

$$g_{rm}\frac{dx^m}{ds}\frac{d^2x^r}{ds^2} + \frac{1}{2}\left(g_{mp,q} + g_{mq,p} - g_{pq,m}\right)\frac{dx^m}{ds}\frac{dx^p}{ds}\frac{dx^q}{ds} = 0 \qquad (3.5.14)$$

But,

$$g_{mp,q}\frac{dx^p}{ds}\frac{dx^q}{ds} = g_{mq,p}\frac{dx^p}{ds}\frac{dx^q}{ds}$$

Then the above equation (3.5.14) becomes,

$$g_{rm}\frac{dx^m}{ds}\frac{d^2x^r}{ds^2} + \frac{1}{2}\left(2g_{mp,q} - g_{pq,m}\right)\frac{dx^m}{ds}\frac{dx^p}{ds}\frac{dx^q}{ds} = 0 \qquad (3.5.15)$$

Also,

$$g_{mp,q}\frac{dx^q}{ds} = \frac{\partial}{\partial x^q}\left(g_{mp}\right)\frac{dx^q}{ds} = \frac{d}{ds}\left(g_{mp}\right)$$

And

$$g_{pq,m}\frac{dx^m}{ds} = \frac{\partial}{\partial x^m}\left(g_{pq}\right)\frac{dx^m}{ds} = \frac{d}{ds}\left(g_{pq}\right)$$

So, the equation (3.5.15) becomes,

$$g_{rm}\frac{dx^m}{ds}\frac{d^2x^r}{ds^2} + \frac{d}{ds}\left(g_{mp}\right)\frac{dx^p}{ds}\frac{dx^m}{ds} - \frac{1}{2}\frac{d}{ds}\left(g_{pq}\right)\frac{dx^p}{ds}\frac{dx^q}{ds} = 0$$

$$or, \qquad g_{rm}\frac{dx^m}{ds}\frac{d^2x^r}{ds} + \frac{d}{ds}\left(g_{mp}\right)\frac{dx^p}{ds}\frac{dx^m}{ds} - \frac{1}{2}\frac{d}{ds}\left(g_{pm}\right)\frac{dx^p}{ds}\frac{dx^m}{ds} = 0$$

$$or, \quad g_{rm}\frac{dx^m}{ds}\frac{dx^r}{ds} + \frac{1}{2}\frac{d}{ds}(g_{pm})\frac{dx^p}{ds}\frac{dx^m}{ds} = 0$$

$$or, \quad g_{rm}\frac{dx^m}{ds}\frac{d^2x^r}{ds^2} + \frac{1}{2}\frac{d}{ds}(g_{rm})\frac{dx^r}{ds}\frac{dx^m}{ds} = 0$$

$$or, \quad 2g_{rm}\frac{dx^m}{ds}\frac{d^2x^r}{ds^2} + \frac{d}{ds}(g_{rm})\frac{dx^r}{ds}\frac{dx^m}{ds} = 0$$

$$or, \quad \frac{d}{dx}\left(g_{rm}\frac{dx^r}{ds}\frac{dx^m}{ds}\right) = 0$$

$$\therefore \quad \left(g_{rm}\frac{dx^r}{ds}\frac{dx^m}{ds}\right) = constant \qquad (By\ integrating) \qquad (3.5.16)$$

Although equation (3.5.16) can be derived from the equation (3.5.11) but later cannot be derived from the former.

For a class of geodesics, equation (3.5.16) is not true and these are the null geodesics which have the property the distance the between the adjacent points variables i.e. $ds = 0$. In this case we cannot parameterize the curve by parameter S (which is always zero) but have to use a distinct parameter λ.

$$i.e.\ x^r = x^r(\lambda)$$

$$\therefore g_{rp}\frac{dx^r}{d\lambda}\frac{dx^p}{d\lambda} = 0$$

Here, $\frac{dx^r}{d\lambda}$ is the tangent vector whose magnitude is zero. For non-zero geodesics one can show that the tangent vector $\frac{dx^r}{d\lambda}$ satisfies,

$$\frac{d^2x^r}{d\lambda^2} + \Gamma^r_{pq}\frac{dx^p}{d\lambda}\frac{dx^q}{d\lambda} = 0 \qquad (3.5.17)$$

which is known as **null geodesic**.

3.6 GEOMETRY OR GEOMETRICS

The geometry that we have studies in schools is known by the name of **Euclidean geometry** after the great mathematician **Euclid** (3rd century B.C.) who collected the known geometrical Knowledge of his day and arranged it in a logical sequence of axioms and theorems. His axioms were like self obvious truths. One and only one straight line passes through two given points or all right angles are equal. These are examples of his self-obvious axioms. But then he introduced one axiom which could not be classified as self-obvious. This has come to be known as the parallel postulate. It is clear that, this is not at all "**self-obvious**" and Euclid himself hesitated a great deal before accepting it as an unproved assumption. We may note that several well-known theorems of our school geometry are based on the validity of this axiom, e.g. the theorem about the sum of the three angles of a triangle being two right angles.

The hesitation which Euclid experienced in accepting the parallel postulate as an unproved assumption troubled later mathematicians for more than 1500 yrs. Geometer after geometer attempted to prove this postulate on the basis of the other axioms of Euclid but with no success. Ultimately, a Russian mathematician, **Lobachevsky** in 1829, first conceived the idea that, it may be possible to prove that the parallel postulate cannot be proved! And proved he succeeded in doing so. He replaced Euclid's postulate by the following. Given a straight line and an outside point, two straight lines can be drawn parallel to the given line and pass through the given point. He did not find any logical flaw following from this assumption and thus, developed a perfectly logical geometry known as "**Lobachevskian geometry**", where the sum of three angles of a triangle is always less than two right angles.

About 25 years later, Riemann developed geometry. He changed Euclid's postulates about straight line and replaced the parallel postulate by the following. Given a straight line and a point outside it, no straight line can be drawn parallel to the given straight line and pass through the concept of parallel straight lines is absent. The geometry is known as "**Riemannian Geometry**". In this geometry, the sum of three angles of a triangle is always greater than two right angles.

3.7 THE METRIC TENSOR AND CONNECTION

One desires to have a notion of distance between any two infinitesimally separated points of a space-time manifold. Such distances should locally reduce to those defined by the special theory of relativity that is, those given by a flat metric with an indefinite signature on the Minkowski space-time. This is because special relativity is the theory which has been shown to be valid by experiments and hence must hold at least when confined to local regions in the space-time which correspond to the measurements of space and time intervals at the laboratory scale for an observer. Thus, the distances between events in a space-time need not necessarily be positive.

This is achieved by assuming the existence of an indefinite metric tensor field defined globally on M as a (0,2) type, symmetric, tensor field. Thus, the metric tensor must act on pairs of vectors to produce a number and it is symmetric in its indices. Choosing a coordinate basis this can be written as

$$g \equiv g_{ij} dx^i \otimes dx^j, \qquad (3.7.1)$$

Where $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j)$. If V and W are any two vectors, this gives $g(V,W) = g_{ij} V^i W^j$. This is often written conventionally in the form of an expression giving the distance between two infinitesimally separated points in space-time as

$$ds^2 = g_{ij} dx^i dx^j \qquad (3.7.2)$$

For a single vector V, $g(V,V)$ gives the magnitude of V, which is $g_{ij} V^i V^j$.

Another property assumed for the metric tensor is that it is non-degenerate, that is, there is no non-zero vector $V \neq 0$ such that $g(V,W) = 0$ for all vectors $W \in T_p$. This amounts to saying that the matrix $[g_{ij}]$ is non-singular and hence there must be an inverse matrix g^{ij} such that

$$g^{ij} g_{jk} = \delta^i_k.$$

Hence, the tensors g^{ij} and g_{ij} provide an isomorphism or a unique correspondence between the space of covariant and contra-variant vectors in the following sense:

$$X_i = g_{ij} X^j, \qquad X^i = g^{ij} X_j.$$

Similarly, we can also write for a second rank tensor T,

$$T^i_j = g^{ik} T_{kj}, \;\; T^j_i = g^{jk} T_{ki}, \;\; T^{ij} = g^{ik} g^{jl} T_{kl}.$$

In particular, we have

$$g^{ik}g_{km} = g^i_m = \delta^i_m,$$

and the Kronecker delta δ^i_m transform as component of a tensor. Thus, δ^i_m and g^i_m are identical tensors.

The tensors T^i_j, T^j_i or T^{ij} are to be regarded as representations of the same geometric object because these are uniquely associated tensors. Such an isomorphism between the covariant and contra-variant arguments is essentially equivalent to the procedure of 'raising' and 'lowering' of indices as pointed out above. In fact, the multi-linear map

$$g: T_p \times T_p \to \mathbb{R}$$

can also be viewed as a linear correspondence from T_p to T_p^* in the sense of the mapping $V \to g(.,V)$. The non-degeneracy of the metric tensor implies that this map is one-one and onto and thus g establishes a one-one correspondence between vectors the dual vectors. The components $V_i = g_{ij}V^j$are the one-form components uniquely associated with the vector components V^j.

Suppose M is an n-dimensional manifold with g being the metric tensor defined on it. Then, at any $p \in M$ one could always choose an orthonormal basis $\{e_i\}$ such that the metric components $g_{ij}s$ have the diagonal form

$$g_{ij} = diag(+1, \dots, +1, -1, \dots - 1).$$

If the metric has the form $g_{ij} = (+1, \dots, +1)$ then it is called **positive definite**. In that case, $g(X,X) = 0$ implies $X = 0$. On the other hand, it is called a Lorentzian metric if the form is

$$g_{ij} = diag(+1, \dots, +1, -1). \tag{3.7.3}$$

Where there are $(n-1)$ terms with +1. This is an indefinite metric in the sense that the magnitude of a non-zero vector could be either positive, negative or zero. Then $x \in T_p$is called **time-like, null, or space-like**, depending on

$$g(X,X) < 0\,, \quad g(X,X) = 0, \qquad g(X,X) > 0 \tag{3.7.4}$$

An indefinite metric divides the vectors in T_p into three disjoint classes, namely the time-like, null, and space-like vectors. The null vectors form a cone in the tangent space T_p which separates the time like vectors from the space like vectors (**Fig. 3.2**).

When the manifold has dimension four, and when it is equipped with a globally defined **Lorentzian** metric tensor field, it is called a **space-time manifold**.

The signature of the metric tensor is defined as the number of its positive eigen values minus the number of negative eigen values. Thus, a space-time is a four-dimensional differentiable manifold with a Lorentzian metric globally defined which has the signature +2.

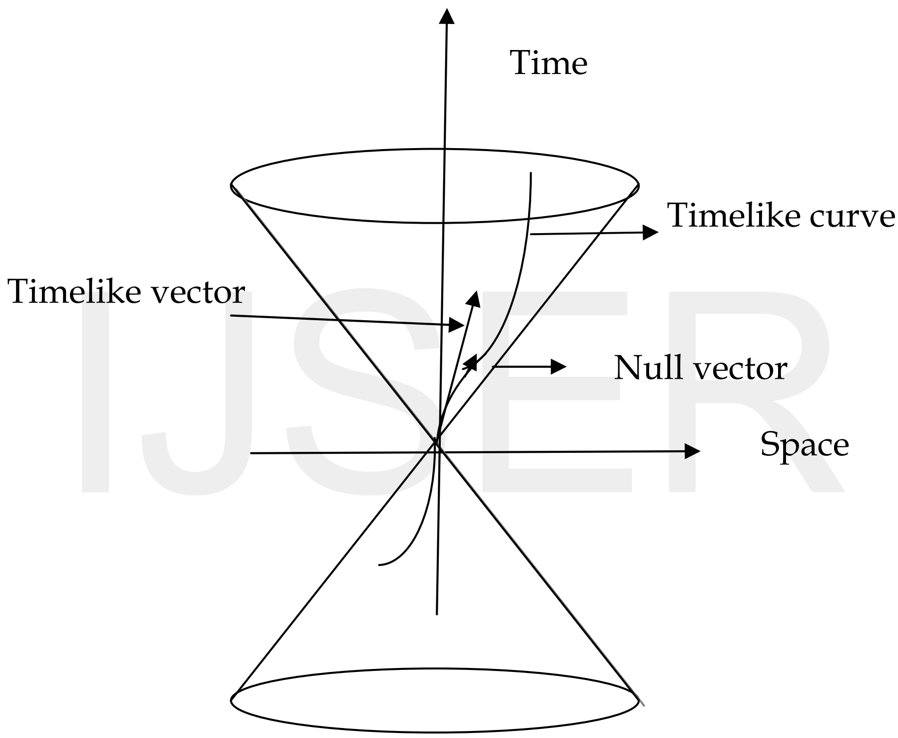

Fig 3.2 : The null cone at a point p in the space-time manifold. The tangent to curve γ is time like at all point, which is thus a time like curve.

In fact, in the special theory of relativity, the space-time admits a global coordinate frame covering the entire manifold so that the metric has the form given by $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ globally, and the metric coefficients are constants throughout the manifold, which is called **Minkowski space-time**. The tangent vector for a particle travelling with a constant velocity less then that of light through a point p in such a space-time is represented by a time like vector at p. The particle must travel within the future light cone at p which satisfies the equation $(X, X) = 0$.

This equation gives the set of all null vectors at p representing the photon paths. Now, according to the special theory of relativity, no material particles and signals could travel at a velocity more than that of light. Thus, the metric determines the causal structure of space-time in the sense that an event p is causally related to another event q if and only if there is a time like or null signal between p and q. All such events lie on or within the double cone at p which is defined by the metric tensor in the above manner.

For a non-flat space-time continuum of the general theory of relativity, the metric coefficients are functions of the space-time coordinates and one has to solve for the metric as a solution of the Einstein field equations. As far as the existence of a Lorentz metric on a space-time is concerned, any c^r paracompact manifold will admit a c^{r-1}Lorentz metric if and only if it admits a non-vanishing c^{r-1} line element field, which is an assignment of a pair of equal and opposite vectors $(V, -V)$ globally on M at each point [14].

Such a line element field is always defined for a non-compact manifold and hence a Lorentz metric always exists for the same. For the reasons, we always take the space-time to be non-compact and without boundary.

Let (M, g) be a space-time γ be a continuous c^1 curve in M. Then γ is called a **time-like, null, or space-like curve** respectively if the tangent vector to γ is time like, null, or space like respectively at all points of γ. A curve which is either time like or null is also sometimes called a **non-space like curve**. The tangent space magnitudes defined by g, namely,

$$X \rightarrow |g(X, X)|^{1/2} dt,$$

can be related to the magnitudes or distances on the manifold as below. Suppose X is the tangent vector along γ such that $g(X, X)$ has the same sign at all points of $\gamma(t)$. Then the arc length between $p = \gamma(t_1)$ and $q = \gamma(t_2)$ along the curve is given by

$$L(\gamma) = s = \int_a^b (|g(X, X)|)^{1/2} dt \qquad (3.7.5)$$

The above as well as the relation (3.7.1) are equivalent to the expression $ds^2 = g_{ij} dx^i dx^j$, which represents the infinitesimal arc length along γ.In Euclidean spaces one has the notion of parallel transport of any given vector X defined by the condition that in going from a point p to another point q, both the magnitude and the direction of X must not change. If along some curve the magnitude and direction of the tangent vector remain unchanged, such a curve is called a straight line along which the tangent vector is parallel transported.

In Euclidean space, if a vector is parallel transported from a point p to another point q along two different curves, the result will be the same, independent of the path taken. However, this will not be case for a general affine manifold (**Fig 3.3**).

For a general differentiable manifold, such a notion of parallel transport of vectors is defined by introducing the concept of a connection on M. Let X be a vector field on M. First we introduce the notion of a derivative operator ∇_X on M which gives the rate of change of vectors or tensor fields along the given vector field X at p for all points of M.

If Y is another vector field at p then the operator ∇_X maps Y into a new vector field $Y \to \nabla_X Y$ such that the following conditions are satisfied:

(1) $\nabla_X(\alpha Y + \beta Z) = \alpha \nabla_X Y + \beta \nabla_X Z; \qquad \alpha, \beta \epsilon \mathbb{R}$

(2) $\nabla_{fX+gZ} Z = f \nabla_X Z + g \nabla_Y Z;$ for real functions f and g

(3) $\nabla_X(fY) = f \nabla_X Y + Y X(f)$

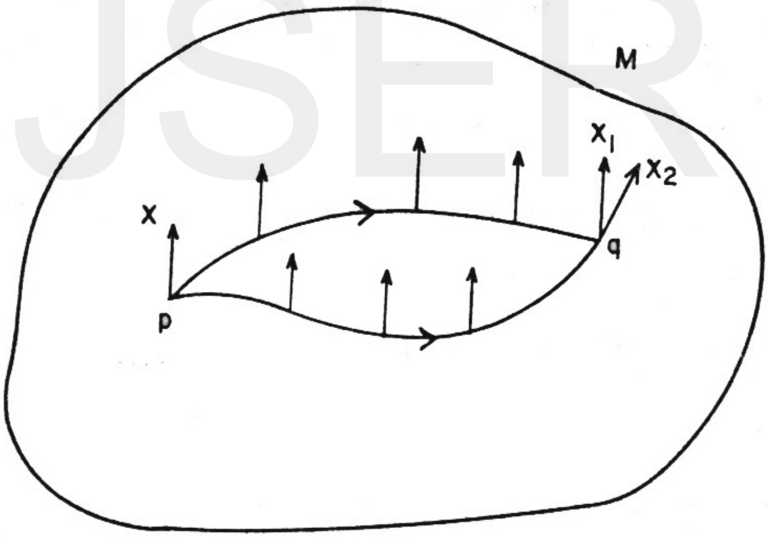

Fig 3.3: In a differentiable manifold, the result of parallel transport of a vector along a curve from point p to q in general depends on the path taken.

A connection ∇ at a point $p \in M$ is a rule which assigns to each vector field X at p a differential operator ∇_X which maps an arbitrary C^r vector field Y ay p into a vector field $\nabla_X Y$ such that the conditions (1), (2), and (3) are satisfied. Thus, ∇Y, called the covariant derivative of Y is defined as a type (1,1) tensor field which gives a vector$\nabla_X Y$ when contracted with the vector X. In such a case, the condition (3) above implies

$$\nabla(fY) = df\otimes Y + f\nabla Y \tag{3.7.6}$$

A C^r connection ∇ on a C^k manifold($k \geq r + 2$)is a rule assigning a connection ∇ to each $p \in M$ such that if Y is a C^{r+1} vector field, then ∇Y is a C^r tensor field of type (1,1). We can write

$$\nabla Y = Y^i_{;j} e^j \otimes e_i \tag{3.7.7}$$

Here $Y^i_{;j}$ is often called the covariant derivative of the vector Y^i. This is completely defined by the n^3connection coefficients Γ^i_{jk} which are defined in the following manner by choosing the vector fields X and Y to be the basis vector fields:

$$\nabla_{e_j} e_k \equiv \Gamma^i_{jk} e_i \tag{3.7.8}$$

It is not difficult to see that the above is equivalent to the condition

$$\langle e^i, \nabla_{e_j} e_k \rangle = \Gamma^i_{jk} \tag{3.7.9}$$

Thus, in a coordinate basis we have

$$\langle dx^i, \nabla_{\partial/\partial x^j}\left(\frac{\partial}{\partial x^k}\right)\rangle = \Gamma^i_{jk}.$$

Consider now the vector $\nabla_X Y$ .Defining

$$\nabla_{\partial/\partial x^i} Y \equiv \nabla_i Y \,,$$

using the rules defining the connection given above, and the relation

$$X(f) = X^i \frac{\partial}{\partial x^i}(f) = X^i \frac{\partial f}{\partial x^i} \;, \tag{3.7.10}$$

we obtain

$$\nabla_X Y = X^i \left(\frac{\partial Y^k}{\partial x^i} + \Gamma^k_{ij} Y^j\right)\left(\frac{\partial}{\partial x^k}\right) \tag{3.7.11}$$

Comparing this with equation (3.7.7), we can write

$$\nabla_X Y = Y^k_{;i} X^i \left(\frac{\partial}{\partial X^k}\right)$$

where

$$Y^k_{;i} \equiv \frac{\partial Y^k}{\partial x^i} + \Gamma^k_{ij} Y^j \qquad (3.7.12)$$

It can be seen that the components of the vector $\nabla_X Y$ are given as $Y^k_{;i} X^i$.

Let us define

$$Y^i_{,j} \equiv \frac{\partial Y^i}{\partial x^j}$$

Then, taking the transformation of coordinates $\{x^i\} \rightarrow \{x^{i'}\}$ when the basis vectors transform as $e_i \rightarrow e_{i'}$, it can be seen that $Y^i_{,j}$ does not transform like the components of a tensor.

Similarly, consider the connection coefficients in the new coordinate system, which are given by

$$\Gamma^{k'}_{i'j'} = \langle e^{k'}, \nabla_{e_{i'}} e_{j'} \rangle.$$

Transforming the dashed vectors to the original coordinate system and using the condition (2) and (3) above gives in a coordinates basis,

$$\Gamma^{k'}_{i'j'} = \frac{\partial x^{k'}}{\partial x^k} \left(\frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \Gamma^k_{ij} + \frac{\partial^2 x^k}{\partial x^{i'} \partial x^{j'}} \right) \qquad (3.7.13)$$

It follows that because of the presence of the second derivative terms in the above, the coefficients Γ^i_{kj} also do not transform like the components of a tensor. Consider, however,

$$\nabla_X Y = \left(Y^i_{;j} X^j\right) \left(\frac{\partial}{\partial x^i}\right) = \left(Y^{i'}_{;j'} X^{j'}\right) \left(\frac{\partial}{\partial x^{i'}}\right)$$

which implies

$$Y^i_{;j} X^j = Y^{i'}_{;j'} \frac{\partial x^{j'}}{\partial x^i} \frac{\partial x^i}{\partial x^{i'}} X^j \qquad (3.7.14)$$

Since the above is true for any arbitrary vector X^j, it follows that $Y^i_{;j}$ are components of a tensor.

Further, if Γ^i_{jk} and $\bar{\Gamma}^i_{jk}$ are components of two different connections on *M*, then it is not difficult to see, using the coordinate transformations, that the quantities

$$\mathrm{C}^i_{jk} = \bar{\Gamma}^i_{jk} - \Gamma^i_{jk}$$

are components of a tensor.

Given a connection ∇ on M, the torsion tensor T is defined by the relation

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{3.7.15}$$

Writing the components,

$$T(X,Y) = \left(\Gamma^i_{jk} - \Gamma^i_{jk}\right) X^j Y^k e_i.$$

This is a type (1, 2) tensor which has the components

$$\mathrm{T}^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{jk}.$$

A connection is called symmetric when the torsion tensor vanishes, that is,

$$\Gamma^i_{jk} = \Gamma^i_{kj}$$

$$or, [X,Y] = \nabla_X Y - \nabla_Y X$$

We shall always work with symmetric connections and assume the torsion tensor to be vanishing.

The notion of connection can be generalized to arbitrary tensor fields to obtain a tensor $\nabla_X T$ Of type(r,s) for any given tensor T of (r,s) type by assuming first that ∇ is linear and obeys the **Leibnitz rule**. That is,

$$\nabla_X(\alpha S + \beta T) = \alpha \nabla_X S + \beta \nabla_X T\,; \qquad \alpha, \beta \in \mathbb{R}$$

And

$$\nabla_X(S \otimes T) = \nabla_X S \otimes T + S \otimes \nabla_X T$$

for any vector field X .

Further, ∇ must agree with our usual notion of directional derivative, that is,

$$\nabla_X f = \langle df, X \rangle = Xf = X^i \frac{\partial f}{\partial X^i} \tag{3.7.16}$$

Finally, ∇ must commute with contractions, that is,

$$(\nabla_a T)^{i_1 \dots l \dots i_r}_{j_1 \dots l \dots j_s} = \nabla_a T^{i_1 \dots l \dots i_r}_{j_1 \dots l \dots j_s} \tag{3.7.17}$$

As earlier, we can write

$$\nabla_X T = T^{i_1 \dots i_r}_{j_1 \dots j_s;a} X^a e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s}, \tag{3.7.18}$$

With

$$\nabla_X T^{i_1 \dots i_r}_{j_1 \dots j_s} = T^{i_1 \dots i_r}_{j_1 \dots j_s;a} X^a.$$

Now, by considering the expansion for $\nabla_i(e_j \otimes e^k)$ it is not difficult to see that

$$\nabla_a e^i = -\Gamma^i_{ac} e_c, \tag{3.7.19}$$

and if ω is a one-form then,

$$\nabla_{e_j} \omega = \omega_{k;j} e^k,$$

with

$$\omega_{k;j} \equiv \frac{\partial \omega_k}{\partial x^j} - \Gamma^i_{jk} \omega_i \tag{3.7.20}$$

In general, we can write for the covariant derivative of a tensor T ,

$$T^{i_1 \dots i_r}_{j_1 \dots j_s;a} = \frac{\partial T^{i_1 \dots i_r}_{j_1 \dots j_s}}{\partial x^a} + \sum_m \Gamma^{i_m}_{he} T^{i_1 \dots e \dots i_r}_{j_1 \dots j_s} - \sum_n \Gamma^e_{h j_n} T^{i_1 \dots i_r}_{j_1 \dots e \dots j_s}. \tag{3.7.21}$$

Finally, we note that given a Lorentzian metric tensor on M, the condition $\nabla_X g = 0$ defines a unique torsion-free connection on M . Then,

$$(\nabla_X g)_{ij} = g_{ij;k} X^k = 0 \tag{3.7.22}$$

which implies that

$$g_{ij;k} = 0.$$

In such a case, that parallel transport of vectors must preserve the scalar product defined by the metric tensor g and the connection coefficients Γ^i_{jk} are determined in terms of the first derivatives of the metric components.

Since all the information on space-time structure is supposed to be contained in the ten metric functions g_{ij}, this is reasonable to expect.

One way to see this is the following. Using equation (3.7.21), we can write for the covariant derivative of the metric,

$$g_{ij;k} = \frac{\partial g_{ij}}{\partial x^k} - \Gamma^m_{il} g_{mj} - \Gamma^m_{jl} g_{mi}$$

Now, using the condition $g_{ij;k} = 0$ and defining

$$g_{mj}\Gamma^m_{il} = \Gamma_{jil},$$

the above can be written as

$$g_{ij;k} \equiv \frac{\partial g_{ij}}{\partial x^k} = \Gamma_{jil} + \Gamma_{ijl}.$$

Using the above and the symmetry property of the connection, we get

$$\Gamma_{jil} = \frac{1}{2}\left(g_{ji;m} - g_{mj;i} + g_{im;j}\right).$$

This result can be seen by specializing to the frame of free fall as well. In such a frame, all the connection coefficients vanish and the metric is locally that of the special theory of relativity. Then, $g_{ij} = \eta_{ij}$ and the partial derivatives of g_{ij} vanish. Thus, from the above equation, for $g_{ij;k}$, we again recover $g_{ij;k} = 0$ .This, being a tensor equation, must hold in all frames in general and we can again proceed as earlier.

3.8 SPACETIME CURVATURE

The measure of the curvature for any given space-time is exhibited in the non-commutation of the tangent vectors when parallel transported along different curves to arrive at the same space-time point. This is given by the Riemann curvature tensor which is defined as a type (1,3) tensor,

$$R: T_p^* \times T_p \times T_p \times T_p \to \mathbb{R}\,.$$

In a coordinate basis one could write the Riemann tensor as

$$R = R^i_{jkl} e_i \otimes e^j \otimes e^k \otimes e^l \qquad (3.8.1)$$

If we define the vector $R(X,Y)Z$ as

$$R(X,Y)Z \equiv \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z \qquad (3.8.2)$$

then the components of the Riemann tensor are given by

$$R^i_{jkl} = \langle e^i, R(e_k, e_l) e_j \rangle \qquad (3.8.3)$$

Working out the components gives

$$R(X,Y)Z = R^i_{jkl} \frac{\partial}{\partial x^i} X^k Y^l Z^j \tag{3.8.4}$$

Now, in order to evaluate equation (3.8.3), note that

$$[\nabla_X(\nabla_Y Z)]^i = \nabla_X\left(Z^i_{;j}Y^j\right) = Z^i_{;jk}Y^jX^k + Z^i_{;j}Y^j_{;k}X^k \tag{3.8.5}$$

Similarly we have

$$[\nabla_Y(\nabla_X Z)]^i = Z^i_{;jk}X^jY^k + Z^i_{;j}X^j_{;k}Y^k \tag{3.8.6}$$

Finally, we have

$$-\nabla_{[X,Y]}Z = -\nabla_{\left(Y^i_{;j}X^j - X^i_{;j}Y^j\right)(\partial/\partial x^i)}Z = -Z^k_l Y^l_{;j}X^j + Z^k_l X^l_{;j}Y^j \tag{3.8.7}$$

Combining equations (3.8.5), (3.8.6) and equation (3.8.7) we obtain

$$R(X,Y)Z = \left(Z^i_{;lk} - Z^i_{;kl}\right)X^kY^l \tag{3.8.8}$$

Comparing equation (3.8.4) and equation (3.8.8) gives

$$Z^i_{;lk} - Z^i_{;kl} = R^i_{jkl}Z^j, \tag{3.8.9}$$

which is the same as

$$\nabla_k\nabla_l Z^i - \nabla_l\nabla_k Z^i = R^i_{jkl}Z^j \tag{3.8.10}$$

The last equation above could also be taken as the defining equation for the components of the curvature tensor. As shown by the left-hand side of equation (3.8.10), the Riemann curvature tensor provides the measure of non-communication of a tangent vector when parallel transported along different curves to arrive at the same space-time point.

In place of the vectors X, Y,and Z let us choose now the basis vectors $e_i s$.

Then,

$$\nabla_{e_j}\nabla_{e_k}e_l = \nabla_{e_j}(\Gamma^a_{kl}e_a) = e_j(\Gamma^a_{kl})e_a + \Gamma^a_{kl}\Gamma^h_{ja}e_h \tag{3.8.11}$$

Consider now the definition of the components of the Riemann tensor as given by equation (3.8.3).

In particular, if a coordinate basis is chosen then $\left[e_i, e_j\right] = 0$ and we can write

$$R^i_{jkl} = \langle e^i, \nabla_{e_k}\nabla_{e_l}e_j\rangle - \langle e^i, \nabla_{e_l}\nabla_{e_k}e_j\rangle$$

Then, using equation (3.8.5) and a coordinate basis, the coordinate components of the Riemann curvature tensor can be given in terms of the coordinate components of the connection as

$$R^i_{jkl} = \frac{\partial \Gamma^i_{lj}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^k} + \Gamma^i_{ka}\Gamma^a_{lj} - \Gamma^i_{la}\Gamma^a_{kj} \qquad (3.8.12)$$

As pointed out the metric tensor g on M there exist a unique, torsion-free connection on M defined by the condition $\nabla_{Xg} = 0$, which is equivalent to the vanishing covariant derivative of the metric tensor, that is, $g_{ij;k} = 0$. Then parallel transport of vectors preserve the scalar product defined by g and $g(V,V) = constant$ along a geodesic γ, where V is the tangent to γ.

Then,

$$\begin{aligned}\nabla_X\big(g(Y,Z)\big) &= Xg(Y,Z) \\ &= \nabla_X\big(g_{ij}Y^iZ^j\big) \\ &= g(\nabla_X Y, Z) + g(\nabla_X Z, Y) \qquad (3.8.13)\end{aligned}$$

Evaluating $Y\big(g(Z,X)\big)$ and $Z\big(g(X,Y)\big)$adding the first and subtracting the second from equation (3.8.13) gives,

$$g(Z,\nabla_X Y) = \frac{1}{2}[Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g(Y,[Z,X]) + g(Z,[X,Y]) - g(X,[Y,Z])] \qquad (3.8.14)$$

Choosing the basis vectors e_i in place of the vectors X, Y, and Z in equation (3.8.14) gives the connection coefficients in terms of derivatives of g_{ij} and the Lie derivatives of the basis vectors,

$$g\left(e_i, \nabla_{e_j} e_k\right) = g_{im}\Gamma^m_{jk} = \Gamma_{ijk}. \qquad (3.8.15)$$

Choosing a coordinate basis with $\left[e_i, e_j\right] = 0$ gives the usual **Christoffel symbols**:

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right). \qquad (3.8.16)$$

Hence it follows from equations (3.8.12) and (3.8.16) that the Riemann tensor components are expressed in terms of the metric tensor and its second derivatives when the connection defined by the metric is used. From now on we shall always mean by the connection this unique connection defined by the metric tensor.

The expression equation (3.8.12) and earlier definitions imply that the Riemann tensor has the symmetry given by

$$R^i_{jkl} = -R^i_{jlk}, \tag{3.8.17}$$

which is equivalent to $R^i_{j(kl)} = 0$. Further, the curvature tensor obeys the *cyclic identity* $R^i_{[jkl]} = 0$ which can be written as

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0. \tag{3.8.18}$$

The covariant derivatives of the Riemann tensor satisfy the Bianchi identities given by $R^i_{j[kl;a]}$=0 which is the same as,

$$R^i_{jkl;a} + R^i_{jla;k} + R^i_{jak;l} = 0. \tag{3.8.19}$$

A straightforward proof would involve writing down each term above explicitly and then substituting from equation (3.8.12) and summing. There are certain additional symmetries valid when the connection is the one induced by the metric. In this case we have

$$\Gamma_{ijk} = g_{il}\Gamma^i_{jk}, \qquad R_{ijkl} = g_{ia}R^a_{jkl}, \quad \Gamma^i_{jk} = g^{li}\Gamma_{ijk}. \tag{3.8.20}$$

The Riemann tensor R_{ijkl} defined by the metric has the symmetry

$$R_{ijkl} = -R_{jikl}, \tag{3.8.21}$$

which means $R_{(ij)kl} = 0$.

Also, in this case the Riemann tensor is symmetric in the pairs of the first two and last two indices,

$$R_{ijkl} = R_{klij} \tag{3.8.22}$$

The space-time (M, g) is said to have a flat connection if and only if $R^i_{jkl} = 0$, that is, all the components of the Riemann tensor must be vanishing. This is the necessary and sufficient condition for a vector at a point p to remain unaltered after parallel transport along an arbitrary closed curve through p. This is subject to the condition that all such curves can be shrunk to zero, in which case the space-time has to be simply connected. In general, the usual concept of parallel transport of vectors breaks down in a space-time manifold in the sense that given a connection, if we parallel transport a given vector along two different space-time curves to arrive at the same point, the resultant vector will be different in each case. However, when all the components of the Riemann tensor vanish, it can be shown that whenever a vector is transported from one point to the other in the space-time, the result is independent of the path taken. In such a case, the connection is also said to happen is the vanishing of all the components of the Riemann tensor. When a symmetric connection is integrable, the manifold is called **flat**.

Further, in the case of the connection being the metric connection, the vanishing of all the Riemann tensor components provides a necessary and sufficient condition for the space-time metric to be flat; that is, there exists a global coordinate system in M such that the metric reduces to the diagonal form with values ±1 everywhere.

The Ricci tensor is defined as a type (0,2) tensor which is obtained by contracting the Riemannian tensor in the following manner

$$R_{jl} = R^{i}_{jil} \tag{3.8.23}$$

As a consequence of symmetry properties discussed above, it follows that the Ricci tensor is symmetric, and also the following holds

$$R^{i}_{ikl} = 0 \tag{3.8.24}$$

A further contraction of the Ricci tensor gives the curvature scalar R, which is defined as

$$R = g^{ij} R_{ij} \tag{3.8.25}$$

The quantity R has the property that it depends only on g_{ij} and on their derivatives only up to the second order. Further, it is linear in the second derivatives of the metric components. The total number of independent scalars that could be constructed from the metric and its derivatives up to second order is 14.As a consequence of various symmetries listed above, the total number of independent components of R_{ijkl} reduces to 20 when the dimension of the manifold is chosen to be four. For example, when the dimension is three, R^{i}_{ikl} has six independent components essentially given by R_{ij}, and when the dimension is two there is only one independent component, which is essentially R.

Another important tensor one could construct from R_{ijkl} is the Weyl tensor , which is also sometimes called the **Weyl conformal tensor**, given as:

$$C_{ijkl} = R_{ijkl} + \{g_{i[l}R_{k]j} + g_{j[k}R_{l]i}\} + \frac{1}{3} R g_{i[k} g_{l]j}. \tag{3.8.26}$$

The symmetry properties of the Weyl tensor follow from the symmetries of the Riemann curvature tensor discussed above in that it possesses the same symmetries as the Riemann tensor. Additionally, it can be verified that the following identically vanishes

$$g^{ik}C_{ijkl} = 0 \qquad (3.8.27)$$

The Weyl tensor is that part of the curvature tensor for which all contractions vanish for any pair of contracted indices,

$$C^{i}_{jil} = 0 \qquad (3.8.28)$$

If the Weyl tensor vanishes throughout the space-time, that is, $C_{ijkl} = 0$ at all points, then one could show that the metric g_{ij} must be conformally flat.

This means that there exists a conformal function $\Omega(x^i), 0 < \Omega < \infty$, such that one could write

$$g_{ij} = \Omega^2 \eta_{ij},$$

Where η_{ij} is the **flat Minkowskian metric**. In fact, the Weyl tensor is conformally invariant in the sense that under a conformal function

$$g_{ij} \rightarrow \bar{g}_{ij} = \Omega^2 g_{ij}$$

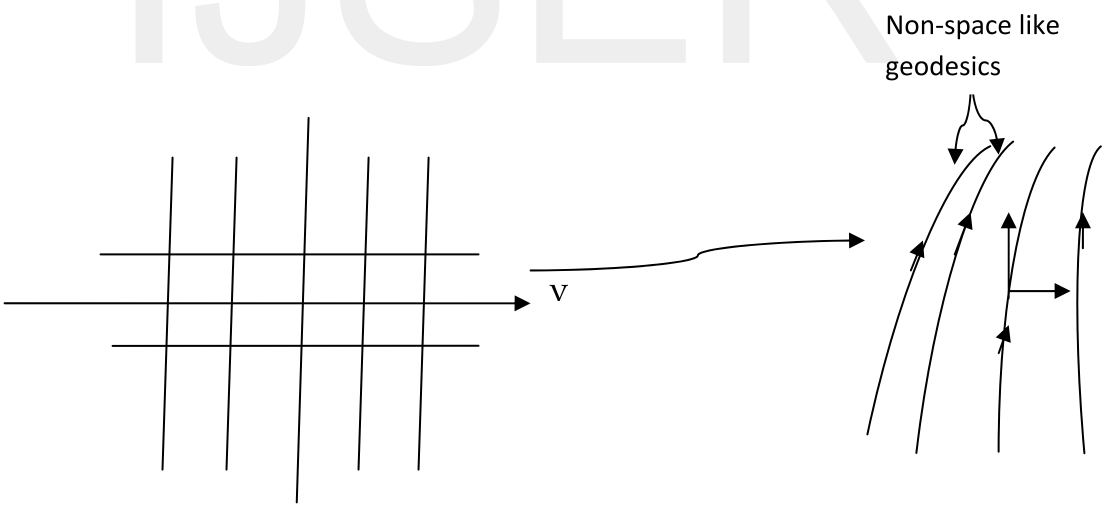

Fig 3.4: A one-parameter family of non-space like geodesics with the tangent vector T and separation vector V

we have

$$\bar{C}^{i}_{jkl} = C^{i}_{jkl} \qquad (3.8.29)$$

It is possible to show that a necessary and sufficient condition for the space-time metric to be conformally flat is that the Weyl tensor must vanish everywhere.

We finally derive here the geodesic deviation equation, which is also called the Jacobi equation. This characterizes the coming together or moving away of space-time geodesics from each other as a result of the space-time curvature.

Consider a smooth one-parameter family of affinely parameterized non-space like geodesics, characterized by the parameters (t, v), where t is the affine parameter along a geodesic and $v = constant$ characterizes different geodesics in the family with $t, v \in \mathbb{R}$ (**fig:3.4**).

Such non-space like geodesics span a two-dimensional sub manifold on which t and v could be chosen as coordinates. The vectors $T = \partial/\partial t$, and $V = \partial/\partial v$ are then coordinate vectors for which are then coordinate vectors for which $[T, V] = 0$. Then, since the torsion tensor is vanishing, we have

$$\nabla_T V = \nabla_V T$$

Which implies $T^i \nabla_i V = V^i \nabla_i T$. Further, T being tangent to the geodesics, $T^i \nabla_i T^j = 0$. Now, define the operator D by $D \equiv T^i \nabla_i$.

Then,

$$DV^j = V^i \nabla_i T^j$$

Taking another derivatives

$$\begin{aligned} D^2 V^j &= DV^i \nabla_i T^j + V^i D(\nabla_i T^j) \\ &= (T^k \nabla_k V^i)(\nabla_i T^j) + V^i T^l \nabla_l \nabla_i T^j \qquad (3.8.30) \end{aligned}$$

However, by the definition of the Riemann curvature tensor,

$$\nabla_l \nabla_i T^j - \nabla_i \nabla_l T^j = R^j_{kli} T^k$$

Substituting this into equation (3.8.29),

$$\begin{aligned} D^2 V^j &= (V^k \nabla_k T^i)(\nabla_i T^j) + \nabla_i \nabla_l T^j V^i T^l + R^j_{kli} T^k V^i T^l \\ &= V^k \left((\nabla_k T^i)(\nabla_i T^j) + (\nabla_k \nabla_l T^j) T^l \right) + R^j_{kli} T^k V^i T^l \\ &= V^k \left(\nabla_k (T^i \nabla_i T^j) \right) + R^j_{kli} T^k V^i T^l \qquad (3.8.31) \end{aligned}$$

$$= R^{j}_{kli} T^{k} V^{i} T^{l} .$$

The equation

$$D^2 V^j = -R^{j}_{kil} T^k V^i T^l \qquad (3.8.32)$$

is called the **Jacobi equation** or **the equation of geodesic deviation**. It is clear from the above that $D^2 V^j = 0$ if and only if all the components of the Riemann tensor are vanishing. On the other hand, whenever some components of the same are non-zero, then the neighbouring non-space like geodesics will necessarily accelerate towards or away from each other.

3.9 EINSTEIN'S EQUATIONS

We discuss in this section the Einstein equations on a space-time manifold. Throughout our discussion, the space-time is modeled by a pair (M, g), where M is a four-dimensional differentiable manifold and g is a Lorentzian metric tensor. Further, M will be assumed to have reasonable topological properties such as para compactness, connectedness, Hausdorff nature and so on, etc. We have been referring to such a model as the space-time manifold. The Einstein equations to be discussed here involve the second derivatives of the metric tensor. Thus, we assume that the metric components are at least C^2 functions of the coordinates. All pairs (M', g') which are diffeomorphic to (M, g) are regarded as equivalent and we study (M, g) which represents this entire equivalence class of space-times with equivalent physical properties.

The principle of local causality and the local conservation of the energy and momentum will be accepted as the basic physical postulates for the space-time manifold (see for example, Hawking and Ellis, 1973). The basic criterion accepted by Einstein while formulating the general theory of relativity was that it is the matter distribution which determines the geometry of the space-time in terms of the Riemann curvature tensor. Next, the motion of any test particle in such a gravitational field is always independent of its own mass and composition. This is the principle of equivalence, which has been verified now to a great degree of accuracy to show that any two objects with different masses and different compositions always arrive at the same time on the surface of the earth when left from the same height. A logical consequence of this fact is that any frame of reference uniformly accelerated with respect to an inertial frame of the special theory of relativity is locally identical to a frame at rest in a gravitational field. Finally, in general relativity, one postulates the principle of general covariance, namely that all the physical laws are expressed as tensor equations so that they are valid in a general frame of reference and are invariant under arbitrary coordinate transformations. When restricted to the frame of free fall, these must produce the laws of special relativistic physics. There are matter fields defined on a space-time such as an electromagnetic field or dust. All such physical fields will be assumed to be represented by a second rank tensor T^{ij}, called the energy momentum tensor, in the sense that T^{ij} would vanish on any open region in the space-time and the derivatives defined with respect to the unique connection defined by the metric tensor. This is because, for any other connection defined on M. Such a stress energy tensor T^{ij} then describes all matter fields such as an electromagnetic field, a scalar field, or a perfect fluid. For example, in the case of dust, which is the matter distribution composed of non-interacting material particles, the field is characterized by the proper density ρ_0 of the flow and the four velocities of the particles given by $dx^i/d\tau$, where τ is the proper time along the time like trajectory describing the particle world line. The simplest second rank tensor constructed from these two quantities is given as

$$T^{ij} = \rho_0 u^i u^j$$

The component T^{00} of this energy momentum tensor is given by

$$T^{ij} = \rho_0 \frac{dx^0}{d\tau}\frac{dx^0}{d\tau}.$$

In a special relativistic frame of reference, this can be interpreted as the relativistic energy density of the matter. One can further show that requiring this tensor to have zero divergence in such a frame implies the conservation of the energy as well as momentum.

Next, a perfect fluid is characterized by an additional scalar quantity, which is the pressure $p = p(x^i)$. In the limit as the pressure vanishes, this must reduce to the dust form of matter. Further, one also demands the conservation laws in a spherical relativistic frame, and that these should reduce to the classical equations of continuity and the Navier-stokes equations in the appropriate limits. Then this energy-momentum tensor is written in a general frame as

$$T^{ij} = (p + \rho)u^i u^j + pg^{ij}$$

which could be taken as the definition of a perfect fluid in the general theory of relativity. In general, one could construct the energy-momentum tensors of various fields by using a variational principle where one has a proposed Lagrangian and the change in action is considered due to the change in the metric.

For an arbitrary frame and for other matter fields such as the electromagnetic field, or a charged scalar field, the principle of local conservation of energy and momentum states that

$$T^{ij}_{;j} = 0 \tag{3.9.1}$$

The equation above for the stress-energy tensor contains considerable information on the matter fields in a space-time. For example, if the space-time contains a Killing vector K^i then the above could be integrated to give a conservation law. The conserved vector in such a case is defined as $P^i = T^{ij}K_j$ and we get $P^i_{;i} = 0$ as a consequence of equation (3.9.2) and the killing equation $K_{i;j} + K_{j;i} = 0$. Then the integration of $P^i_{;i}$ over a compact region implies that the total flux over a closed surface of the energy-momentum is zero in the direction of the killing vector (see for example, Hawking and Ellis, 1973). Even when the space-time does not admit a killing vector, given any point p one could set up a Riemannian normal coordinate system at p so that the metric components have the Minkowskian values and the connection coefficients Γ^i_{jk} vanish at p. One could then choose a small enough neighbourhood of p so that the values of g^{ij} and Γ^i_{jk}differ by an arbitrarily small amount from values at p. Using this fact it could be shown that isolated test particles should move along time like geodesics.

Further, all matter fields are supposed to obey the postulate of local causality, which is given by the statement that the equations governing the matter fields are such that given any $p \in M$, there is an open neighbourhood U of p in which a signal can be sent between any two points of U if and only if there exists a non-spacelike curve between these points. This principle is valid in the special theory of relativity and is also accepted in general relativity. The general theory of relativity is a theory of gravitation defined on a space-time manifold where the force of gravity is described in terms of the space-time curvature. These curvatures are in turn generated by the matter fields, as governed by the Einstein equations which we discuss in this section.

The above principles effectively imply that it is the space-time metric, and the quantities derived from it, that must appear in the equations for physical quantities and that these equations must reduce to the flat space-time case when the metric is Minkowskian. This is the basic content of the general theory of relativity where the space-time manifold is now allowed to have topologies other than $\mathbb{R}^4$ and the metric g_{ij} could be non-flat. In general relativity, the matter fields expressed by the stress-energy tensor are related to the non-flat nature of space –time by means of the Einstein equations, which are the basic equations satisfied by the space-time metric. In Einstein's theory, one does not discuss the physical interaction of matter fields on a fixed background metric prescribed in advance. Actually, g_{ij}s is treated as dynamical variables which depend on the matter content of the space-time and are to be solved from the Einstein equations.

An important indicator towards obtaining this relationship between the matter content and space-time geometry is provided by the Newtonian theory where the gravitational field is described by a potential ϕ. The tidal acceleration between nearby particles is given in terms of the separation between them and second derivatives of ϕ. In a curved space-time manifold, such tidal accelerations are described by the Jacobi equation (3.8.32) in terms of the Riemannian curvature tensor.

Further, we must recover the Poisson equation

$$\nabla^2\phi = 4\pi\rho \tag{3.9.2}$$

in the Newtonian limit. Now, both in the special and general theory of relativity the matter content is described by the stress-energy tensor T_{ij} and the mass-energy density ρ corresponds to the quantity $T_{ij}V^iV^j$. Thus, each side of the Poisson's equation corresponds to the Riemann tensor as expressed in the Jacobi equation and $T_{ij}V^iV^j$ respectively. Another important indicator for this comparison is provided by the Bianchi identities (3.8.19).

Einstein proposed the field equations

$$G_{ij} \equiv R_{ij} - \frac{1}{2}g_{ij}R = 8\pi T_{ij} \tag{3.9.4}$$

In this case, in fact the contracted Bianchi identities imply the local conservation of energy and momentum through the Einstein equations. Taking the trace of the equation (3.9.4) we get

$$R = -8\pi T.$$

Substituting this back in equation (3.9.4) gives the alternative from of the Einstein equations

$$R_{ij} = 8\pi\left(T_{ij} - \frac{1}{2}Tg_{ij}\right) \tag{3.9.5}$$

It is clear by considering the definition of the Ricci tensor that the Einstein equations depend on the derivatives of g_{ij} up to the second order and that they are highly non-linear in $g_{ij}s$. It may be noted, however, that the Einstein equations are linear in the second derivatives of g_{ij}.in fact, the quantities R_{ij} and Rg_{ij} are the only second-rank symmetric tensors which are linear in the second derivatives of the metric and involve only up to second derivatives of g_{ij}. Actually, the Einstein equations are a coupled system of non-linear second order partial differential equations for $g_{ij}s$. This makes the task of solving them extremely difficult. One generally needs to impose several symmetry assumptions on the space-time in order to work out the metric components as a solution to the Einstein equations.

Given the energy momentum tensor T^{ij} , the field equations may be viewed as the set of differential equations to determine the gravitational potentials g^{ij} to determine the resulting geometry. A particularly important case here is that of vacuum solutions when $T^{ij} = 0$.

On the other hand, one could arbitrarily specify the ten metric potentials, then the one could compute the Einstein tensor G_{ij} and then the field equations determine the energy-momentum tensor $T_{ij.}$ however, in this case, the resulting T^{ij} turns out to be unphysical most of the time in that it may violate the energy conditions ensuring the positively of mass-energy density. Such a violation of the energy conditions is rejected on physical grounds in that all observed classical fields obey such a positivity of energy density, which is closely connected with the physical features of gravitation theory.

In general, the field equations are ten equations connecting the total of twenty quantities which are ten components g_{ij} and the other ten components of $T_{ij.}$Thus, the field equations are the condition placing constraints on the simultaneous choice of these twenty quantities. If part of the gravitational potentials and the matter contents are determined from physical conditions, then such conditions are used to determine the matter and geometry fully. In particular, if one considers the vacuum equations

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R = 0$$

then there are ten equations to determine the ten quantities g_{ij}.

However, the Bianchi identities

$$\nabla_j G^{ij} = 0$$

place four differential constraints on these equations which are not all independent. Thus, there is indeterminacy in that there are fewer equations as compared to the unknowns to be determined.

Further, there is an intrinsic gauge freedom available in the general theory of relativity which does not allow a complete determination of the metric potentials. This is given by the coordinate freedom which allows a transformation from one set of coordinates x^i to any other set of coordinates $x^{i'}$. One could, however, use this coordinate freedom to impose conditions on the metric components. For example, choosing the normal coordinates gives $g_{00} = 1$ and $g_{0\propto} = 0$, $\propto = 1,2,3$ in this co-ordinate system. This leaves six other components to be determined from the field equations. This issue is closely connected with the Cauchy problem in general relativity where the basic problem is, given an initial data on a regular space like hyper surface one would like to determine its unique evolution in the feature or past.

Finally, we discuss here the Einstein equations with a cosmological term. It may be noted that the most general second rank tensor which can be constructed out of G_{ij} and g_{ij} so that it is divergence free and involves the derivatives of the metric tensor up to second order only is the linear combination $G_{ij} + \Lambda g_{ij}$(Lovelock, 1972), where Λ is a constant.

Thus, addition of such a constant multiple of g_{ij} to the Einstein tensor preserves all the required properties of equations (3.9.4) discussed above. Einstein historically introduced the cosmological term Λ in his equations in order to generate static cosmological solutions, and wrote the equations as

$$R_{ij} - \frac{1}{2} g_{ij} R + \Lambda g_{ij} = 8\pi T_{ij} \qquad (3.9.6)$$

If $\Lambda \neq 0$ then one does not obtain the Newtonian theory in the limit of slow motions and weak fields; however, if the magnitude of Λ is very small then such departures will be quite negligible and approximate agreement with the Newtonian theory is obtained. It is seen that for an empty space-time with $T_{ij} = 0$, the Einstein equations simplify to

$$R_{ij} = \Lambda g_{ij}$$

3.10 MINKOWSKI SPACETIME

The Minkowski space time is mathematically the manifold $M = \mathbb{R}^4$ with the Lorentzian metric

$$ds^2 = -\,dt^2 + dx^2 + dy^2 + dz^2 \qquad (3.10.1)$$

with $-\infty < t, x, y, z < \infty$ giving the range of the co-ordinates. This is a flat space-time with all the components of the Riemann tensor $R^i_{jkl} = 0$, and hence the simplest empty spacetime solution to Einstein equations

$$G_{ij} = 8\pi T_{ij} = 0$$

which underlies the physics of special theory of relativity. The vector $\partial/\partial t$ provides a time orientation for this model. If we use the spherical polar co-ordinates (t, r, θ, ϕ) given by $x = r sin\theta sin\phi, y = r sin\theta cos\phi, and\ z = r cos\theta$ then equation (1) becomes

$$ds^2 = -\,dt^2 + dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \qquad (3.10.2)$$

The range of the coordinates r, θ, ϕ is $0 < r < \infty, 0 < \theta < \pi\ and\ 0 < \phi < 2\pi$. Two such coordinate neighbourhoods are needed to cover all of the Minkowski space-time.

As discussed all the components of the Riemannian curvature tensor vanish for the Minkowski space-time which is a flat space-time. In other coordinate system(t, r, θ, ϕ), the connection coefficients Γ^i_{jk}will not all vanish (for example, $\Gamma^1_{22} = r$); however, all the Riemann curvature components will still be vanishing.

The **Lorentz transformations** on the Minkowski space-time are defined as the set of those metric preserving isometries which are linear and homogeneous transformations. Physically, these represent the change of reference frame from one inertial observer to another inertial observer. Thus, the Lorentz transformations are defined by the coordinate change

$$x^i \to x^{i'} = L^i_j x^j.$$

From the above and the fact that these are metric preserving isometries, it follows that det $L^i_j = \pm 1$ and hence the matrix is non-singular. If det $L^i_j = +1$ and further $L^0_0 \geq 1$, then the Lorentz transformation preserves both the orientations in space as well as time. The set of all the Lorentz transformations form a group where the identity map is given by δ^i_j and the inverse is defined by the inverse matrix. ***The Lorentz group is a subgroup of the Poincare group***

of transformations which are general inhomogeneous mappings that leave the Minkowskian metric invariant. Such a general mapping consists of a Lorentz transformation together with an arbitrary translation in space and time.

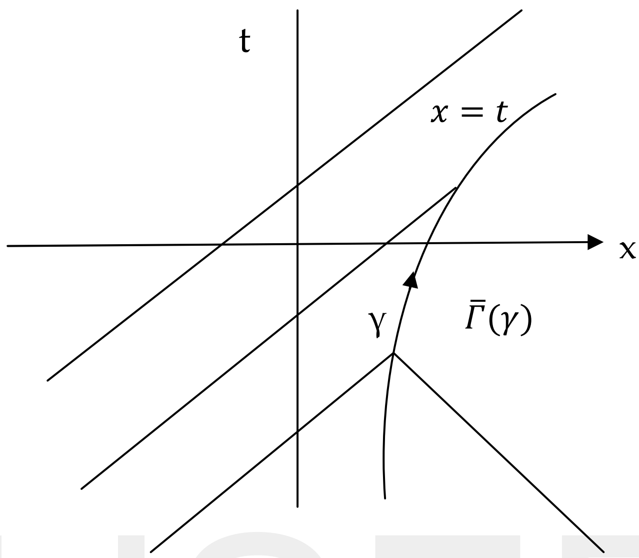

Fig 3.5: The pasts of the time-like curve γ and the null hypersurface $x = t$

The geodesics of Minkowski space-time are the straight lines of the underlying Euclidian geometry. Given an event in M, the lines at 45^0 to the time axis through that event give null geodesics in M. Such null geodesics from the boundary of the chronological future or past $I^{\pm}(p)$ of an event p, which, contains all possible timelike material particle trajectories through p including time-like geodesics. The causal future $J^{+}(p)$ is the closure of $I^{+}(p)$ in Minkowski space, which includes all the events in M which are either timelike or null related to p by means of future directed non-space like curves from p. The family of space like hyper surface $t = const.$ in the Minkowski space-time gives a family of Cauchy surfaces which covers all of M. (***A Cauchy surface is a space-like hyper surface in the space-time such that all in extendible non-space-like curves in M meet this surface once and only once.*** However, all space like hyper surfaces in M not be Cauchy surfaces. For example, the family given by

$$-t^2 + x^2 + y^2 + z^2 = A = const.$$

with $A < 0$ are inextendible spacelike surfaces which are not Cauchy surfaces. All these surfaces are fully contained inside the chronological past or the future of the origin and there are time-like geodesics outside this past or future cone which do not meet any of these surfaces.

To understand the global properties and structure of infinity of the Minkowski space-time, we can use the procedure given by **Penrose**(1968) and **Geroch**, **Kronheimer** and **Penrose** (1972). An arbitrary event p in the Minkowski space-time is uniquely determined either by its chronological future $I^+(p)$ or past $I^-(p)$. If a future directed non-spacelike curve γ has a future end point at p, we have $I^-(\gamma) = I^-(p)$. (By definition $I^-(\gamma)$ is the union of all $I^-(q)$ with q being a point on the curve γ.) On the other hand, if γ is future inextendible without any future end point, the set $I^-(\gamma)$ determines a **'point at infinity'** of M. (A future or past inextendible curve, in the context of Minkowski space-time, is a trajectory which goes off to the infinity in future or past without stopping anywhere.)Two such curves γ_1, and $\gamma_2$determine the same ideal point or a point at infinity if $I^-(\gamma_1) = I^-(\gamma_2)$. Such a procedure defines future ideal points. Past ideal points are defined dually using past inextendible non-space like curves.

In the Minkowski space-time, there are future directed inextendible time-like curves γ which have the same past, which is the entire space-time M, that is $I^-(\gamma) = M$. Hence, all such time-like curves determine a single future ideal point i^+, called the **future time-like infinity**. The past time-like infinity i^- is similarly defined. If we choose γ to be a future endless null curve, it is possible to have a situation where $I^-(\gamma)$ is not the entire Minkowski space-time. Certain timelike curves also have this property. For example, consoder the past of the time-like hyperbola

$$t = sinh\lambda, \quad x = cosh\lambda, \quad y, z = 0, \quad -\infty < \lambda < \infty \qquad (3.10.3)$$

Then, $I^-(\gamma)$ lies completely to the past of the null hypersurface $x = t$. It can be shown in general (see Geroch, Kronheimer and Penrose, 1972) that if for a non-spacelike curve γ, if $I^-(\gamma) \neq M$ and if γ is future endless, then there exists a null hypersurface S_γ, the half-space below which coincides with $I^-(\gamma)$ **(Fig. 3.5).**

If we denote the collection of ideal points so defined by $\mathfrak{I}^+$, then there is a one-one and onto correspondence between the points of $\mathfrak{I}^+$ and such null hypersurfaces. Any such null hypersurface is determined by the value of the time t at which it intersects the time axis and by the direction of null vector at the point of intersection. Since the set of all possible light rays directions at any point is equivalent to the two-sphere S^2, it follows that $\mathfrak{I}^+$is a three-dimensional manifold with topology $S^2 \times \mathbb{R}$.

That three-dimensional null hypersurface $\mathfrak{I}^+$ and $\mathfrak{I}^-$are called the future and past null infinities respectively for the Minkowski space-time. As we shall discuss a general space-time also would admit such a boundary construction under certain conditions such as being asymptotically flat and empty. One can show for the Minkowski space-time that all complete null hypersurfaces are flat and so are like the surfaces $\{x = t\}$, in which case the topological structure of the null infinity is clearly $\mathfrak{I}^+ = S^2 \times \mathbb{R}$. It is not clear, however, that the null infinities will necessarily have the same topological structure even in the case of a general space-time.

It is possible to introduce a differential structure as well as a metric on $\mathfrak{I}^+$. To see this, we first note that a convenient way to attach the ideal point boundary $\mathfrak{I}^+$to M is to use a suitable conformal factor Ω to obtain a transform of the original space-time metric η_{ij},

$$g_{ij} = \Omega^2 \eta_{ij}, \quad \Omega > 0, \tag{3.10.4}$$

which leaves the causal structure of M invariant because the null geodesics of η_{ij} and the unphysical metric g_{ij} are the same up to a reparametrization as discussed earlier. Thus, the past of any non-space like curve γ is unchanged and there is a natural correspondence between ideal points in two space-times. Since light cones are unaltered by a conformal transformation, the boundary attachment in this manner is coordinate independent.

In the metric (3.10.2), one could introduce the **advanced** and **retarded** null coordinates given by

$$v = t + r, \qquad u = t - r, \tag{3.10.5}$$

which gives a reference frame based on null cones, which is most suitable to analyse the radiation fields **(Fig. 3.6)**

$$ds^2 = -dudv + \frac{1}{4}(u - v)^2(d\theta^2 + sin^2\theta d\phi^2) \tag{3.10.6}$$

with $-\infty < v < \infty$ and $-\infty < u < \infty$. Now, the information at future null infinity corresponds to taking limit as $v \to \infty$, which amounts to moving in future along $u =$ const. light cones. Similarly, past null infinity corresponds to $u \to \infty$.This procedure could be made precise in a coordinate independent way.We can compactify the Minkowski space-time M by means of a conformal transformation of equation (3.10.6) given by

$$\Omega^2 = (1 + v^2)^{-1}(1 + u^2)^{-1} \tag{3.10.7}$$

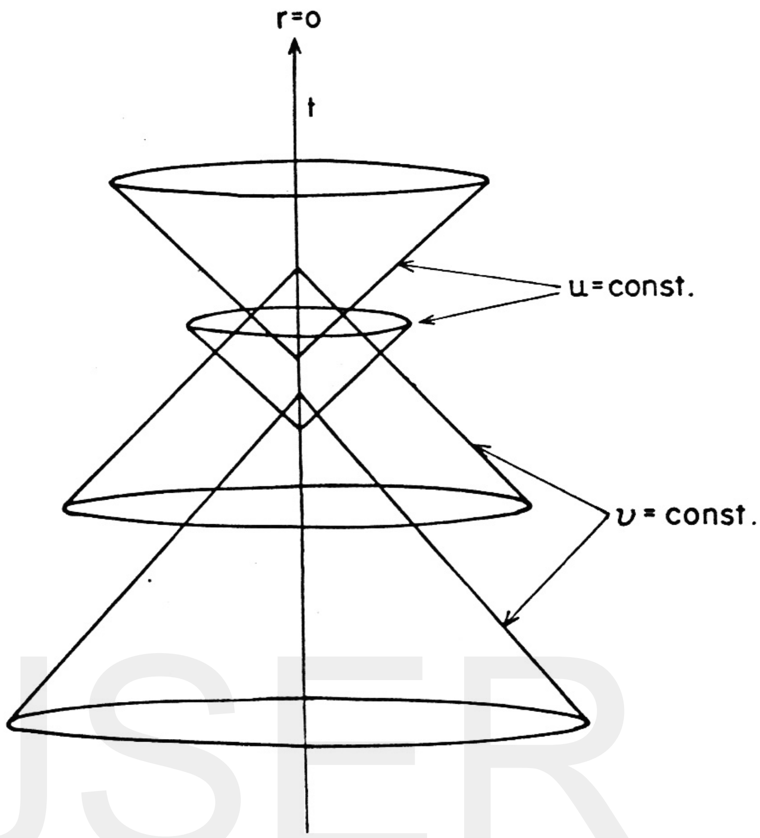

Fig 3.6: In the Minkowski space-time, the future light cone are given as the null surfaces $u = const$.Similarly, the past light cones are given as $v = const$.

and then by adding closure to add the null infinities. We also introduce new coordinates p, q by

$$v = tanp, \qquad u = tanq. \tag{3.10.8}$$

Then, the corresponding ranges for p and q are

$$-\frac{\pi}{2} < p < \frac{\pi}{2}, \qquad -\frac{\pi}{2} < q < \frac{\pi}{2}$$

And the metric $\overline{g_{ij}}$ on the unphysical space-time $\bar{M}$, after the conformal transformation, is given by

$$d\bar{s}^2 = -dpdq + sin^2(p-q)(d\theta^2 + sin^2\theta d\phi^2) \tag{3.10.9}$$

It is possible to see now that the metric (3.10.9), with the coordinate ranges given above, is a manifold embedded as a part of the Einstein static universe to see this, write

$$T = p + q, \quad R = p - q, \tag{3.10.10}$$

then equation (3.10.9) becomes in (T, R, θ, ϕ) coordinates

$$d\bar{s}^2 = -dT^2 + dR^2 + sin^2R(d\theta^2 + sin^2\theta d\phi^2), \tag{3.10.11}$$

With the coordinate ranges

$$-\pi < T + R < \pi, \quad -\pi < T - R < \pi \tag{3.10.12}$$

This is precisely the natural Lorentz metric on $S^3 \times \mathbb{R}$, which is the Einstein static universe, except that the coordinate ranges are now restricted by equation (12). In this picture, the future null infinity $\mathfrak{I}^+$ is given by $T = \pi - R$ for $0 < R < \pi$ and the past null infinity is given by $T = -\pi + R$ for $0 < R < \pi$.

This conformal structure of infinity for the Minkowski space-time is symmetric space-time can be depicted by a similar diagram which is called a **Penrose Diagram**. As mentioned above, $\mathfrak{I}^+$ is topologically $S^2 \times \mathbb{R}$ and this discussion on Minkowski space-time will be useful when we define general asymptotically flat space-times.

Finally, in order to have a better insight into the asymptotic structure of the Minkowski space-time, we work out below the light cone cuts of future null infinity for the Minkowski space-time. The light cone evolves from an arbitrary apex in the space-time to future null infinity and its intersection with $\mathfrak{I}^+$ is obtained. Both the light cone and $\mathfrak{I}^+$ are three-dimensional null hypersurfaces in M and hence their intersection is a two-surface at $\mathfrak{I}^+$. It will be shown that the knowledge of such cuts yields considerable information about the interior space-time and the metric in the neighbourhood of the apex point.

First we introduce a coordinate system on M which is more suitable for this purpose. Using the retarded time u, and a complex stereographic coordinate ζ and its complex conjugate $\bar{\zeta}$ on the sphere defined by

$$\zeta = e^{i\phi}\cot(\theta/2), \tag{3.10.13}$$

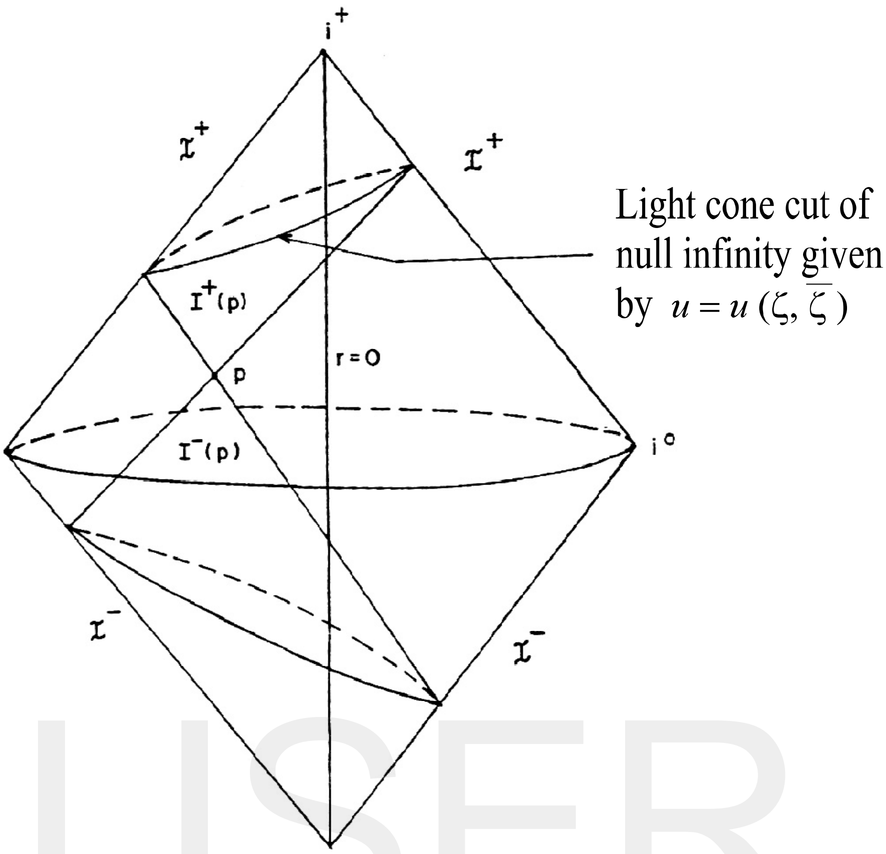

Fig 3.7: Conformal infinity in the Minkowski space-time. The future and past null infinities $\mathfrak{T}^{\pm}$ are both topologically $S^2 \times \mathbb{R}$. Every null geodesic in the space time meets $\mathfrak{T}^{+}$in future and $\mathfrak{T}^{-}$ in the past. Here the point i^{+} denotes the future timelike infinity, and past timelike infinity is similarly defined where every timelike geodesic terminates in future and past respectively. The spacelike infinity is denoted by i^0 . The light cone cuts of null infinity for an event p are shown.

the Minkowski metric (3.10.2), when written in the $(u, r, \zeta, \bar{\zeta})$ system is given by

$$ds^2 = -du^2 - 2dudr + r^2 \frac{d\zeta\, d\bar{\zeta}}{{P_0}^2} \qquad (3.10.14)$$

where $P_0 = (1 + \zeta\bar{\zeta})/2$. Following Exton, Newman and Penrose (1969) we define $u' = (1/\sqrt{2})u$ and $r' = \sqrt{2}r$, which are more convenient variables for the study of asymptotic structure.

Since null cones and null geodesics are conformally invariant freedom available. Suppressing the primes, we conformally transform equation (3.10.14) to

$$d\bar{s}^2 = \Omega^2 ds^2 = -4\ell^2 du^2 + 4du d\ell + \frac{d\zeta\, d\overline{\zeta}}{{P_0}^2} \qquad (3.10.15)$$

Where we have introduced a new variable $\ell = 1/\sqrt{2}\,r$ and the conformal factor is chosen to be $\Omega = \sqrt{2}\ell$. In the limit as $r \to \infty$, we have $\ell \to 0$ and the null infinity $\mathfrak{I}^+$ is defined by the condition $\ell = 0$. Future directed null cones are characterized by the values of u, ζ and $\overline{\zeta}$ and so the coordinates $(u, \zeta, \overline{\zeta})$ can be used as coordinates on $\mathfrak{I}^+$, which are called the Bondi coordinates on $\mathfrak{I}^+$.

In these coordinates, a hypersurface of $\mathfrak{I}^+$ has the metric

$$ds^2 = \frac{1}{{P_0}^2} d\zeta\, d\overline{\zeta}. \qquad (3.10.16)$$

Thus, $\mathfrak{I}^+$ is a null hypersurface which is generated by the null curves $\zeta, \overline{\zeta} =$ const. For the manifold $(\bar{M}, \bar{g})$ given by the conformal compactification as above, it is easy to see that

$$\frac{\partial \Omega}{\partial x^i} = (0,1,0,0) \qquad (3.10.17)$$

And

$$\bar{g}^{ij} \frac{\partial \Omega}{\partial x^i} \frac{\partial \Omega}{\partial x^j}\bigg|_{\mathfrak{I}^+} = 0 \qquad (3.10.18)$$

Thus, Ω is differentiable on $\bar{M}$, the new unphysical manifold with boundary, and $\partial \Omega / \partial x^i$ is a null vector. The factor Ω is smooth everywhere and $\Omega = 0$ on $\mathfrak{I}^+$ which is a **null hypersurface**. Similarly we could discuss the past null infinity $\mathfrak{I}^-$, which has a similar structure. Next, we work out the complete light cone at any given apex point in the space time. The null geodesic equations of the space-time are given by with the dot denoting derivative with respect to s, the affine parameter)

$$\begin{aligned} 2\ell^2 \dot{u} - \dot{\ell} &= 1, \\ \ddot{u} + \ell \dot{u}^2 &= 0, \\ \ddot{\zeta}(1 + \zeta\bar{\zeta}) - 2\bar{\zeta}\dot{\zeta}^2 &= 0, \\ \ddot{\bar{\zeta}}(1 + \zeta\bar{\zeta}) - 2\zeta\dot{\bar{\zeta}}^2 &= 0, \end{aligned} \qquad (3.10.19)$$

$$4\ell^2\dot{u}^2 - 4\dot{u}\dot{\ell} - \frac{\dot{\zeta}\dot{\bar{\zeta}}}{{P_0}^2} = 0,$$

Where the last equation corresponds to $ds^2 = 0$.

Restricting ourselves presently to the equatorial plane $\theta = \frac{\pi}{2}$ for the sake of simplicity, equation (3.10.19) can be written as

$$2\ell^2\dot{u} = 1,$$

$$\ddot{u} + \ell\dot{u}^2 = 0,$$

$$\ddot{\phi} = 0, \quad \dot{\phi} = b, \tag{3.10.20}$$

$$\ell^2\dot{u}^2 - \dot{u}\dot{\ell} = \dot{\phi}^2 = b^2/4\,.$$

We shall now eliminate the parameter s from the above equations. For that, one could use the first and last equations in the above to obtain $\dot{\ell}^2 = 1 - b^2\ell^2$. Thus we have

$$\dot{\ell} = \pm\sqrt{(1 - b^2\ell^2)},$$

$$ds = \pm\frac{d\ell}{\sqrt{1 - b^2\ell^2}} \tag{3.10.21}$$

We note that $\dot{\ell} < 0$ corresponds to a null ray moving away from the origin.

Next, if $\dot{\ell} > 0$ initially, then the ray moves initially towards the origin of the coordinate system $(r = 0, \ell = \infty)$and after reaching a minimum value r_m, where $\dot{\ell} = \sqrt{1 - b^2{\ell_m}^2} = 0$, it begins to move outwards and again $\dot{\ell} < 0$. For the sake of definiteness we shall choose here rays such that initially $\dot{\ell} < 0$; however by considering the other sheet $\dot{\ell} > 0$ as well, we can span the full light cone of null rays from our starting point.

The above equations can be written as

$$du = -\frac{d\ell}{2\ell^2\sqrt{1 - b^2\ell^2}} + \frac{d\ell}{2\ell^2},$$

$$d\phi = -\frac{bd\ell}{\sqrt{1 - b^2\ell^2}}. \tag{3.10.22}$$

Suppose now the apex of the cone has the values $\ell = \ell_0, u = u_0, \phi = \phi_0$. Then integrating the above equations from ℓ_0 to an arbitrary ℓ gives the equations for one sheet of the light cone.

For the sake of simplicity we choose the apex on the $\phi = 0$ axis, that is, $\ell \to \ell_0$, the above integration provides us with a portion of the cut at infinity of the rays in the equatorial plane, integration of equation (3.10.22) from ℓ_0 to 0 yields

$$u - u_0 = \frac{1 - \sqrt{1 - b^2{\ell_0}^2}}{2\ell_0}, \tag{3.10.23}$$

$$\phi = sin^{-1}(b\ell_0) \tag{3.10.24}$$

The initial direction b ranges from 0 to ℓ_0^{-1}. Note that for a fixed apex one has a one-to-one relation between the initial direction b and the final angular position ϕ on the future null infinity. By eliminating b from equations (3.10.23) and (3.10.24), we obtain the equatorial plane portion of the light cone cut which is given by

$$u = u_0 = \left({}^{1}/_{2\ell_0}\right)(1 - cos\phi). \tag{3.10.25}$$

For the sheet $\dot{\ell} < 0$ which we have been considering, $cos\phi$ will be positive because ϕ will always be in the first or fourth quadrant in this situation. For the other sheet corresponding to $\dot{\ell} > 0$, initially $cos\phi$ will be negative.

The portion of the light cone cut given by equation (3.10.25) describes, as we have mentioned. Only an S^1 worth of null rays intersecting $\mathfrak{T}^+$ since we have restricted ourselves to the equatorial plane. However, because of spherical symmetry the fyll cut, which is topologically S^2, can be generated by rotating this plane.

"God not only plays dice, He also sometimes throws the dice where they cannot be seen."

STEPHEN HAWKING

CHAPTER 4

SYMMETRIC SPACES

4.1 INTRODUCTION

Euclid implicitly assumed that metric relations are unaffected by translations or rotations. Real gravitational fields do not usually have such a high degree of symmetry, but they often admit some group of approximate symmetry transformations, and when they do, we can use this information to help solve the Einstein equations, or even to do without a solution. I shall give only a very brief introduction to do elaborate mathematical theory of symmetric spaces, with special attention to the maximally symmetric spaces that are of special interest of cosmology.

The initial difficultly here is: How can we use some supposed symmetry of a metric space to gain information about the metric, when we need to know the metric before we can establish a coordinate system in which to define the symmetry? In order to avoid this impasse, we shall have to learn ways to describing symmetries in a covariant language, which does not depend on any particular choice of coordinate system. Once this language is established, it becomes a matter of mathematical manipulation to determine those properties of a metric that follow from its symmetries.

4.2 KILLING VECTORS

A metric $g_{ij}(x)$ is said to be **form-invariant** under a given coordinate transformation $x \rightarrow x'$, when the transformed metric $g'_{ij}(x)$ is the same function of its argument x'^{i} as the original metric $g_{ij}(x)$ was of its argument x^{i}, that is

$$g'_{ij}(y) = g_{ij}(y) \quad \text{for all } y \tag{4.2.1}$$

[This is different from the condition for a scalar, which is that $S'(x') = S(x)$.] At any given point the transformed metric is given by the relation

$$g'_{ij}(x') = \frac{\partial x^{\rho}}{\partial x'^{i}} \frac{\partial x^{\sigma}}{\partial x'^{j}} g_{\rho\sigma}(x)$$

or equivalently

$$g_{ij}(x) = \frac{\partial x'^{\rho}}{\partial x^{i}} \frac{\partial x'^{\sigma}}{\partial x^{j}} g'_{\rho\sigma}(x')$$

When (4.2.1) is valid, we can replace $g'_{\rho\sigma}(x')$ with $g_{\rho\sigma}(x')$ and so obtain the fundamental requirement for a form invariance of the metric :

$$g_{ij}(x) = \frac{\partial x'^{\rho}}{\partial x^i}\frac{\partial x'^{\sigma}}{\partial x^j} g_{\rho\sigma}(x') \qquad (4.2.2)$$

Any transformation $x \to x'$ that satisfies (4.2.2) is called an **isometry**.

In general, Eq. (4.2.2) is a very complicated restriction on the function $x'^i(x)$. It can be greatly simplified by descending to the special case of an infinitesimal coordinate transformation :

$$x'^i = x^i + \varepsilon\xi^i(x) \qquad with\ |\varepsilon| \ll 1 \qquad (4.2.3)$$

To first order in ε, Eq. (4.2.2) now reads

$$0 = \frac{\partial \xi^i(x)}{\partial x^\rho} g_{i\sigma}(x) + \frac{\partial \xi^j(x)}{\partial x^\sigma} g_{\rho j}(x) + \xi^i(x)\frac{\partial g_{\rho\sigma}(x)}{\partial x^j} \qquad (4.2.4)$$

This can be rewritten in terms of derivatives of the covariant components $\xi_\sigma \equiv g_{i\sigma}\xi^i$:

$$0 = \frac{\partial \xi_\sigma}{\partial x^\rho} + \frac{\partial \xi_\rho}{\partial x^\sigma} + \xi^i\left[\frac{\partial g_{\rho\sigma}}{\partial x^i} - \frac{\partial g_{i\sigma}}{\partial x^\rho} - \frac{\partial g_{\rho i}}{\partial x^\sigma}\right]$$

$$= \frac{\partial \xi_\sigma}{\partial x^\rho} + \frac{\partial \xi_\rho}{\partial x^\sigma} - 2\xi_i\Gamma^i_{\rho\sigma}$$

or, more compactly,

$$0 = \xi_{\sigma;\rho} + \xi_{\rho;\sigma} \qquad (4.2.5)$$

Any four-vector field $\xi_\sigma(x)$ that satisfies Eq. (4.2.5) will be said to form a **killing vector** of the metric $g_{ij}(x)$.

The problem of determining all infinitesimal isometries of a given metric is now reduced to the problem of determining all Killing vectors of the metric. Any linear combination of Killing vectors (with constant coefficient) is a Killing vector, so it is the space of vector fields spanned by the Killing vectors that really determines the infinitesimal isometrics of a metric.

The Killing condition (4.2.5) is much more restrictive then it looks, for it allows us to determine the whole function $\xi_i(x)$ from given values of ξ_σ and $\xi_{\sigma;\rho}$ at some point X. For the commutator of two covariant derivatives,

$$\xi_{\sigma;\rho;i} - \xi_{\sigma;i;\rho} = -R^\lambda_{\sigma\rho i}\xi_\lambda \qquad (4.2.6)$$

and the cyclic sum rule for the curvature tensor,

$$R^{\lambda}_{\sigma\rho i} + R^{\lambda}_{i\sigma\rho} + R^{\lambda}_{\rho i\sigma} = 0 \qquad (4.2.7)$$

By adding (4.2.6) and its two cyclic permutations, we find that any vector ξ_i must satisfy the relation

$$0 = \xi_{\sigma;\rho;i} - \xi_{\sigma;i;\rho} + \xi_{i;\sigma;\rho} - \xi_{i;\rho;\sigma} + \xi_{\rho;i;\sigma} - \xi_{\rho;\sigma;i} \qquad (4.2.8)$$

For a Killing vector, (4.2.5) and (4.2.8) give

$$0 = \xi_{\sigma;\rho;i} - \xi_{\sigma;i;\rho} - \xi_{i;\rho;\sigma}$$

and thus (4.2.6) becomes

$$\xi_{i;\rho;\sigma} = -R^{\lambda}_{\sigma\rho i}\xi_{\lambda} \qquad (4.2.9)$$

Hence, given ξ_{λ} and $\xi_{\lambda;j}$ at some point$X$, we can determine the second derivatives of $\xi_{\lambda}(x)$ at X from Eq. (4.2.9), and we can find successively higher derivatives of ξ_{λ} at X by taking derivatives of Eq. (4.2.9). All the derivatives of ξ_{λ} at X will thus be determined as linear combinations of $\xi_{\lambda}(X)$ and $\xi_{\lambda;j}(X)$. The function $\xi_{\lambda}(X)$ can then (when it exists) be constructed as a Taylor series in $x^{\lambda} - X^{\lambda}$ within some finite neighborhood of X, and will again be linear in the initial values $\xi_{\lambda}(X)$, $\xi_{\lambda;j}(X)$. Thus any particular Killing vector $\xi^{n}_{\rho}(x)$of the metric $g_{ij}(x)$ can be expressed as

$$\xi^{n}_{\rho}(x) = A^{\lambda}_{\rho}(x;X)\xi^{n}_{\lambda}(X) + B^{\lambda j}_{\rho}(x;X)\xi^{n}_{\lambda;j}(X) \qquad (4.2.10)$$

where $A^{\lambda}_{\rho}$and $B^{\lambda j}_{\rho}$are functions that of course depend on the metric and on X, but do not depend on the initial values $\xi_{\lambda}(X)$ and $\xi_{\lambda;j}(X)$, and hence are the same for all Killing vectors. Each Killing vector $\xi_{\rho}(x)$ of a given metric in uniquely specified by the values of $\xi_{\rho}(x)$ and $\xi_{\rho;\sigma}(X)$ at any particular point X. A set of Killing vectors $\xi^{n}_{\rho}(x)$ is said to be ***independent*** if they do not satisfy any linear relations of the form

$$\sum_{n} c_n \xi^{n}_{\rho}(x) = 0 \qquad (4.2.11)$$

with *constant* coefficient c_n. Equation (4.2.10) tells us that there can be at most $N(N+1)/2$ independent Killing vectors in N-dimensions. For consider any M Killing vectors $\xi^{n}_{\rho}(x)$. For each n, there are N quantities $\xi^{n}_{\rho}(x)$ and $N(N-1)/2$ independent quantities $\xi^{n}_{\rho;j}(X)$ [recall Eq. (4.2.5)], so we can think of the quantities $\xi^{n}_{\rho}(x)$and $\xi^{n}_{\rho;j}(X)$ as the components of M vectors in an $N(N+1)/2$

dimensional space. if $> N(N+1)/2$, then these M vectors cannot be linearly independent, so they must satisfy relations of the form

$$\sum_n c_n \xi_\rho^n(X) = \sum_n c_n \xi_{\rho;j}^n(X) = 0$$

Equation (4.2.10) then tells us that the Killing vectors $\xi_\rho^n(x)$ satisfy the relations (4.2.11) everywhere, and are therefore not independent Killing vectors.

This result is significant only because we defined independent Killing vectors as vectors that are not subject to any linear relations with *constant* coefficients. At some given point X in an N-dimensional space, any set of more than N Killing vectors will of course be subject to one or more linear relations such as (4.2.11). However, the coefficients c_n in these linear relations need not be constant in x^i. The above theorem says that any set of more than $N(N+1)/2$ Killing vectors will be subject to linear relations with constant coefficients.

A metric space is said to be **homogeneous** if there exist infinitesimal isometries (4.2.3) that carry any given point X into any other point in its immediate neighborhood. That is, the metric must admit Killing vectors that at any given point take all possible values. In particular, in an N-dimensional space we can choose a set of N Killing vectors $\xi_\lambda^{(i)}(x;X)$with

$$\xi_\lambda^{(i)}(x;X) = \delta_\lambda^i$$

There are evidently independent, because any relation of the form $c_i \xi_j^{(i)}(x;X) = 0$ would at $x = X$ imply that all c_λ vanish.

A metric space is said to be **isotropic** about a given point X if there exist infinitesimal isometries (4.2.3) that leave the point X fixed, so that $\xi^\lambda(X) = 0$, and for which the first derivatives $\xi_{\lambda;j}(X)$ take all possible values, subject only to the anti-symmetry condition (4.2.5). In particular, in N dimensions we can choose a set of $N(N-1)/2$ Killing vectors $\xi_\lambda^{(ij)}(x;X)$ with

$$\xi_\lambda^{(ij)}(x;X) \equiv -\xi_\lambda^{(ji)}(x;X)$$

$$\xi_\lambda^{(ij)}(x;X) \equiv 0$$

$$\xi_{\lambda;\rho}^{(ij)}(X;X) \equiv \left[\frac{\partial}{\partial x^\rho} \xi_\lambda^{(ij)}(x;X)\right]_{x=X} \equiv \delta_\lambda^i \delta_\rho^j - \delta_\rho^i \delta_\lambda^j$$

These are independent, because any relation of the form $c_{ij}\xi_\lambda^{(ij)}(x;X) = 0$ with $c_{ij} = -c_{ji}$ would at X imply that $c_{\lambda\rho} - c_{\rho\lambda} = 2c_{\lambda\rho} = 0$.

We shall also have to deal with spaces that are isotropic about *every* point. in this case there are Killing vectors $\xi_\lambda^{(ij)}(x;X)$ and $\xi_\lambda^{(ij)}(x;X+dX)$ that satisfy the above initial conditions at X and at $X+dX$, respectively. Any linear combination of these will be a Killing vector, and so $\partial\xi_\lambda^{(ij)}(x;X)/\partial X^\rho$ will also be a Killing vector of the metric. In order to evaluate this Killing vector at $x = X$ we need only recall that $\xi_\lambda^{(ij)}(X;X)$ vanishes, and therefore

$$0 = \frac{\partial}{\partial X^\rho}\xi_\lambda^{(ij)}(X;X) = \left[\frac{\partial}{\partial x^\rho}\xi_\lambda^{(ij)}(x;X)\right]_{x=X} + \left[\frac{\partial}{\partial X^\rho}\xi_\lambda^{(ij)}(x;X)\right]_{x=X}$$

This gives

$$\left[\frac{\partial}{\partial X^\rho}\xi_\lambda^{(ij)}(x;X)\right]_{x=X} = -\delta_\lambda^i\delta_\rho^j + \delta_\rho^i\delta_\lambda^j$$

It is now obvious that we can construct a Killing vector $\xi_\lambda(x)$ that takes any arbitrary value a_λ at $x = X$; we need only take

$$\xi_\lambda(x) = \frac{a_j}{N-1}\frac{\partial}{\partial X^\rho}\xi_\lambda^{(\rho j)}(x;X)$$

Hence *any space that is isotropic about every point is also homogeneous*.

A metric that admits the maximum number $N(N+1)/2$ of Killing vectors is said to be ***maximally symmetriic***.

In particular, a space that is both homogeneous and isotropic about some given point X will admit the $N(N+1)/2$ Killing vectors $\xi_\lambda^{(i)}(x;X)$ and $\xi_\lambda^{(ij)}(x;X)$. These Killing vectors are obviously independent, for if they satisfy a linear relation

$$0 = c_\mu\xi_\lambda^{(i)}(x;X) + c_{\mu\nu}\xi_\lambda^{(ij)}(x;X)$$

$$c_{ij} = -c_{ji}$$

then differentiating with respect to x^ρ and setting $x = X$ gives $c_{\lambda\rho} = 0$, and setting $x = X$ then gives $c_\lambda = 0$. Thus a homogeneous space that is isotropic about some point is maximally symmetric. It then also follows that space that is isotropic about every point is maximally symmetric.

We can also prove the converse, that an ***maximally symmetric space is necessarily homogeneous and isotropic about all points***.

If there are $N(N+1)/2$ independent Killing vectors $\xi_\lambda^n(x)$, then we can think of the quantities $\xi_\rho^n(X), \xi_{\lambda;j}^n(X)$ as forming a square matrix, with $N(N+1)/2$ rows labeled by n, and $N(N+1)/2$ columns labeled by the N values of ρ and the $N(N-1)/2$ values of λ and v with $\lambda > j$.

Furthermore, this matrix must have a nonvanishing determinate, because any relation of the form

$$\sum_n c_n \xi_\rho^n(X) = \sum_n c_n \xi_{\rho;j}^n(X) = 0$$

would with (4.2.10) imply that $\sum_n c_n \xi_\rho^n(X)$ vanishes, contrary to our assumption that these Killing vectors are independent. It must therefore be possible, for any "row vector" with "components" a_μ and $b_{ij} = -b_{ji}$, to find a solution of the equations

$$\sum_n d_n \xi_i^n(X) = a_i$$

$$\sum_n d_n \xi_{i;j}^n(X) = b_{ij}$$

Hence we can find a Killing vector $\xi_i(x)$ for which $\xi_i(X)$ takes the value a_i and $\xi_{i;j}(X)$ takes the value b_{ij}, by choosing

$$\xi_i(x) = \sum_n d_n \xi_i^n(x)$$

But a_i is arbitrary, so the space is homogeneous, and $b_{\mu v}$ is arbitrary (except that $b_{ij} = -b_{ji}$), so the space is isotropic about X.

As an example of a maximally symmetric space, consider an N-dimentional flat space, with vanishing curvature tensor. We can then choose Cartesian coordinates with a constant metric and vanishing affine connection. In this coordinate system, equation (4.2.9) reads

$$\frac{\partial^2 \xi_i}{\partial x^\rho \partial x^\sigma} = 0$$

The solution is

$$\xi_i(x) = a_i + b_{ij}x^j$$

with a_i and b_{ij} constant. This satisfies the Killing vector condition (4.2.5) if and only if

$$b_{ij} = -b_{ji}$$

We can thus choose a set of $N(N+1)/2$ Killing vectors as follows:

$$\xi_i^{(j)}(x) \equiv \delta_i^j$$

$$\xi_i^{(j\lambda)}(x) \equiv \delta_i^j x^\lambda - \delta_i^\lambda x^j$$

and the general Killing vector is

$$\xi_i(x) = a_j \xi_\mu^{(j)}(x) + b_{j\lambda}\xi_i^{(j\lambda)}(x)$$

The N vectors $\xi_i^{(j)}(x)$ represent translations, whereas the $N(N-1)/2$ vectors $\xi_i^{(j\lambda)}$ represent infinitesimal rotations (or, for a Minkowski space, Lorentz transformations). Thus any flat metric admits $N(N+1)/2$ independent Killing vectors, and is therefore maximally symmetric.

Of course, not all metrics admit the maximum number of Killing vectors. Whether (4.2.9) is soluble for a given set of initial data $\xi_\lambda(X), \xi_{\lambda;\rho}(X)$ depends of the integrability of this equation, which in turn depends on the metric. One integrability condition we shall use below follows from the general formula for commutators of covariant derivatives of tensors :

$$\xi_{\rho;i;\sigma;j} - \xi_{\rho;i;j;\sigma} \equiv -R^\lambda_{\rho\sigma j}\xi_{\lambda;i} - R^\lambda_{i\sigma j}\xi_{\rho;\lambda}$$

Equation (4.2.9) will satisfy this condition if and only if

$$R^\lambda_{\sigma\rho i}\xi_{\lambda;j} - R^\lambda_{j\rho i}\xi_{\lambda;\sigma} + \left(R^\lambda_{\sigma\rho i;j} - R^\lambda_{j\rho i;\sigma}\right)\xi_\lambda = -R^\lambda_{\rho\sigma j}\xi_{\lambda;i} - R^\lambda_{i\sigma j}\xi_{\rho;\lambda}$$

or, using (4.2.5),

$$\left[-R^\lambda_{\rho\sigma j}\delta_i^\chi + R^\lambda_{i\sigma j}\delta_\rho^\chi - R^\lambda_{\sigma\rho i}\delta_j^\chi + R^\lambda_{j\rho i}\delta_\sigma^\chi\right]\xi_{\lambda;k} = \left[R^\lambda_{\sigma\rho i;j} - R^\lambda_{j\rho i;\sigma}\right]\xi_\lambda \qquad (4.2.12)$$

These conditions are of course empty for a flat space, but in general they will impose linear relations among the ξ_λ and $\xi_{\lambda;k}$ at any given point. Alternatively, if we know something about the killing vectors admitted by an unknown metric, then we can use (4.2.12) to learn something about its curvature tensor. In this way, we shall able in the following sections to deduce the form of a maximally symmetric metric from its isometries.

It should be emphasized that the existence of a definite number of independent Killing vectors does not depend on a particular choice of coordinate system. If $\xi^i(x)$ is a Killing vector of a metric $g_{ij}(x)$, then by performing a coordinate transformation $x^i \to x'^i$ we obtain a metric

$$g'_{ij}(x') = \frac{\partial x^\rho}{\partial x'^i}\frac{\partial x^\sigma}{\partial x'^j} g_{\rho\sigma}(x)$$

and, since (4.2.5) is generally covariant, this obviously has a Killing vector

$$\xi'^i(x') = \frac{\partial x'^i}{\partial x^j}\xi^j(x)$$

If M Killing vectors $\xi_i^n(x)$ are independent, then so are the M Killing vectors $\xi_i^{n'}(x')$ for any linear relation among the $\xi^{n'}$would imply a linear relation among the ξ^n. Thus the maximal symmetry of a given space is an inner property, not depending on how we choose the coordinate system. In particular, it follows that any space with vanishing curvature tensor is maximally symmetric; the converse, however, is not true. It is also easy to see that the homogeneity or isotropy of a given space is independent of the choice of coordinates. As far as these simple symmetries are concerned, we have accomplished the task laid out in the introduction to this chapter, that of describing symmetries of the metric in a generally covariant language.

4.3 MAXIMALLY SYMMETRIC SPACES: UNIQUENESS

We know show that the maximally symmetric spaces are uniquely specified by a "**curvature constant**" K, and by the numbers of eigenvalues of the metric that are positive or negative. That is, given two maximally symmetric metrics with the same k and the same numbers of eigen values of each sign, it will always be possible to find a coordinate transformation that carries one metric into the other. Armed with this theorem, we shall be able in the next section to carry out an exhaustive study of maximally symmetric spaces by simply constructing such metrics in one convenient coordinate system.

We showed in the last section that at any given point x in a maximally symmetric space, we can find Killing vectors for which $\xi_\lambda(x)$ vanishes and for which $\xi_{\lambda;k}(x)$ is an arbitrary matrix. It follows then that the coefficient of $\xi_{\lambda;k}(x)$ in Eq.(4.2.12) must have a vanishing anti-symmetric part, that is,

$$\begin{aligned} -R^{\lambda}_{\rho\sigma j}\delta^{k}_{i} + R^{\lambda}_{i\sigma j}\delta^{k}_{\rho} - R^{\lambda}_{\sigma\rho i}\delta^{k}_{j} + R^{\lambda}_{j\rho i}\delta^{k}_{\sigma} \\ = -R^{k}_{\rho\sigma j}\delta^{\lambda}_{i} + R^{k}_{j\sigma i}\delta^{\lambda}_{\rho} - R^{k}_{\sigma\rho i}\delta^{\lambda}_{j} + R^{k}_{j\rho i}\delta^{\lambda}_{\sigma} \end{aligned} \quad (4.3.1)$$

We also showed that at any given point x in a maximally symmetric space, there exist Killing vectors for which $\xi_{\lambda}(x)$ takes any values we like, so (4.2.12) and (4.3.1) require that

$$R^{\lambda}_{\sigma\rho i;j} = R^{\lambda}_{j\rho i;\sigma} \quad (4.3.2)$$

We actually only need to use (4.3.1), because we have shown in the last section that a space is isotropic about every point and hence satisfies (4.3.1) must also be homogeneous, and hence must also satisfy (4.3.2).

Our first step in the proof is to use Eq. (4.3.1) to drive a formula for the curvature tensor. Contracting k with μ yeilds

$$-NR^{\lambda}_{\rho\sigma j} + R^{\lambda}_{\rho\sigma j} - R^{\lambda}_{\sigma\rho j} + R^{\lambda}_{j\rho\sigma} = -R^{\lambda}_{\rho\sigma j} + R_{\sigma\rho}\delta^{\lambda}_{j} - R_{j\rho}\delta^{\lambda}_{\sigma}$$

(Recall that $R^{k}_{k\sigma j}$ vanishes, $-R^{k}_{\sigma\rho k}$ is the Ricci tensor $R_{\sigma\rho}$, and in N dimensions, $\delta^{k}_{k} = N$.) Using the cyclic sum rule $R_{\lambda ij\kappa} + R_{\lambda\kappa ij} + R_{\lambda j\kappa i} = 0$ and the antisymmetry of $R^{\lambda}_{\sigma\rho j}$, we find

$$(N-1)R_{\lambda\rho\sigma j} = R_{j\rho}g_{\lambda\sigma} - R_{\sigma\rho}g_{\lambda j} \quad (4.3.3)$$

But this must be antisymmetric in λ and ρ, so

$$R_{j\rho}g_{\lambda\sigma} - R_{\sigma\rho}g_{\lambda j} = -R_{j\lambda}g_{\rho\sigma} + R_{\sigma\lambda}g_{\rho j}$$

Contracting λ with j, we find

$$R_{\sigma\rho} - NR_{\sigma\rho} = -R^{\lambda}_{\lambda}g_{\rho\sigma} + R_{\rho\sigma}$$

The Ricci tensor thus takes the form

$$R_{\sigma\rho} = \frac{1}{N}g_{\sigma\rho}R^{\lambda}_{\lambda} \quad (4.3.4)$$

Inserting this in (4.3.3) gives our formula for the curvature tensor

$$R_{\lambda\rho\sigma j} = \frac{R^{\lambda}_{\lambda}}{N(N-1)}\{g_{j\rho}g_{\lambda\sigma} - g_{\sigma\rho}g_{\lambda j}\} \quad (4.3.5)$$

This formula satisfies (4.3.1), so there is nothing further to be learned from that condition.

In a space that is isotropic about every point, equations (4.3.4) and (4.3.5) will hold everywhere, and we can use the Bianchi identities to say something about the dependence of the curvature scalar ${R^\lambda}_\lambda$ on position. Using (4.3.4) in $\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{;\mu} = 0$, we have

$$0 = \left[R^\sigma_\rho - \frac{1}{2}\delta^\sigma_\rho R^\lambda_\lambda\right]_{;\sigma} = \left(\frac{1}{N} - \frac{1}{2}\right)\left(R^\lambda_\lambda\right)_{;\sigma}$$

$$or, \qquad 0 = \left(\frac{1}{N} - \frac{1}{2}\right)\frac{\partial}{\partial x^\sigma}R^\lambda_\lambda \tag{4.3.6}$$

Hence any space of three or more dimensions, in which (4.3.4) holds everywhere, will have ${R^\lambda}_\lambda$ $constant$. It is convenient to introduce a curvature constant K in place of ${R^\lambda}_\lambda$, with

$$R^\lambda_\lambda \equiv -N(N-1)K \tag{4.3.7}$$

Using this in (4.3.4) gives the Ricci tensors and the Riemann-Christoffel tensor here as

$$R_{\sigma\rho} = -(N-1)Kg_{\sigma\rho} \tag{4.3.8}$$

$$R_{\lambda\rho\sigma j} = \mathrm{K}\{g_{\sigma\rho}g_{\lambda j} - g_{j\rho}g_{\lambda\sigma}\} \tag{4.3.9}$$

In differential geometry a space with these properties is called a **space of constant curvature**.

Incidentally, the curvature tensor in two dimensions is always of the form (4.3.5), so it is not surprising that in this case (4.3.6) does not allow us to draw any conclusions about the constancy of ${R^\lambda}_\lambda$. However, by using (4.3.2) one can show that the quantity K in (4.3.9) is also constant for maximally symmetric spaces of dimensionality$N = 2$.

Now suppose that we are given two metrics $g_{ij}(x)$ and ${g'}_{ij}(x')$, both having the same numbers of positive and negative eigenvalues, and both satisfying the condition (4.3.9) for a maximally symmetric space, that is,

$$R_{\lambda\rho\sigma j} = \mathrm{K}\left(g_{\sigma\rho}g_{\lambda j} - g_{j\rho}g_{\lambda\sigma}\right) \tag{4.3.10}$$

with the same curvature constant K. We shall show that $g_{ij}(x)$ and $g'_{ij}(x')$ must be equivalent, in the sense that there is a transformation $x \to x'$ that converts $g_{ij}(x)$ and $g'_{ij}(x')$, that is, for which

$$g'_{ij}(x')\frac{\partial x'^i}{\partial x^\rho}\frac{\partial x'^j}{\partial x^\sigma} = g_{\rho\sigma}(x)$$

We shall prove this by actually constructing $x'^i(x)$ as a power series in x^i . First, note that the equality in the numbers of positive and negative eigenvalues of g_{ij} and g'_{ij} means that we can find a nonsingular matrix $d^i{}_\rho$ for which

$$g'_{ij}(0)d^i{}_\rho d^j{}_\sigma = g_{\rho\sigma}(0) \tag{4.3.13}$$

Thus we can satisfy (4.3.12) to zero order in x with

$$x'^i = d^i{}_\rho x^\rho$$

Now we proceed by mathematical induction. Suppose that we succeed in satisfying (4.3.12) to order $n-1$ in x^i with a polynomial

$$x'^i(x) = d^i{}_\rho x^\rho + \sum_{m=2}^{n}\frac{1}{m!}d^i{}_{\rho_1\cdots\rho_m}x^{\rho_1}\dots x^{\rho_m} \tag{4.3.14}$$

We want to add a term of order $n+1$ in x^μ so that (4.3.14) holds to order n. This condition will be satisfied if the derivatives of (4.3.12) holds in order $n-1$, that is, if

$$\frac{\partial^2 x'^i}{\partial x^\rho \partial x^\lambda}\frac{\partial x'^j}{\partial x^\sigma}g'_{ij}(x') + \frac{\partial^2 x'^j}{\partial x^\sigma \partial x^\lambda}\frac{\partial x'^i}{\partial x^\rho}g'_{ij}(x') + \frac{\partial x'^i}{\partial x^\rho}\frac{\partial x'^j}{\partial x^\sigma}\frac{\partial x'^\kappa}{\partial x^\lambda}\frac{\partial g'_{ij}(x')}{\partial x'^\kappa} = \frac{\partial g_{\rho\sigma}(x)}{\partial x^\lambda} \qquad \textit{in order } x^{n-1}$$

This will be satisfied if (and, in fact, only if)

$$\frac{\partial^2 x'^i}{\partial x^\rho \partial x^\lambda}\frac{\partial x'^j}{\partial x^\sigma}g'_{ij}(x') = g_{\sigma\tau}(x)\Gamma^\tau_{\lambda\rho}(x) - \frac{\partial x'^i}{\partial x^\rho}\frac{\partial x'^j}{\partial x^\sigma}\frac{\partial x'^\kappa}{\partial x^\lambda}g'_{j\eta}(x')\Gamma'^\eta_{i\kappa}(x') \qquad \textit{in order } x^{n-1}$$

This only needs to hold in order $n-1$ in x^μ, so we can use (4.3.12), which was assumed to hold to this order, to convert it into an equivalent requirement

$$\frac{\partial^2 x'^i}{\partial x^\rho \partial x^\lambda} = \frac{\partial x'^i}{\partial x^\kappa}\Gamma^\kappa_{\lambda\rho}(x) - \frac{\partial x'^j}{\partial x^\rho}\frac{\partial x'^\kappa}{\partial x^\lambda}\Gamma'^i_{j\kappa}(x') \qquad in\ order\ x^{n-1} \qquad (4.3.15)$$

We can use (4.3.14), which is correct to order x^n, to calculate the term on the right-hand side of order x^{n-1}. Let us write the result as

$$\left[\frac{\partial x'^i}{\partial x^\kappa}\Gamma^\kappa_{\lambda\rho}(x) - \frac{\partial x'^j}{\partial x^\rho}\frac{\partial x'^\kappa}{\partial x^\lambda}\Gamma'^i_{j\kappa}(x')\right]_{order\ n-1} = \frac{1}{(n-1)!}c^i_{\lambda\rho\sigma_1\ldots\sigma_{n-1}}x^{\sigma_1}\ldots x^{\sigma_{n-1}} \qquad (4.3.16)$$

the coefficients $c^i_{\lambda\rho}$ ... depending in a complicated way on the functions $g_{ij}(x)$ and $g'_{ij}(x')$ and on the previously determined coefficients $d^i_{\rho_1\ldots\rho_m}$.Then (4.3.15) will be satisfied in order $n-1$ if we add to (4.3.14) a term

$$[x'^i(x)]_{order\ n+1} = \frac{1}{(n+1)!}c^i_{\lambda\rho\sigma_1\ldots\sigma_{n-1}}x^\lambda x^\rho x^{\sigma_1}\ldots x^{\sigma_{n-1}} \qquad (4.3.17)$$

provided that the coefficient $c^i_{\lambda\rho\sigma_1\ldots\sigma_{n-1}}$ is totally symmetric in all its lower indices. These coefficients are obviously symmetric under interchange of λ and ρ or among the σ_m indices, so the only condition that needs to be satisfied is that they are symmetric between λ and any σ_m, or equivalently, that the derivative of (4.3.16) with respect to x^σ should be symmetric between λ and σ:

$$\frac{\partial}{\partial x^\sigma}\left(\frac{\partial x'^i}{\partial x^\kappa}\Gamma^\kappa_{\lambda\rho}(x) - \frac{\partial x'^j}{\partial x^\rho}\frac{\partial x'^\kappa}{\partial x^\lambda}\Gamma'^i_{j\kappa}(x')\right) = \frac{\partial}{\partial x^\lambda}\left(\frac{\partial x'^i}{\partial x^\kappa}\Gamma^\kappa_{\sigma\rho}(x) - \frac{\partial x'^j}{\partial x^\rho}\frac{\partial x'^\kappa}{\partial x^\sigma}\Gamma'^i_{j\kappa}(x')\right) \ in\ order\ x^{n-2} \qquad (4.3.18)$$

Since (4.3.12) is assumed to hold to order x^{n-1}, its derivative, Eq. (4.3.15), will hold to order x^{n-2}, so we can use (4.3.12) and (4.3.15) to rewrite (4.3.18) as the equivalent requirement

$$\frac{\partial x'^i}{\partial x^\kappa}R^\kappa{}_{\rho\lambda\eta}(x) = \frac{\partial x'^j}{\partial x^\rho}\frac{\partial x'^\kappa}{\partial x^\lambda}\frac{\partial x'^\sigma}{\partial x^\eta}R'^i{}_{j\kappa\sigma}(x) \ \ in\ order\ x^{n-2} \qquad (4.3.19)$$

Now for the first time we use equations (4.3.10) and (4.3.11), which allow (4.3.19) to be replaced with the equivalent requirement

$$\frac{\partial x'^i}{\partial x^\eta}g_{\lambda\rho}(x) - \frac{\partial x'^i}{\partial x^\lambda}g_{\rho\eta}(x) = \frac{\partial x'^j}{\partial x^\rho}\left(\frac{\partial x'^\kappa}{\partial x^\lambda}\frac{\partial x'^i}{\partial x^\eta}g'_{j\kappa}(x') - \frac{\partial x'^i}{\partial x^\lambda}\frac{\partial x'^\sigma}{\partial x^\eta}g'_{j\sigma}(x')\right) \ in\ order\ x^{n-2} \qquad (4.3.20)$$

This condition is satisfied, because (4.3.12) was assumed to hold to order

x^{n-1}. To recapitulate, this implies that (4.3.19) holds in order x^{n-2}, which implies that (4.3.18)holds in orderx^{n-2}, which implies that the coefficients $c^{i}_{\lambda\rho\sigma_1\ldots\sigma_m}$are totally symmetric in their lower indices, which implies that (4.3.17) satisfies (4.3.15), which implies that by adding (4.3.17) to (4.3.14) we can satisfy (4.3.12) to order x^n. Thus, if (4.3.12) can be satisfied to order x^{n-1} by a polynomial $x^{'}(x)$ of order n, it can be satisfied to order x^n by a polynomial $x^{'}(x)$ of order $n+1$, and therefore a function $x^{'}(x)$ satisfying (4.3.12) exactly can be built up as a power series, as was to be proven.

4.4 MAXIMALLY SYMMETRIC SPACES: CONSTRUCTION

Maximally symmetric spaces are essentially unique, so we can learn all about them by constructing examples with arbitrary curvature K in any way we like.

This is one rather obvious way to carry out this construction. (See Figure 4.1) Consider a flat $(N+1)$-dimensional space, with metric given by

$$-d\tau^2 \equiv g_{AB}dx^A dx^B = C_{ij}dx^i dx^j + K^{-1}dz^2 \qquad (4.4.1)$$

where C_{ij} is a constant $N \times N$ matrix and K is some constant. We can embed a non-Euclidean N-dimensional space in this larger space by restricting the variables x^i and z to the surface of a sphere (or pseudosphere) :

$$KC_{ij}x^i x^j + z^2 = 1 \qquad (4.4.2)$$

On this surface, dz^2 is given by

$$dz^2 = \frac{K^2\left(C_{ij}x^i dx^j\right)^2}{z^2}$$

$$= \frac{K^2\left(C_{ij}x^i dx^j\right)^2}{\left(1 - KC_{ij}x^i x^j\right)}$$

and therefore (4.4.1) gives

$$-d\tau^2 = C_{ij}dx^i dx^j + \frac{K\left(C_{ij}x^i dx^j\right)^2}{\left(1 - KC_{\rho\sigma}x^\rho x^\sigma\right)} \qquad (4.4.3)$$

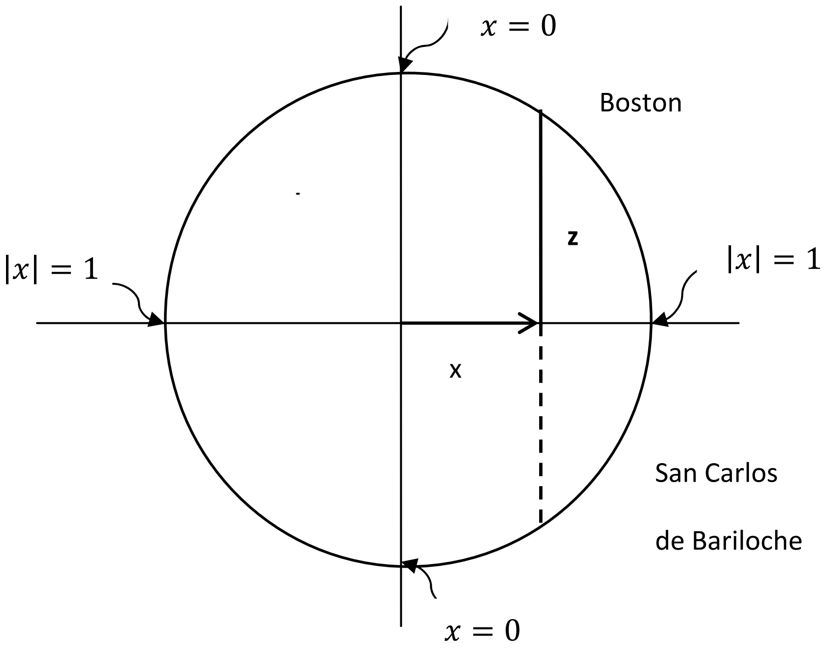

Fig: 4.1 Representation of points on a sphere by projection onto the equatorial plane. Note that two points on a sphere correspond to each projected point with given coordinates x^i.

The metric is then

$$g_{ij}(x) = C_{ij} + \frac{K}{\left(1 - KC_{\rho\sigma}x^{\rho}x^{\sigma}\right)} C_{i\lambda}x^{\lambda}C_{jk}x^{k} \tag{4.4.4}$$

A flat space appears here as the special case $K = 0$.

This construction makes it obvious that (4.4.4) admits an $(N(N+1)/2)$ parameter group of isometries, for both the $(N+1)$-dimensional line element (4.4.1) and the "**embedding**" condition (4.4.2) are manifestly invariant under rigid "**rotations**" of the $(N+1)$-dimensional space, that is, under the transformations

$$x^i \to x'^i = R^i_j x^j + R^i_z z \tag{4.4.5}$$

$$z \to z' = R^z_i z^i + R^z_z z \tag{4.4.6}$$

where the $R^A{}_B$ are constant, with

$$C_{ij}R^i_\rho R^j_\sigma + K^{-1}R^z_\rho R^z_\sigma = C_{\rho\sigma} \tag{4.4.7}$$

$$C_{ij}R^i_\rho R^j_z + K^{-1}R^z_\rho R^z_z = 0 \tag{4.4.8}$$

$$C_{ij}R^i_z R^j_z + K^{-1}(R^z_z)^2 = K^{-1} \tag{4.4.9}$$

It is convenient to distinguish two classes of simple transformations satisfying (4.4.7) − (4.4.9)

(A) $$R^i_j = \mathcal{R}^i_j \qquad R^i_z = R^z_i = 0 \qquad R^z_z = 1 \tag{4.4.10}$$

where $\mathcal{R}^i_j$ is any $N \times N$ matrix with

$$C_{ij}\mathcal{R}^i_\rho \mathcal{R}^j_\sigma = C_{\rho\sigma} \tag{4.4.11}$$

These are just rigid "rotations" about the origin:

$$x'^i = \mathcal{R}^i_j x^j \tag{4.4.12}$$

(B) $$R^i_z = a^i \quad R^z_i = -KC_{ij}a^j \quad R^z_z = \left(1 - KC_{\rho\sigma}a^\rho a^\sigma\right) \tag{4.4.13}$$

$$R^i_j = \delta^i_j - bKC_{j\rho}a^\rho a^i \tag{4.4.14}$$

where a^i is arbitrary except that $R^z{}_z$ must be real, that is,

$$KC_{\rho\sigma}a^\rho a^\sigma \leq 1 \tag{4.4.15}$$

and

$$b \equiv \frac{1 - \left(1 - KC_{\rho\sigma}a^\rho a^\sigma\right)^{1/2}}{KC_{\rho\sigma}a^\rho a^\sigma} \tag{4.4.16}$$

These are "**quasitranslations**" with

$$x'^i = x^i + a^i\left[\left(1 - KC_{\rho\sigma}x^\rho x^\sigma\right)^{1/2} - bKC_{\rho\sigma}x^\rho a^\sigma\right] \tag{4.4.17}$$

In particular, these transformations take the origin $x^i = 0$ into a^i.

This existence of isometries (4.4.7) that take the origin into any point (at least within a finite region) means that this space is ***homogeneous***; any point is geometrically like any other point. (Our coordinate system hides this property, just as a polar projection map of the earth hides the fact that the curvature of the earth is about the same in Massachusetts as at the North Pole.) also, the existence of isometries (4.4.10) that include all rigid "**rotations**" about the origin means that this space is *isotropic* about the origin. Since the metric is homogeneous, and isotropic about the origin, it is isotropic about every point, and maximally symmetric.

We can construct the Killing vectors for this metric by letting the finite transformations (4.3.5), (4.3.6) approach the unit transformation. First, consider the transformations (A), and let

$$\mathcal{R}^i_j = \delta^i_j + \varepsilon\Omega^i_j, \qquad |\varepsilon| \ll 1$$

$$C_{i\sigma}\Omega^i_\rho + C_{\rho i}\Omega^i_\sigma = 0 \tag{4.4.18}$$

Comparing with (4.2.3), the corresponding Killing vectors are

$$\xi^i_\Omega(x) = \Omega^i_j x^j \tag{4.4.19}$$

Next, consider the transformations (B), and let

$$a^i = \varepsilon\alpha^i, \ \ |\varepsilon| \ll 1$$

Comparing with (4.2.3), the corresponding Killing vectors are

$$\xi^i_\alpha(x) = \alpha^i\left[1 - KC_{ij}a^i a^j\right]^{1/2} \tag{4.4.20}$$

There are $N(N-1)/2$ independent parameters $\Omega^i{}_j$ [that is, N^2 elements $\Omega^i{}_j$, subject to the $N(N+1)/2$ conditions (4.4.18) and N parameters a^i, so this metric admits $N(N+1)/2$ independent Killing vectors, verifying maximal symmetry.

The geodesics of this metrics take a remarkably simple form. From (4.4.4) we can readily calculate that the affine connection is

$$\Gamma^i_j = Kx^i g_{j\lambda} \tag{4.4.21}$$

so the differential equation for a geodesic is

$$\frac{d^2x^i}{d\tau^2} + Kx^i = 0 \tag{4.4.22}$$

The solutions are thus linear combinations of $\tau\sqrt{K}$ and $cos\left(\tau\sqrt{K}\right)$ for $K > 0$, or of $sinh\left(\tau\sqrt{-K}\right)$ and $cosh\left(\tau\sqrt{-K}\right)$ for $K < 0$.

We can uncover the inner properties of this space by calculating the curvature tensor; a straightforward computation gives the Riemann-Christoffel tensor for the metric (4.4.4) as

$$R_{\kappa j\rho\sigma} = K\left[C_{\kappa\sigma}C_{j\rho} - C_{\kappa\rho}C_{j\sigma}\right] + K^2\left[1 - KC_{i\lambda}x^i x^\lambda\right]^{-1}\left[C_{\kappa\sigma}x_j x_\rho - C_{\kappa\rho}C_{\kappa\sigma}x_j x_\sigma + C_{j\rho}x_\kappa x_\sigma - C_{j\sigma}x_\rho x_\kappa\right]$$

(where $X_j \equiv C_{ji}X^i$), or

$$R_{\kappa j\rho\sigma} = K\left[g_{\rho j}g_{\kappa\sigma} - g_{\sigma j}g_{\kappa\rho}\right]$$

in agreement with Eq. (4.3.9).

Hence the constant K introduced in Eqs (4.4.1) and (4.4.2) is the same as the curvature constant introduced in the last section.

Since K is an invariant parameter, we cannot by a coordinate transformation convert the metric (4.4.4) into a similar metric with a different K.

In contrast, Eq. (4.4.3) makes it obvious that by a linear transformation

$$x^i = A^i_j x'^j$$

we can convert the metric (4.4.4) into a similar metric with the same K and with C_{ij} changed into

$$C'_{ij} = A^\rho_i A^\sigma_j C_{\rho\sigma}$$

In this way C_{ij} can be changed into any real symmetric matrix we like, as long as we do not change the numbers of its positive and negative eigenvalues. Also, the numbers of eigenvalues of each sign of the matrix C_{ij} are the same as for the matrix g_{ij} at the point $x = 0$, and hence the same everywhere, since all points are equivalent.

An N-dimensional metric that allows the introduction of locally Euclidean (as opposed, say, to Minkowskian) coordinate systems will have all its eigenvalues positive, so for $K \neq 0$ we can take C_{ij} as $|K|^{-1}$ times the unit matrix, in which case (4.4.3) becomes

$$ds^2 = K^{-1}\left[dx^2 + \frac{(x.dx)^2}{1 - x^2}\right] \qquad for\ K > 0 \qquad (4.4.23)$$

or,

$$ds^2 = |K|^{-1}\left[dx^2 + \frac{(x.dx)^2}{1 - x^2}\right] \qquad for\ K > 0 \qquad (4.4.24)$$

For $K = 0$, we take C_{ij} as just the unit matrix, and (4.4.3) gives

$$ds^2 = dx^2 \quad \text{for } K = 0 \tag{4.4.25}$$

(We are using an obvious N-dimensional vector notation. Also, we have replaced $-d\tau^2$ with a proper length ds^2, because for the moment we are doing geometry rather than physics.) Let us explore the global properties of these space.

For $K > 0$, our most convenient approach is to go back to the interpretation of (4.4.23) as the metric of the curved space embedded by Eq.(4.4.2) in the flat space (4.4.1); that is, (4.4.23) describes the surface

$$x^2 + z^2 = 1 \tag{4.4.26}$$

in the flat space with

$$ds^2 = K^{-1}[dx^2 + dz^2] \tag{4.4.27}$$

Obviously this metric simply describes the surface of a sphere of radius $K^{-1/2}$ in an $(N+1)$-dimensional Euclidean space. (To make the coordinates x and z truly Euclidean, we should define $x' = K^{-1/2}x$ and $z' = K^{-1/2}z$, in which case (4.4.26) reads $x'^2 + z'^2 = K^{-1}$.) Indeed, in two dimensions we can introduce angular coordinates θ, φ by:

$$x_1 = sin\theta cos\varphi \quad x_2 = sin\theta sin\varphi$$

and (4.4.27) then becomes the familiar line element on a sphere of radius $K^{-1/2}$:

$$ds^2 = K^{-1}[d\theta^2 + sin^2\theta d\varphi^2] \tag{4.4.28}$$

In general, the range of the variables x is

$$x^2 \leq 1$$

However, each x actually corresponds to *two* point, corresponding to the two roots of Eq. (4.4.26) for z. (For instance, in two dimensions the components of x are the coordinates of points on a sphere projected on a tangent plane; in a polar projection map of the earth, Boston will appear at the same point as San Carlos de Bariloche, Argentina.) The volume of the N-dimensional space described by (4.4.23) is therefore

$$V_N = 2\int_{x^2\le 1}\sqrt{g}dx_1 \dots dx_N = 2K^{-N/2}\int_{x^2\le 1}\frac{dx_1 \dots dx_N}{[1-x^2]^{1/2}}$$

A straightforward calculation gives

$$V_N = \frac{2\pi^{(N+1)/2}}{\Gamma((N+1)/2)}K^{-N/2}$$

For instance, $V_1 = 2\pi K^{-\frac{1}{2}}$, which is just the perimeter of a circle of radius $K^{-1/2}$ and $V_2 = 4\pi K^{-1}$, which is just the area of a sphere with radius $K^{-1/2}$. A three-dimensional space of constant positive curvature has the volume

$$V_3 = 2\pi^2 K^{-3/2}$$

We can also calculate the circumference of such spaces, using for the geodesics the solutions of Eq. (4.4.22), which now reads

$$\frac{d^2x}{ds^2} + Kx = 0 \qquad (4.4.30)$$

The solutions that pass through the point $x = 0$ are

$$x = esin(sK^{1/2}) \qquad (4.4.31)$$

where, in order to satisfy (4.4.23),

$$e^2 = 1 \qquad (4.4.32)$$

As we go out along a geodesic from the "**North pole**"$x = 0$, we reach the "**Equator**" $x = e$ at $s = \pi K^{-1/2}/2$, we reach the "**South pole**" $x = 0$ at $s = \pi K^{-1/2}$, we reach the opposite point $x = -e$ of the "**equator**" at $s = 3\pi K^{-1/2}/2$, and we return to our starting point at $s = 2\pi K^{-1/2}$.

Thus the distance from any point around the whole space and back to itself along a geodesic is

$$L = 2\pi K^{-1/2} \qquad (4.4.33)$$

for spaces of constant positive curvature and arbitrary dimensionally. This calculation shows very clearly that the space described by (4.4.23) is *finite*, but it is not *bounded*; when we come to the apparent singularity at $x^2 = 1$, we continue right through, but with z given by the root of Eq. (4.4.26) of opposite sign.

For $K < 0$ the metric (4.4.24) does not even have an apparent singularity, and there is nothing to restrict the coordinates x to any finite range. This can be seen even more definitely by calculating the geodesics, which are now given by Eqs. (4.4.30) and (4.4.24) as

$$x = esinh\left(s(-K)^{1/2}\right) \tag{4.4.34}$$

$$e^2 = 1 \tag{4.4.35}$$

We can obviously go out along this geodesic an unlimited distance from the origin. For $N = 2$, this space is just that discovered by Gauss, Bolyai, and Lobachevski. In order to put the metric in the form of Klein's model, it is necessary to introduce a new set of coordinates x'^i, defined by $x' = x(1 + x^2)^{-1/2}$.] We see from (4.4.1) and (4.4.2) that this geometry describes the surface

$$-x^2 + z^2 = 1 \tag{4.4.36}$$

in a flat space, with

$$ds^2 = [K]^{-1}[dx^2 - dz^2] \tag{4.4.37}$$

The minus sign in (4.4.7) means that this flat space is not Euclidean. It is therefore understandable that the Gauss-Bolyai-Lobachevski geometry could not be discovered until geometers had learned to think of curved surfaces, not as sub-spaces of an ordinary Euclidean space, but as space characterized by their own inner metric relations.

Finally, let us return to space-time, and consider the structure of a four-dimensional maximally symmetric metric with three positive and one negative eigenvalue. In this case, we can set

$$C_{ij} = \eta_{ij} \tag{4.4.38}$$

and the metric is

$$-d\tau^2 = dx^2 - dt^2 + \frac{K(x.dx - tdt)^2}{1 - K(x^2 - t^2)} \tag{4.4.39}$$

For $K > 0$, we can introduce coordinates in which the metric appears *spatially* flat, by setting

$$t = \frac{1}{\sqrt{K}}\left[\frac{Kr'^2}{2}cosh\left(K^{1/2}t'\right) + \left(1 + \frac{Kr'^2}{2}\right)sinh\left(K^{1/2}t'\right)\right]$$

$$x = x' exp\left(K^{1/2} t'\right) \quad (4.4.40)$$

Then (4.4.39) becomes

$$d\tau^2 = dt'^2 - exp\left(2K^{1/2} t'\right) dx'^2 \quad (4.4.41)$$

We can also introduce coordinates in which the metric appears time-independent, by setting

$$t'' = t' - \frac{1}{2K^{1/2}} ln\left[1 - Kx'^2 exp\left(2K^{1/2} t'\right)\right]$$

$$x'' = x' exp\left(2K^{1/2} t'\right) \quad (4.4.42)$$

Then (4.4.41) becomes

$$d\tau^2 = (1 - Kx''^2) dt'^2 - dx''^2 - \frac{K(x''.dx'')^2}{1 - Kx''^2} \quad (4.4.43)$$

This metric was first discussed in this form by de-Sitter.

Once again, it should be stressed that the maximally symmetric metric (4.4.4), although derived by an apparently arbitrary procedure, actually represents the most general possible maximally symmetric metric, because the uniqueness theorem of the last section tells us that any other maximally symmetric metric can be converted into the form (4.4.4) by a suitable coordinate transformation.

4.5 TENSOR IN A MAXIMALLY SYMMETRIC SPACE

The assumption of maximal symmetry can be applied, not only to the metric of a space, but to any tensor fields that inhabit the space. a tensor field $T_{ij...}$ is said to be **form invariant** under a transformation $x \rightarrow x'$if $T'_{ij...}(x')$ is the same function of its argument x'^i as $T_{ij...}(x)$ was of its argument x^i, that is,

$$T'_{ij...}(y) = T_{ij...}(y) \qquad for\ all\ y \quad (4.5.1)$$

At any given point, the transformed tensor is given by the usual formula

$$T_{ij...}(x) = \frac{\partial x'^\rho}{\partial x^i} \frac{\partial x'^\sigma}{\partial x^j} \ldots T'_{\rho\sigma...}(x')$$

so the form-invariance condition (4.5.1) reads

$$T_{ij...}(x) = \frac{\partial x'^{\rho}}{\partial x^{i}} \frac{\partial x'^{\sigma}}{\partial x^{j}} \ldots T_{\rho\sigma...}(x') \qquad (4.5.2)$$

For an infinitesimal transformation

$$x'^{i} = x^{i} + \varepsilon \xi^{i}(x) \quad |\varepsilon| \ll 1$$

the condition (4.5.2) becomes, to first order in ε

$$0 = \frac{\partial \xi^{\rho}(x)}{\partial x^{i}} T_{\rho j}(x) + \frac{\partial \xi^{\sigma}(x)}{\partial x^{j}} T_{i\sigma...}(x) + \cdots + \xi^{\lambda}(x) \frac{\partial}{\partial x^{\lambda}} T_{\lambda...}(x) \qquad (4.5.3)$$

(That is, the *Lie derivative* of $T_{ij...}$ with respect to ξ^{λ} vanishes)

A tensor in a maximally symmetric space, which satisfies (4.5.3) for all $N(N+1)/2$ independent Killing vectors $\xi^{\lambda}(x)$, will be called **maximally form-invariant.**

For a scalar $S(x)$, Eq. (4.5.3) reads simply

$$\xi^{\lambda}(x) \frac{\partial}{\partial x^{\lambda}} S(x) = 0 \qquad (4.5.4)$$

If the scalar is maximally form-invariant, then $\xi^{\lambda}(x)$, can at any given point be choose to have any value we like, and (4.5.4) therefore requires that S be constant:

$$\frac{\partial S}{\partial x^{\lambda}} = 0 \qquad (4.5.5)$$

For any other maximally form-invariant tensor, it is convenient first to chosen a Killing vector $\xi^{\lambda}(x)$ that at a given point X satisfies

$$\xi^{\lambda}(x) = 0$$

and for which the quantities

$$\xi_{\sigma;i}(X) = g_{\sigma\rho}(X) \left(\frac{\partial \xi^{\rho}(x)}{\partial x^{i}} \right)_{x=X}$$

from an arbitrary anti-symmetric matrix. Equation (4.5.3) then reads, at $x = X$:

$$0 = \xi_{\sigma;i} \{ \delta_{i}^{\tau} T_{j}^{\sigma} \ldots + \delta_{j}^{\tau} T_{i}^{\sigma} \ldots + \cdots \}$$

Since $\xi_{\sigma;\tau}$ is an arbitrary antisymmetric matrix, its coefficient must be symmetric in σ and τ:

$$\delta_i^\tau T_j^\sigma \dots + \delta_j^\tau T_i^\sigma \dots + \cdots = \delta_i^\sigma T_j^\tau \dots + \delta_j^\sigma T_i^\tau \dots + \cdots \quad (4.5.6)$$

Since X was arbitrary, this must hold everywhere.

For a maximally form-invariant vector $A_i(x)$, Eq. (4.5.6) reads

$$\delta_j^\tau A^\sigma = \delta_i^\sigma A^\tau$$

Contracting τ with i, we find that in N dimensions

$$NA^\sigma = A^\sigma$$

so, except for the trivial case $N = 1$,we must have

$$A^\sigma = 0 \quad (4.5.7)$$

For a maximally form-invariant tensor B_{ij} of second rank, Eq. (4.5.6) reads

$$\delta_i^\tau B_j^\sigma + \delta_j^\tau B_i^\sigma = \delta_i^\sigma B_j^\tau + \delta_j^\sigma B_i^\tau$$

Contracting τ with i gives

$$NB_j^\sigma + B_j^\sigma = B_j^\sigma + \delta_j^\sigma B_i^i$$

or, lowering the σ index,

$$(N-1)B_{\sigma j} + B_{j\sigma} = g_{\sigma j} B_i^i \quad (4.5.8)$$

Subtracting the same equation with j and σ interchanged yields

$$(N-2)\left(B_{\sigma j} - B_{j\sigma}\right) = 0$$

so as long as $N \neq 2$, the tensor $B_{\sigma j}$ must be symmetric:

$$B_{\sigma j} = B_{j\sigma} \quad (4.5.9)$$

(In two dimensions, $B_{\sigma j}$ can have an antisymmetric part proportional to $g^{-1/2}\varepsilon_{\sigma j}$). Using (4.5.9) in (4.5.8) gives now for $N \geq 3$ (and for the symmetric part of $B_{\sigma j}$ for $N = 2$)

$$B_{\sigma j} = f g_{\sigma j} \quad (4.5.10)$$

where

$$f \equiv \frac{1}{N} B_i^i$$

To determine the dependence of f on the coordinates, we can use (4.5.10) back in the form-invariance condition (4.5.3):

$$0 = \frac{\partial \xi^\rho}{\partial x^i} f g_{\rho j} + \frac{\partial \xi^\sigma}{\partial x^j} f g_{i\sigma} + \xi^\lambda \frac{\partial}{\partial x^\lambda}\left(f g_{\rho j}\right)$$

But $g_{\mu\varphi}$ satisfies the Killing condition (4.2.4), so this becomes

$$0 = g_{ij}\xi^\lambda \frac{\partial f}{\partial x^\lambda}$$

In a maximally symmetric space we can at any given point ξ^λ to have any value we like, and therefore

$$\frac{\partial f}{\partial x^\lambda} = 0 \qquad (4.5.11)$$

Thus ***the only maximally form-invariant tensor of second rank is the metric tensor, times a possible constant.***

"Although we have omitted the singular points from the definition of space time, we can still recognize the 'holes' left where they have been cut by the existence of incomplete geodesics."

STEPHEN HAWKING

CHAPTER 5

SPHERICALLY SYMMETRIC SCHWARZSCHILD SOLUTION AND ITS PROPERTIES

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5.1 INTRODUCTION

The purpose of this chapter is to discuss several useful space-times are exact solution of Einstein equations and to review some of other properties. In this connection we have accommodated here the symmetry, invariance property and conservation laws which Euclid implicitly assumed that metric relations are unaffected by transformations or rotations. The gravitational fields do not usually have such a high degree of symmetry, but they often admit some group of approximate symmetry transformation and when they do, we can use this information to help solve the Einstein's equations or even to do without a solution.

The Einstein's equations form a highly non-linear system of differential equations and due to their complexity, a completely general solution is not known. Thus, the known exact solutions usually assume a rather high degree of symmetry such as the spherical or axial symmetry, and the existence of necessary killing vector fields on the space-time, and to that external represent an idealized situation. However, such space-time examples provide a good idea of what is possible within the frame work of the general theory of relativity. The spherically symmetric and asymptotically flat space-times outside the sun and stars and could be used to obtain conclusions relevant for the experimental verification of the general theory of relativity. Such solutions could also possibly represent the outcome of a complete gravitational collapse of a massive star. The other models discussed here also have interesting implications, particularly on the issue of the fate of gravitational collapse and cosmology. A large number of exact solutions to the Einstein equations are known which are obtained under various symmetry conditions and studied mostly locally. The examples discussed here aim at either studying certain global properties of interest or reviewing results.

5.2 SYMMETRY AND CONSERVATION LAWS

Spherical symmetry has played an important role in the development of general relativity. The exact solution of Einstein field equation which provided the decisive experimental verification of the theory, namely the Schwarzschild external solution and the Robertson-Walker- Friedman cosmological solution, were found under the assumption that space time was spherically symmetric if colloquially expressed, it is possible to rotate it leaving its metric unchanged. In more precise terms for every rotation R (a 3×3 rotation matrix) in the rotation group SO, there is an isometry of the space time φ(R) and the isometrics, constitute what is called an action of the isometrics corresponds of the composition of the corresponding rotations.

$$\varphi(R)\varphi(S) = \varphi(RS) \tag{5.2.1}$$

In analytic mechanics, one knows the symmetries of a Lagrangian or Hamiltonian result in conservation laws. That is, there is a conserved quantity, wherever symmetry exists. These general principles also exists in the general theory of relativity and are used to deduce, from the symmetries of Schwarzschild space-time, constants of motion for the trajectories of freely falling particles in the gravitational field outside a star. The same constants of motion are obtained in a differential geometry, where a killing vector is the standard tool for the description of symmetry.

In general the spherical symmetry of a space-time can be defined vigorously in terms of the killing vectors; there must be three linearly independent space like killing vector fields x^1, x^2 and x^3 in the space time which satisfy the commutator relations,

$$[X^1, X^2] = X^3, [X^2, X^3] = X^1, [X^3, X^1] = X^2$$

And their orbits must be closed. Using the properties, one could then again derive line element for a spherically symmetric space-time.

5.3 THE CENTRALLY SYMMETRIC GRAVITATIONAL FIELD

Let us consider a gravitational field possessing central symmetry. The central symmetry of the field means that the space-time metric, that is, the expression for the interval ds, must be the same for all points located at the same distance from the centre. In Euclidean space this distance is equal to the radius vector, in a non Euclidean space, such as we have in the presence of a gravitational field, there is no quantity which has all the properties of the Euclidean radius vector (for example to be equal both to the distance from the centre and to the length of the circumference divided by 2π). Therefore, the choice of a "**radius vector**" is now arbitrary.

Spherical symmetric means that there exists a privileged point called the origin, such that the system is invariant under spatial rotations about O. If time is fixed and consider a point p at a distance a from O travels a 2-sphere centered on O and in produce an axial coordinate ϕ and azimuthally coordinate θ on the sphere. Then the coordinate ranges will cover all points on the 2-sphere $0 \leq \theta \leq \pi$ and $-\pi \leq \phi \leq \pi$.

Then the line element of the 2-sphere is

$$ds^2 = a^2(d\theta^2 + sin^2\theta d\phi^2) \tag{5.3.1}$$

The line reduces to the form (5.3.1) on a 2-sphere $t = constant, r = constant$. Spherical symmetry requires that, the line element does not vary when θ and ϕ are varied, so that θ and ϕ only occur in the line element in the form $(d\theta^2 + sin^2\theta d\phi^2)$.

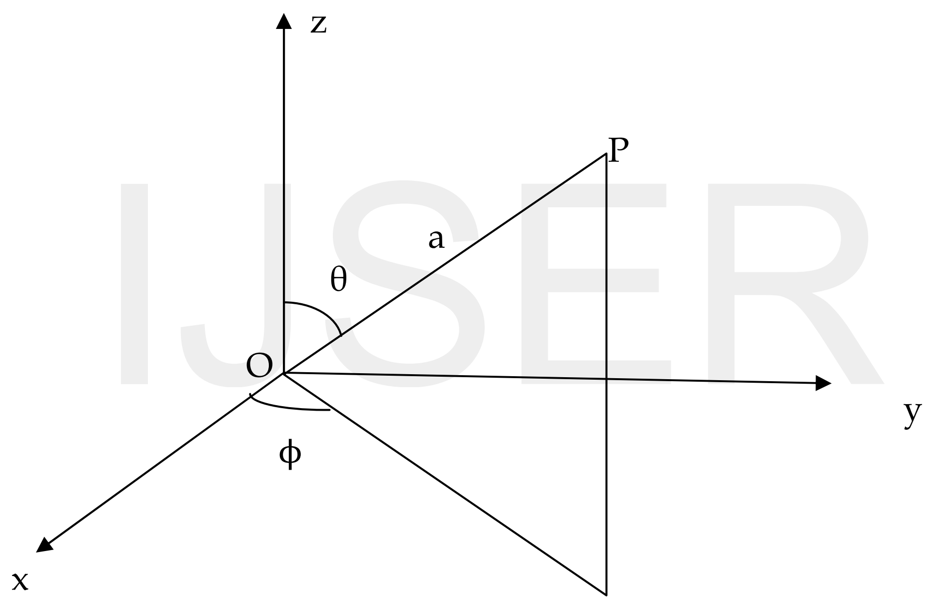

Fig: 5.1 The standard spherical co-ordinate θ and ϕ

Now there exists a spatial coordinate system

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$$

In which the line element has the form of the most general centrally symmetric expression for ds^2 as

$$ds^2 = h(r,t)dr^2 + k(r,t)(d\theta^2 + sin^2\theta d\phi^2) + l(r,t)dt^2 + a(r,t)drdt \tag{5.3.2}$$

Where a, h, k, l are certain functions of the radius vector r and the time t, but because of the arbitrariness in the choice of a reference system in the general theory of relativity. We can still subject the coordinates to any transformation which does not destroy the central symmetry of ds^2; this means that we can transformation the coordinate to the formulas $r = f_1(r', t'),\ t = f_2(r', t')$ where f_1, f_2 are any function of the new coordinate r′, t′. Making use of these possibilities, we choose the coordinate r and the time t in such a way that first of all the coefficient $a(r, t)$ of dr, dt in the expression of ds^2 vanishes and secondly the coefficient $k(r, t)$ becomes equal simply to $-r^2$. The latter condition implies that the radius vector r is defined in such a way that the circumference of a circle with centre of the origin of coordinate is equal to $2\pi r$(the element of arc of a circle in the plane $\theta = \frac{\pi}{2}$ is equal to $dl = rd\theta$. It will be convenient to write the quantities h and l is exponential form as $-e^{\lambda} and\ e^{\nu}$respectively, where λ and v are some functions of r and t. Thus we obtain the following expression for

$$ds^2 = e^{\nu}dt^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) - e^{\lambda}dr^2 \qquad (5.3.3)$$

Where, $\nu = \nu(t, r),\ \lambda = \lambda(t, r)$.

5.4 SCHWARZSCHILD GEOMETRY

The Schwarzschild solution represents the geometry exterior to a spherically symmetric massive body such as a star and has been used extensively to verify the predictions of the general theory of relativity experimentally. This is the empty exterior solution where the Ricci tensor vanishes and which is matched at the boundary to the interior solution inside the body. In (t, r, θ, ϕ)coordinates, the metric can be given in the form

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \qquad (5.4.1)$$

Where, $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$. Here the coordinate t is time-like and the other three coordinates r, θ, ϕ are space-like. The radial coordinate r has the property that the two-sphere given by $t = const., r = const.$ has the two-metric given by

$$ds^2 = r^2(d\theta^2 + sin^2\theta d\phi^2).$$

It follows that the area of any such two-sphere would be $4\pi r^2$.

The coordinate r is restricted by the condition $r > 2m$ because the above metric has an apparent singularity at $r = 2m$. The coordinate t has the range $-\infty < t < \infty$. The solution above is generated by solving the vacuum Einstein field equations for a spherically symmetric space-time and the quantity m appears as the constant of integration. The value of this constant can be determined by considering the weak field Newtonian limit of general relativity. If ϕ is the Newtonian gravitational potential, then in non-relativistic units,

$$g_{00} \simeq 1 + \frac{2\phi}{c^2} = 1 - \frac{2GM}{c^2 r},$$

Where, G is the Newtonian constant of gravity, c is the velocity of light, and M is the point mass at the origin which gives rise to the Newtonian potential ϕ. This determines the constant of integration m in the Schwarzschild solution as

$$m = \frac{GM}{c^2}.$$

Thus, the Schwarzschild solution is interpreted as describing the gravitational field of a point particle with mass m (in relativistic units, $G = c = 1$) situated at the center.

Apart from predicting small observable departures from the Newtonian gravity, the Schwarzschild solution of Einstein equations is important for the theory of black holes as well. Sufficiently massive stars unable to support themselves against the pull of self gravity must undergo a complete gravitational collapse when they have exhausted their internal nuclear fuel. The final fate of a spherically symmetric homogeneous dust collapse must be a Schwarzschild configuration which contains a space-time singularity hidden within the event horizon. This gives rise to a black hole in the space-time, which is a region from which no causal signals can reach a far-away observer. This scenario forms the basis of much of the theory and applications of the modern black hole physics.

The Schwarzschild metric is static in the sense that $\partial/\partial t$ is a timelike Killing vector which is a gradient. The metric components g_{ij} here are independent of time. Also, there are no mixed terms in equation (5.4.1) involving both space and time; hence there is no rotation inherent in the space-time. To make this more precise, we discuss here briefly the stationary and static solutions of the field equations briefly. A solution will be called stationary if the time does not enter explicitly in the metric potentials. In such a case, a coordinate system will exist in which the metric components will be time-independent; that is, if x^0 is the timelike coordinate,

$$\frac{\partial g_{ij}}{\partial x^0} = 0$$

Defining a vector $X^i = \delta^i_0$, it is seen that the Lie derivative $L_x g_{ij} = 0$ in this coordinate system. Since this is a tensor, it follows that this Lie derivative will vanish in all coordinate systems and hence X^i is a killing vector. On the other hand, if the space time admits a time-like killing vector field, then it is possible to choose a coordinate systems admits a time-like Killing vector field, then it is possible to choose a coordinate system adapted to it such that the Lie derivative $L_x g_{ij} = 0$ in this frame. Then, the metric is again stationary. Thus, a space-time is called **stationary** if and only if it admits the existence of a time-like Killing vector field.

We note that if a space time is stationary, that does not mean that the metric components cannot evolve in time. It is just the time does not enter explicitly in the solution. However, the stronger requirement of staticity means that there is no time evolution of the system, which is time-symmetric about any origin of time. In such a case, one would expect that in the coordinate system adapted to the time-like Killing vector field, the metric would admit no cross terms as well such as g_{ij} with $i \neq j$. The reason is, in such a case under a time reversal $x^0 \to -x^0$, the sign of those pieces of ds^2 containing the cross terms in g_{ij} will be reversed. However, the staticity assumption means that ds^2 must remain invariant under time reversal about any origin of time. This implies that the cross terms must vanish in the expression for ds^2. Thus, a **static** space-time is characterized by the existence of a time-like Killing vector field for the space-time, and the additional requirement that in the coordinate system adapted to this vector field the metric components are time independent and no cross terms appear in the line element ds^2. Such a property of the Killing vector field is characterized by its being hypersurface orthogonal. (A vector field X^i is called **hypersurface orthogonal** if and only if $X_{[\ i}\nabla_j X_{k]} = 0$.)Thus, a space time is static if and only if it admits a timelike Killing vector field which is hypersurface orthogonal. It is also possible to check directly from the form of the metric (5.4.1) that the metric is time symmetric under a change $t \to -t$ and is also invariant under time translations.

As pointed out above, the Schwarzschild metric (5.4.1) above is the solution of the vacuum Einstein equations with the assumption of spherical symmetry on the space time. It is clear that the coordinate system (t, r, θ, ϕ)provides a frame in which the metric components are time independent. Thus the solution is stationary. Further, defining the vector field with the components

$$X_i = (1 - 2m/r, 0,0,0),$$

which can be seen to be hypersurface orthogonal to the family of space-like hypersurface $t =$ const. thus the solution is seen to be static. We thus have the **Birkhoff theorem** (1923), namely that a spherically symmetric vacuum solution of the Einstein equations must be necessarily static. An important implication of this theorem is that even when a spherically symmetric star undergoes pulsations or changes in shape, while maintaining the spherical symmetry, it cannot radiate any disturbances in the exterior such as gravitational waves. It is thus shown by the Birkhoff theorem that any spherically symmetric solution of Einstein equations with $R_{ij} = 0$ is necessarily the Schwarzschild solution. Hence, the Schwarzschild exterior solution can be used to describe the outside metric for several situations such as a spherically symmetric star which is either static or which undergoes radial pulsations , or a radial spherically symmetric gravitational collapse.

The spherical symmetry of the Schwarzschild space time M is exhibited by the fact that the metric components g_{00} and g_{11} are functions of r alone and not of θ and ϕ, and as implied by the angular part of the metric. Specifically, the isometry group of M contains a subgroup which is isomorphic to the group $SO(3)$ and the orbits of this subgroup are two-dimensional spheres (see for example, Hawking and Ellis, 1973). These isometries are interpreted as rotations and thus the metric remains invariant under rotations in general for any spherically symmetric space-time. The parameter m here serves as the source of the gravitational field and setting $m = 0$ gives the flat Minkowski space time. As pointed out above, the comparison with Newtonian theory shows that m is to be treated as the gravitational mass of the body producing the field as measured from infinity. The space time here is asymptotically flat because as r tends to infinity we recover a flat space time metric and the gravitational field diminishes to zero.

Generally, equation (5.4.1) is taken to represent the outside metric for a star with $r > r_0$ where r_0 gives the boundary of the star. The metric inside $r < r_0$ is a different interior metric determined by the matter distribution T_{ij} inside the star and is matched at the boundary $r = r_0$ with equation (5.4.1).

However, in the case of a complete gravitational collapse, when all the mass collapse at $r = 0$, it is necessary to consider the metric (5.4.1) as an empty space-time solution for all the values of r. Clearly this metric has singularities at $r = 0$ and $r = 2m$ and hence it represents only one of the patches $0 < r <$

$2m$ or $2m < r < \infty$. If we confine to the manifold M given by the later range of values of r, it is necessary to determine if M is extendible; that is, if there exists a bigger space-time $(M^{'}, g^{'})$ with M embedded in $M^{'}$and $g = g'$ on M. That this should be possible is indicated by the fact that even through the form of the above metric is singular at $r = 2m$, the curvature scalars are all well-behaved at this point and so this could be merely a singularity due to an inappropriate choice of coordinates. A decision on whether a given space-time manifold is maximal or not can be made by looking at the geodesics of the space-time. In a maximal manifold, one would require all the geodesics to be extended in both the directions to an infinite value of their affine parameter, or they must terminate at an intrinsic singularity of the space time which is not removable.

On the other hand, if we take equation (5.4.1) to be describing the patch $0 < r < 2m$, then it is seen that as r tends to zero, the curvature scalar

$$R^{ijkl}R_{ijkl} = \frac{48m^2}{r^6},$$

diverges and it follows that the point $r = 0$ is a **real space time** singularity. It is not possible to extend the space time across this singularity in a continuous manner. Such a maximal extension of the manifold (5.4.1) with $2m < r < \infty$ was obtained by Kruskal (1960) and Szekeres (1960). We describe this procedure below, which uses suitably defined advanced and retarded null coordinates. Using the condition for null geodesics, that is,

$$g_{ij}k^i K^j = 0,$$

the radial null geodesics in the Schwarzschild space time (5.4.1) are given by

$$\left(\frac{dt}{dr}\right)^2 = \left(\frac{r}{r-2m}\right)^2 \tag{5.4.2}$$

Define r^* as

$$r^* \equiv \int \frac{dr}{1 - 2m/r} = r + 2mlog(r/2m - 1) \tag{5.4.3}$$

The radial null geodesics above satisfy

$$t = \pm r^* + const. \tag{5.4.4}$$

The null coordinates u and v are now defined by

$$u = t - r^*, \quad v = t + r^*. \tag{5.4.5}$$

Thus, $r^* = (v - u)/2$, which is given in terms of r by equation (6.3.3). Now r can be viewed as defined implicitly in terms of u and v. Then, using above equations, the metric (6.3.1) can be written as

$$ds^2 = -\frac{2me^{-r/2m}}{r} e^{(v-u)/4m} du dv + r^2 d\Omega^2. \qquad (5.4.6)$$

Now $r \to 2m$ corresponds to $u \to \infty$ or $v \to \infty$. Define new coordinates U and V now by

$$U = -e^{-u/4m}, \qquad V = e^{v/4m}, \qquad (5.4.7)$$

which gives the non-singular part of the metric as

$$ds^2 = -\frac{32m^3 e^{-r/2m}}{r} dU dV. \qquad (5.4.8)$$

There is no singularity now at $U = 0$ and $V = 0$, which corresponds to the value $r = 2m$. Now a final transformation of the form $T = (U + V)/2$ and $X = (V - U)/2$ gives the Schwarzschild metric in the Kruzkal-Szekeres form

$$ds^2 = -\frac{32m^3 e^{-r/2m}}{r} (-dT^2 + dX^2) + r^2 d\Omega^2. \qquad (5.4.9)$$

The coordinate transformation between the original coordinates $(t, r)$and new coordinates (T, X) is given by

$$X^2 - T^2 = \left(\frac{r}{2m} - 1\right) e^{r/2m}, \qquad (5.4.10)$$

$$t = 4m \tanh^{-1}(T/X). \qquad (5.4.11)$$

The quantity r in equation (5.4.9) is determined implicity by equation (5.4.10). The condition $r > 0$ specifies the allowed range of coordinates, which is $X^2 - T^2 > -1$.

The structure of this maximal Schwarzschild manifold is shown in **Fig: 5.2**, which is also called the **kruskal extension of the Schwarzschild space time**. The radial null geodesics are 45^0 lines in the X, T coordinates

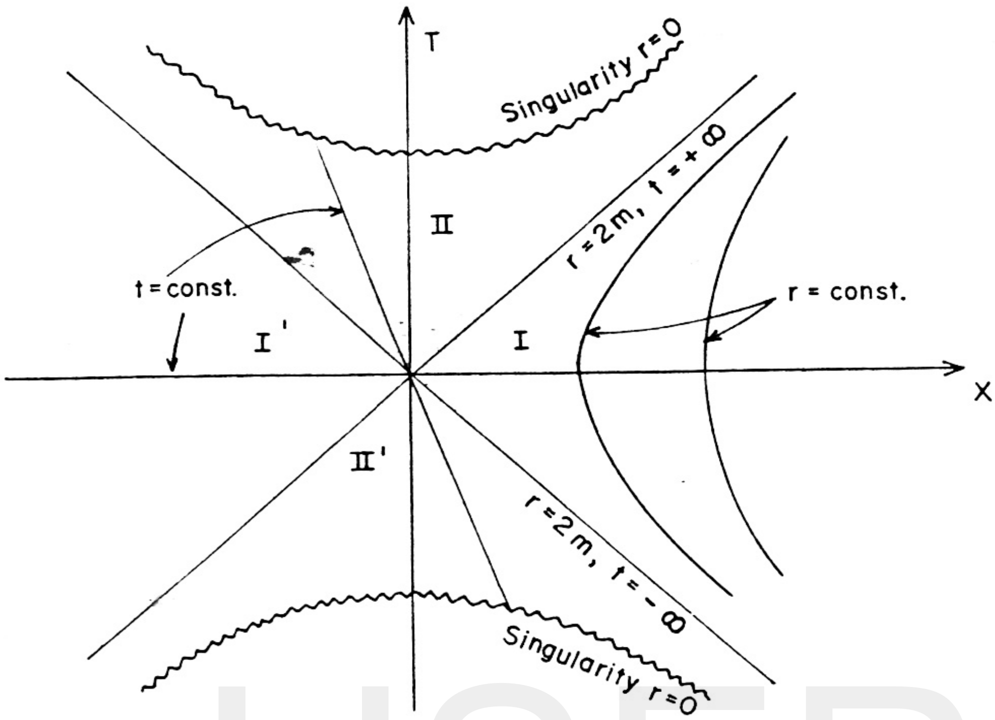

Fig 5.2: The Kruskal extension of the Schwarzschild geometry

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The physical singularity at $r = 0$ corresponds to the values $X = \pm(T^2 - 1)^{1/2}$, and we note that there is no singularity in the metric now at $r = 2m$. The original Schwarzschild solution for $r > 2m$ corresponds to the region I here which is interpreted as the exterior gravitational field of a collapsing body. Region I is asymptotically flat and so is region I', which has identical properties as region I. (Note however, that the Kruskal representation is not best suited to study asymptotic properties and it is best to use a conformal compactification of the metric (5.4.1) for that purpose.) There is no causal communication between regions I and I'; any observer or photon from region I either goes away to infinity or crosses the null line $X = T$ and enters region II. Once a radially in falling observer is inside region II, there is no escape from it and within a finite proper time the observer must fall into the singularity and it can never cross into region I.

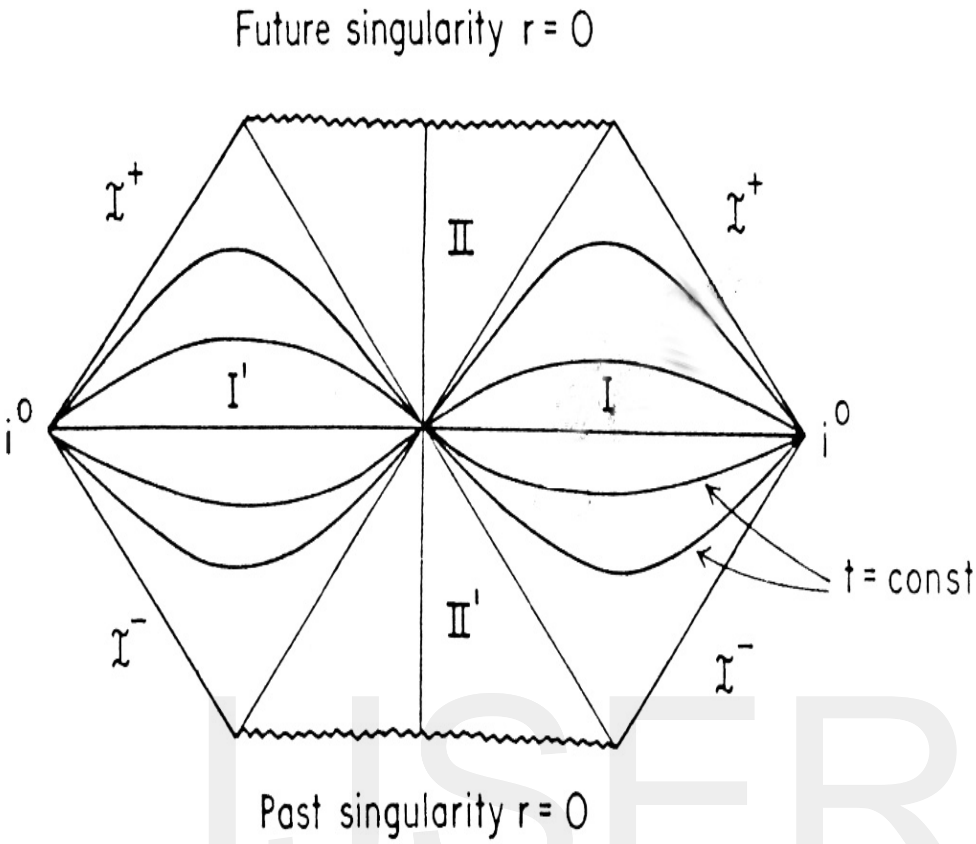

Fig 5.3: A conformal picture of the Kruskal geometry

Hence, region II is termed a ***black hole.*** A particle emitted by the singularity at $X = -(T^2 - 1)^{1/2}$ must leave this region within a finite proper time. Each point of **Fig.5.2** represents a two-sphere in the space time. If a source at a point p in the region $r > 2m$ emits a flash of light, there will be two two-spheres formed, one by the outgoing wave front and the other by ingoing wave front. The outgoing sphere will have a greater area as compared to the ingoing one. However, if the source p lies in the limit $r < 2m$, both the outgoing and ingoing spheres will have areas less than that of p. Then we say that p is a closed **trapped surface**. Such surfaces play an important role towards showing the existence of space-time singularities. Just as in the Minkowski case, one could construct a conformal compactification of the above Kruskal extension of the Schwarzschild geometry, which is more convenient as far as the investigation of asymptotic structure is concerned. Such a conformal diagram of Kruskal geometry is given in **Fig: 5.3.**

Whereas the regions I and II of the extended Schwarzschild manifold have a clear physical interpretation as discussed above, the physical relevance of regions I' and II' is not very obvious. Neither is it possible to rule them out easily. While points in region II' are time-reversed closed trapped surfaces, region I' is another asymptotically flat universe on the other side of the Schwarzschild '**throat**'. This is clear from considering the spatial geometry of the hypersurface $t = 0$. The two-spheres $r = const.$ are almost flat Euclidian for large values of r, but for small r, their area decreases to minimum corresponding to that of the value $r = 2m$, and then it increases again as the two spheres expand in the other region of asymptotically flat three-space. However, if we consider a complete gravitational collapse of a spherically symmetric homogeneous dust cloud, the regions I' and II' are no longer relevant as they are replaced by the interior metric which is not vacuum Schwarzschild, T_{ij} being non-zero there. The situation is shown in **Fig: 5.4(a)** and a conformal diagram of such a collapse is given in **Fig: 5.4(b).**

The uncovered portions of regions I and II represents the vacuum Schwarzschild geometry exterior to the collapsing matter. The portion of region II indicates that a Schwarzschild black hole is always produced in the complete gravitational collapse which fully covers the resulting space time singularity of infinite curvature and density. This situation has a great significance for the cosmic censorship hypothesis and the black hole formation. The interior metric in this case is exactly that of a closed Friedman model.

In the extended Schwarzschild manifold, the surface $r = 2m$ is a null hypersurface and each point there is a two-sphere of area $16\pi m^2$. Note that in equation(5.4.1), the component $g_{00} = (1 - 2m/r) > 0$ for $r > 2m$, however, $g_{00} < 0$ for $r < 2m$. Thus, it is no longer possible to use t as a time coordinate as the coordinates t and r reverse their roles and space time is no longer static. Thus, $r = 2m$ surface is called a 'static limit' as well. The vector $\partial/\partial t$ with components $\xi^i = \delta_0^i = (1,0,0,0)$ gives

the time translation, leaving the g_{ij}, unchanged as it does not involve the time coordinate. Thus, ξ is a Killing vector which leaves the space time geometry unchanged. We have $\xi^2 = g_{ij}\xi^i\xi^j = g_{00}$ and for the Schwarzschild metric, ξ^2 vanishes on $r = 2m$. Hence, at the static limit the timelike Killing vector becomes null. In the Kruskal diagram also it is seen that ξ^i vanishes at $X = T = 0$ and this leads to the odd labeling of lines $X = \pm T$ as '$t = \pm\infty$'.

The Schwarzschild geometry provides an illustration of the basic principle which Einstein used to formulae his gravitation theory, namely that matter tells the space time in its vicinity how to curve. To see this, consider the Schwarzschild solution in a space-like surface $t = constant$ and in the equatorial plane $\theta = \pi/2$. The metric of this two-dimensional curved surface is described by the metric

$$ds^2 = \frac{dr^2}{(1 - 2m/r)} + r^2 d\phi^2$$

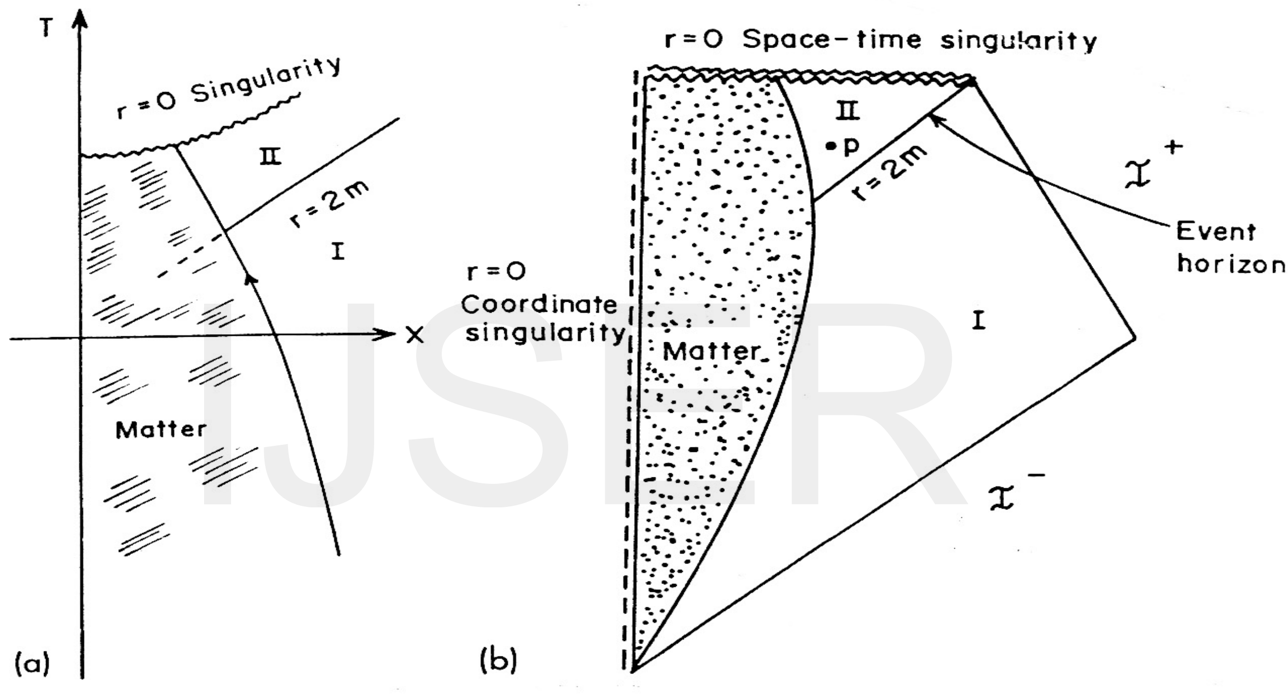

Fig 5.4(a) Complete gravitational collapse of a homogenous dust cloud represented in the Kruskal picture. The regions I' and II' of fig: 5.2 are completely covered by the matter now and so is a part of the region II represents the formation of a black hole. The points in region I and II of the figure represent two-spheres in the space-time and any point, such as p in Fig 5.4(b) within the horizon is a closed trapped surface. (b) A Penrose diagram for this collapse scenario.

The geometry of such a curved surface can be visualized as embedded in the ordinary Euclidean space. Here, the region $0 < r < r_b$is to be considered as filled by the matter which represents the spherical star with a boundary at $r = r_b$, and the curved surface would then represent the geometry outside such a star.

Consider now a static observer along a Killing direction, for whom the four-velocities are $u^i = \xi^i/|\xi|$. Suppose now a static source with four-velocity u_1^i emits a photon with four-momentum p^i(so $p_{i;j}p^j = 0$ with a suitable parametrization) and is observed by a static observer with four-velocity u_2^i (fig:16). Now, take the directional derivative of $\xi^i p^i$ along the geodesic tangent p^i,

$$(\xi^i p^i)_{;j}p^j = \xi_{i;j}p^i p^j + \xi_i p^i_{;j}p^j = 0 \quad (5.4.11)$$

The first term vanishes because ξ^i is a killing vector and the second term vanishes because of the geodesic equation. The ratio of energies measured at these two points by static observers is given by

$$\frac{E_1}{E_2} = \frac{{(u^i p_i)_1}^{1/2}}{{(u^i p_i)_2}^{1/2}} \quad (5.4.12)$$

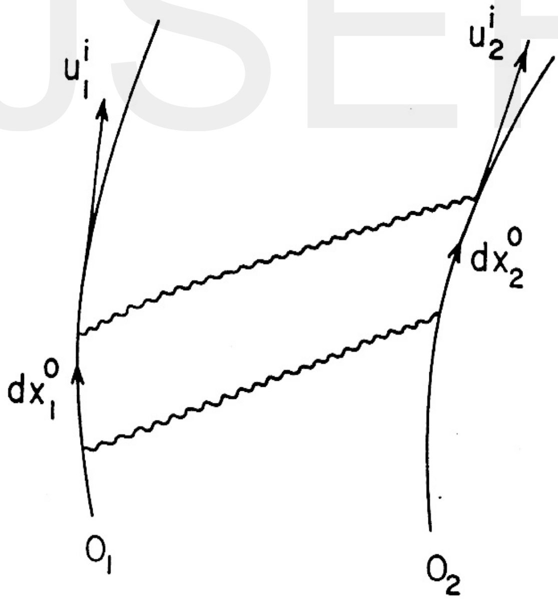

Fig 5.5: The static source $o_1$emits light rays which are received by the static observer o_2

Using $u^i = \xi^i/|\xi|$ and the implication of equation of equation (5.4.11) that $\xi^i p_i = constant$ along the geodesic, we get

$$\frac{E_1}{E_2} = \frac{{(\xi^i \xi_i)_1}^{1/2}}{{(\xi^i \xi_i)_2}^{1/2}} \tag{5.4.13}$$

Since $\xi^2 = g_{00}$, this is the gravitational red-shift formula for a static source and observer in terms of the metric components. It is now seen that if the observer remains at a finite radius but the source approaches $r = 2m$, the red-shift tends to infinity. Thus, as a particle falls into the black hole approaching $r = 2m$, the light rays emitted by it are infinitely red-shifted as observed by a distant static observer in the outside space time.

As pointed above, the Schwarzschild space time is asymptotically flat. For a source situated outside $r = 2m$, part of the photon trajectories emitted with decreasing r values will enter the black hole and fall into the singularity. All other null geodesics will escape to infinity to intersect $\mathfrak{I}^+$. If a source is located below $r = 2m$, no null geodesic can come out of the black hole and all end up in the singularity in future. As in the case of Minkowski space time, we now work out the light cone cuts of future null infinity from an arbitrary apex in the Schwarzschild region $r > 2m$ (Joshi, Kozameh and Newmann, 1983). This process leads to obtaining all the null geodesic and the full light cone from a given point in the space time. Such null trajectories in Schwarzschild geometry are of considerable importance as they are used to verify the general relativity theory experimentally by means of effects such as bending of light rays near a star, the time delay of light and other such effects.

The Schwarzschild metric in (u, r, θ, ϕ) coordinates, where $u = t - r - 2mlog(r - 2m)$ is the retarded time, is given as

$$ds^2 = -\left(1 - \frac{2m}{r}\right) du^2 - 2dudr + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{5.4.14}$$

As in the case of the Minkowski space time, we make necessary coordinate transformations, using stereographic coordinates $\zeta, \bar{\zeta}$ and conformally transform the metric by $\Omega = r^{-1} = \sqrt{\ell}$ which gives

$$d\bar{s}^2 = \Omega^2 ds^2 = -4\left(\ell^2 - 2\sqrt{2}m\ell^3\right)du^2 + 4dud\ell + \frac{d\zeta d\bar{\zeta}}{P_0^2} \tag{5.4.15}$$

The new coordinate ℓ is now finite at infinity and $\mathfrak{I}^+$ is described by the hypersurface $\ell = 0$, which corresponds to $r = \infty$. The Lagrangian for the geodesic is written as

$$L = 2\left(\ell^2 - 2\sqrt{2}m\ell^3\right)\dot{u}^2 - 2\dot{u}\dot{\ell} - \frac{\zeta}{2P_0^2}, \tag{5.4.16}$$

where the dot denotes differentiation with respect to an affine parameter s along null geodesics. The equations for null geodesics are then given as:

$$
\begin{aligned}
&2\left(\ell^2 - 2\sqrt{2}m\ell^3\right)\dot{u} - \dot{\ell} = 1,\\
&\ddot{u} + 2\left(\ell - 3\sqrt{2}m\ell^2\right)\dot{u} = 0,\\
&\ddot{\zeta}(1+\zeta\bar{\zeta}) - 2\bar{\zeta}\dot{\zeta}^2 = 0, \qquad (5.4.17)\\
&\ddot{\bar{\zeta}}(1+\zeta\bar{\zeta}) - 2\dot{\bar{\zeta}}^2\zeta = 0,\\
&4\left(\ell^2 - 2\sqrt{2}m\ell^3\right)\dot{u}^2 - 4\dot{u}\dot{\ell} - \frac{\dot{\zeta}\dot{\bar{\zeta}}}{P_0^2} = 0,
\end{aligned}
$$

where the last equation corresponds to $ds^2 = 0$. Through, in principle, all the null geodesics of the space time are obtained from equations (5.4.17), we first consider only those in the equatorial plane $\theta = \pi/2$. From a fixed apex this yields an S^1 worth of null geodesics. Using these and the spherical symmetry of the space time we can generate all the null geodesics from an arbitrary apex by a rigid rotation. For $\theta = \pi/2$ we have $\zeta = e^{i\phi}$, which gives $\ddot{\phi} = 0$, that is $\dot{\phi} = b$. Further the equation for $\ddot{u}$, follows as an identity from other equations. Combining the first and the last of the above equations we can then write

$$
\begin{aligned}
&\dot{u} = \frac{1+\dot{\ell}}{2\left(\ell^2 - 2\sqrt{2}m\ell^3\right)},\\
&ds = \pm\frac{d\ell}{\sqrt{A}}, \qquad (5.4.18)\\
&d\phi = \pm\frac{b\,d\ell}{\sqrt{A}},
\end{aligned}
$$

where the cubic A is given by

$$A = 2\sqrt{2}mb^2\ell^3 - b^2\ell^2 + 1.$$

Before integrating the above, we note that the null rays coming from an arbitrary apex are divided into two sets (the two sheets of A), that is, those given initially by $\dot{\ell} < 0$ and $\dot{\ell} > 0$. For the first set, the geodesics continue with a decreasing ℓ (increasing r) until intersection with $\mathfrak{I}^+$. For the rays which begin with $\dot{\ell} > 0$ that is, those rays with initially increasing ℓ (decreasing r), some reach a maximum ℓ (when $A = 0$) and then begin to move outwards, and eventually also intersect $\mathfrak{I}^+$. We shall not be concerned with the later rays.

For a fixed apex (say at $\ell = \ell_0 < 1/3\sqrt{2}m$), the null rays, on each sheet, are characterized by the value of the impact parameter b. For the first sheet, the range of b is from $b = 0$ to a maximum b_m, where the b_m is determined by $A = b_m^2\left(2\sqrt{2}m\ell_0^3 - \ell_0^2\right) + 1 = 0$, which yields the value for b_m. For the second sheet $\left(\dot{\ell} > 0\right)$, the range is again from some b_m to $b = 0$, but now there is a critical value b_c such that for all $b < b_c$ the rays continue past the horizon. To determine b_c, we want the smallest b so that A has a real positive root ℓ_c. By plotting A against ℓ it is easily calculated that ℓ_c is a double root and $\ell_c = 1/3\sqrt{2}m$ with $b_c = 3\sqrt{6}m$. Thus, on the second sheet, the range for b is $b_c < b < b_m$. Note that a ray beginning at $\ell = \ell_0$ with $b = b_c$ approaches asymptotically the well-known (unstable) orbit $\ell = \ell_c$.

We can now integrate equations (5.4.18) from a fixed apex to the value $\ell = 0$, which represents the future null infinity, and this gives part of the light cone cut at the infinity. Using that and the spherical symmetry of the space time gives the full light cone of the future null infinity.

5.5 SPHERICALLY SYMMETRIC COLLAPSE

In order to understand the possible final fate of a massive gravitationally collapsing star, we consider here the spherically symmetric collapse situation. Such symmetry represents a high degree of idealization of the physical situation but the advantage is that one can solve it analytically to get exact result when matter is taken in the form of a homogeneous dust cloud. It is also possible that many salient features of the situation, including notion of a black hole. In fact, the basic motivation for the idea and theory of black holes comes from the case of homogeneous dust cloud collapse. Independently of the interior solution, the metric exterior to such a spherical body must be the Schwarzschild space time and no gravitational radiation will be present, which follows form **Birkhoff's theorem** is that the only vacuum, spherically symmetric gravity field must be static. That the Schwarzschild geometry is relevant to gravitational collapse follows from Birkhoff's(1923)[10] theorem: Let the geometry of a given region of space time **(1)** be spherically symmetric and **(2)** be a solution to the Einstein field equation in vacuum, then that geometry is necessarily a piece of the Schwarzschild geometry. The external field of any electrically neutral, spherical star satisfies the conditions of Birkhoff's theorem, whether the star is static, vibrating or collapsing. Therefore the external field must be a piece of the Schwarzschild geometry.

In order to consider spherically symmetric space time, if any point at a distance a from the origin O. The system must be invariant under rotation around O. Such rotation will generates a two sphere **(Fig: 5.1)** around O, the line element on which must be given by

$$ds^2 = a^2(d\theta^2 + sin^2\theta d\phi^2) \tag{5.5.1}$$

This is the line element for a two sphere given by $t = constant,\ r = constant$ in a general spherically symmetric space time. Further as the metric must be invariant under the reflection $\theta \to \pi - \theta$ and $\phi \to -\phi$, there must not be any cross terms in the metric $d\theta$ and $d\phi$. As the line element must not change with any change in θ and ϕ, the must occur in the metric only in the form of metric given above. Then in the (t, r, θ, ϕ)coordinate system, the metric has the form

$$ds^2 = adt^2 - 2Bdtdr - Cdr^2 - D(d\theta^2 + sin^2\theta d\phi^2) \tag{5.5.2}$$

Here the quantities A, B, C and D are the function of r and t to be determined. Introducing now a new radial coordinate by the transformation, $t \to r' = D^{1/2}$

Then (5.5.2) becomes,

$$ds^2 = A'(t,r')dt^2 - 2B'(t,r')dtdr' - C'(t,r')dr'^2 - r'^2(d\theta^2 + sin^2\theta d\phi^2) \tag{5.5.3}$$

Consider the differential,

$$A'(t,r')dt - B'(t,r')dr'$$

Now, we define a new time coordinates t' by requiring that,

$$dt' = F(t,r')[A'(t,r')dt - B'(t,r')dr']$$

Where $F(t,r')$is a suitable integrating factor.

Squaring, we get,

$$dt'^2 = F^2(t,r')\left[A'^2(t,r')dt^2 - 2A'(t,r')B'(t,r')dr'dt + B'^2(t,r')dr'^2\right]$$

$$\Rightarrow\ F^2A'^2(t,r')dt^2 - 2F^2A'^{(t,r')}B'^{(t,r')}dr'dt = dt'^2 - B'^2(t,r')dr'^2$$

$$\Rightarrow\ A'^{(t,r')dt^2} - 2B'^{(t,r')}dr'dt = A'^{-1}F^{-2}dt'^2 - A'^{-1}B'^{-2}(t,r')dr'^2$$

The line element (5.5.3) reduces to,

$$ds^2 = A'^{-1}(t,r')F^{-2}dt'^2 - A^{-1}B'^2(t,r') + C'(t,r')dr'^2 - r'^2(d\theta^2 + sin^2\theta d\varphi^2)$$

Hence,

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2) \tag{5.5.4}$$

Where, we have dropped the primes. Here $\nu = \nu(r,t)$and $\lambda = \lambda(r,t)$and the quantities $e^{\nu} = \frac{1}{F^2 A'}$ and $e^{\lambda} = C' - \frac{B'^2}{A'}$ appearing in the metric are always positive.

However, equation (5.5.4) is the spherically symmetric gravitational field of a space time which can also be derived in terms of the killing vectors.

Consider now a spherically symmetric massive star, collapsing gravitationally when it was exhausted its internal nuclear fuel. We need to consider the interior solution for such a star.

Of course there is no unique interior solution available which basically depends on the properties of matter, the equation of state obeyed by the matter, and the physical processes taking place within the stellar interior. However, assuming the matter to be pressure less dust $(i.e. p = 0)$ allows one to solve the problem analytically, which provides many important insights.

In this case, the energy momentum tensor is given by $T^{ij} = \rho u^i u^j$ and one needs to solve the Einstein equations for the spherically symmetric form of the metric given above. Solving the Einstein equation determines the metric potentials completely and the interior geometry of the star which is collapsing dust ball, is described by the same line element as that of the closed homogeneous and isotropic. Friedman models given by

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-r^2} + r^2 d\Omega^2\right], [as, k = 1]$$

Where $d\Omega^2$ represent the metric on a two sphere. The geometry outside the star is vacuum and is of necessity the Schwarzschild space time as implied by the Birkhoff's theorem. It is possible to show that the interior geometry of the dust cloud matches correctly at the boundary of the star $r = r_b$ with exterior Schwarzschild space time.

When the collapse is complete, the space time settles to a vacuum Schwarzschild geometry for the range of coordinate $0 < r < \infty$ (with coordinate singularity at $r = 2M$, when can be removed by going to the Kruskal expansion). Here M can be identified with the total mass of the star.

5.6 SPHERICALLY SYMMETRIC SCHWARZSCHILD SOLUTION

Now we would like to solve Einstein's vacuum field equation under the spherical symmetry assumption. According to Birkhoff's theorem, this solution is the famous Schwarzschild solution which was found in 1916. The line element will be the form,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad (5.6.1)$$

Here the coordinates are,

$$x^\mu = (1,0,0,0) = (t, r, \theta, \varphi)$$

Comparing (5.6.1) with (5.5.4) we get the covariant metric as,

$$g_{ab} = diag\left(e^\nu, e^{-\lambda}, -r^2, -r^2 sin^2\theta\right) \qquad (5.6.2)$$

Also, the contravariant form of (5.6.1) is,

$$g^{ab} = diag\left(e^{-\nu}, e^{-\lambda}, -r^{-2}, -r^{-2} cosec^2\theta\right) \qquad (5.6.3)$$

Now the christoffel symbols of second kind can be calculated from the equation

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd}\left(g_{ad,b} + g_{db,a} - g_{ab,d}\right) \qquad (5.6.4)$$

The above equation can be reduced only for the diagonal components of g^{cd} as follows

$$\Gamma^c_{ab} = \frac{1}{2} g^{cc}\left(g_{ac,b} + g_{cb,a} - g_{ab,c}\right) \qquad (5.6.5)$$

Thus the non vanishing components of Christoffel symbols of second kind can be calculated from the equation (5.6.5) as follows:

$$\Gamma^1_{00} = \frac{1}{2} g^{11}\left(g_{01,0} + g_{10,0} - g_{00,1}\right)$$

$$= \frac{1}{2}\left(-e^{-\lambda}\right)\frac{\partial}{\partial r}(-g_{00})$$

$$= \frac{1}{2}\left(-e^{-\lambda}\right)\frac{\partial}{\partial r}(-e^{\nu})$$

$$\Gamma^1_{00} = \frac{1}{2} e^{\nu-\lambda}\frac{\partial \nu}{\partial r} \qquad (5.6.5a)$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11}\left(g_{11,1} + g_{11,1} - g_{11,1}\right)$$

$$= \frac{1}{2}\left(-e^{-\lambda}\right)\frac{\partial}{\partial r}\left(-e^{\lambda}\right)$$

$$\Gamma^1_{11} = \frac{1}{2}\frac{\partial \lambda}{\partial r} \qquad (5.6.5b)$$

$$\Gamma^1_{22} = \frac{1}{2}g^{11}\left(g_{21,2} + g_{12,2} - g_{22,1}\right)$$

$$= -\frac{1}{2}\left(-e^{-\lambda}\right)\frac{\partial}{\partial r}\left(-r^2\right)$$

$$\Gamma^1_{22} = -re^{-\lambda} \qquad (5.6.5c)$$

$$\Gamma^1_{33} = \frac{1}{2}g^{11}\left(g_{31,3} + g_{13,3} - g_{33,1}\right)$$

$$= -\frac{1}{2}\left(-e^{-\lambda}\right)\frac{\partial}{\partial r}\left(-r^2 sin^2\theta\right)$$

$$\Gamma^1_{33} = -re^{-\lambda} sin^2\theta \qquad (5.6.5d)$$

$$\Gamma^0_{10} = \frac{1}{2}g^{00}\left(g_{00,1} + g_{10,0} - g_{01,0}\right)$$

$$= \frac{1}{2}\left(e^{-\nu}\right)\frac{\partial}{\partial r}\left(e^{\nu}\right)$$

$$\Gamma^0_{10} = \frac{1}{2}\frac{\partial \nu}{\partial r} \qquad (5.6.5e)$$

$$\Gamma^2_{12} = \frac{1}{2}g^{22}\left(g_{22,1} + g_{21,2} - g_{12,2}\right)$$

$$= \frac{1}{2}\left(-\frac{1}{r^2}\right)\frac{\partial}{\partial r}\left(-r^2\right)$$

$$\Gamma^2_{12} = \frac{1}{r} \qquad (5.6.5f)$$

$$\Gamma^3_{13} = \frac{1}{2}g^{33}\left(g_{31,1} + g_{33,1} - g_{13,3}\right)$$

$$= \frac{1}{2}\left(-\frac{1}{r^2 sin^2\theta}\right)\frac{\partial}{\partial r}\left(-r^2 sin^2\theta\right)$$

$$\Gamma^3_{13} = \frac{1}{r} \qquad (5.6.5g)$$

$$\Gamma^2_{33} = \frac{1}{2} g^{22}\left(g_{23,3} + g_{32,3} - g_{33,2}\right)$$

$$= -\frac{1}{2}\left(-\frac{1}{r^2}\right)\frac{\partial}{\partial \theta}\left(-r^2 sin^2\theta\right)$$

$$\Gamma^2_{33} = -sin\theta cos\theta \tag{5.6.5h}$$

$$\Gamma^3_{23} = \frac{1}{2} g^{33}\left(g_{33,2} + g_{23,2} - g_{23,3}\right)$$

$$= \frac{1}{2}\left(-\frac{1}{r^2 sin^2\theta}\right)\frac{\partial}{\partial \theta}\left(-r^2 sin^2\theta\right)$$

$$\Gamma^3_{23} = cot\theta \tag{5.6.5i}$$

Now, from Ricci tensor, we have,

$$R_{ab} = R^c_{abc} = \Gamma^c_{db}\Gamma^d_{ac} - \Gamma^c_{dc}\Gamma^d_{ab} + \frac{\partial}{\partial x^b}\left(\Gamma^c_{ac}\right) - \frac{\partial}{\partial x^c}\left(\Gamma^c_{ab}\right) \tag{5.6.7}$$

Substituting $a = b = c = 0,1,2,3$ in equation (5.6.7) respectively we obtain,

$$R_{00} = \Gamma^c_{d0}\Gamma^d_{c0} - \Gamma^c_{dc}\Gamma^d_{00} + \frac{\partial}{\partial x^0}\left(\Gamma^c_{c0}\right) - \frac{\partial}{\partial x^c}\left(\Gamma^c_{00}\right)$$

$$= \left(\Gamma^0_{do}\Gamma^d_{00} + \Gamma^1_{d0}\Gamma^d_{10}\right) - \left(\Gamma^0_{d0} + \Gamma^1_{d1} + \Gamma^2_{d2} + \Gamma^3_{d3}\right)\Gamma^d_{00} + \frac{\partial}{\partial t}\left(\Gamma^0_{00}\right) - \frac{\partial}{\partial x^1}\left(\Gamma^1_{00}\right)$$

$$= \left(2\Gamma^0_{10} - \Gamma^0_{10} - \Gamma^1_{11} - \Gamma^2_{12} - \Gamma^3_{13}\right)\Gamma^1_{00} - \frac{\partial}{\partial r}\left(\Gamma^1_{00}\right)$$

$$= \frac{1}{2} e^{\nu-\lambda}\frac{\partial \nu}{\partial r}\left(\frac{1}{2}\frac{\partial \nu}{\partial r} - \frac{1}{2}\frac{\partial \lambda}{\partial r} - \frac{1}{r} - \frac{1}{r}\right) - \frac{\partial}{\partial r}\left(\frac{1}{2} e^{\nu-\lambda}\frac{\partial \nu}{\partial r}\right)$$

$$= \frac{1}{2} e^{\nu-\lambda}\left(\frac{1}{2}\left(\frac{\partial \nu}{\partial r}\right)^2 - \frac{1}{2}\frac{\partial \nu}{\partial r}\frac{\partial \lambda}{\partial r} - \frac{2}{r}\frac{\partial \nu}{\partial r}\right) - \frac{1}{2} e^{\nu-\lambda}\frac{\partial^2 \nu}{\partial r^2} - \frac{1}{2} e^{\nu-\lambda}\frac{\partial \nu}{\partial r}\left(\frac{\partial \nu}{\partial r} - \frac{\partial \lambda}{\partial r}\right)$$

$$= \frac{1}{2} e^{\nu-\lambda}\left(\frac{1}{2}\left(\frac{\partial \nu}{\partial r}\right)^2 - \frac{1}{2}\frac{\partial \nu}{\partial r}\frac{\partial \lambda}{\partial r} - \frac{2}{r}\frac{\partial \nu}{\partial r} - \frac{\partial^2 \nu}{\partial r^2} - \left(\frac{\partial \nu}{\partial r}\right)^2 + \frac{\partial \nu}{\partial r}\frac{\partial \lambda}{\partial r}\right)$$

$$\therefore R_{00} = -\frac{1}{2} e^{\nu-\lambda}\left(\frac{1}{2}\left(\frac{\partial \nu}{\partial r}\right)^2 - \frac{1}{2}\frac{\partial \nu}{\partial r}\frac{\partial \lambda}{\partial r} + \frac{2}{r}\frac{\partial \nu}{\partial r} + \frac{\partial^2 \nu}{\partial r^2}\right) \tag{5.6.8}$$

$$R_{11} = \Gamma^c_{d1}\Gamma^d_{c1} - \Gamma^c_{dc}\Gamma^d_{11} + \frac{\partial}{\partial x^1}\left(\Gamma^c_{c1}\right) - \frac{\partial}{\partial x^c}\left(\Gamma^c_{11}\right)$$

$$= (\Gamma^0_{d1}\Gamma^d_{01} + \Gamma^1_{d1}\Gamma^d_{11} + \Gamma^2_{d1}\Gamma^d_{21} + \Gamma^3_{d1}\Gamma^d_{31}) - (\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13})\Gamma^1_{11} + \frac{\partial}{\partial x^1}(\Gamma^0_{01} + \Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{31}) - \frac{\partial}{\partial x^1}(\Gamma^1_{11})$$

$$= (\Gamma^0_{01}\Gamma^0_{01} + \Gamma^1_{11}\Gamma^d_{11} + \Gamma^2_{21}\Gamma^2_{21} + \Gamma^3_{31}\Gamma^3_{31}) - (\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13})\Gamma^1_{11} + \frac{\partial}{\partial x^1}(\Gamma^0_{01} + \Gamma^2_{21} + \Gamma^3_{31})$$

$$= \left(\frac{1}{2}\frac{\partial \nu}{\partial r}\right)^2 + \left(\frac{1}{2}\frac{\partial \lambda}{\partial r}\right)^2 + \left(\frac{1}{r}\right)^2 + \left(\frac{1}{r}\right)^2 - \frac{1}{2}\frac{\partial \lambda}{\partial r}\left(\frac{1}{2}\frac{\partial \nu}{\partial r} + \frac{1}{2}\frac{\partial \lambda}{\partial r} + \frac{1}{r} + \frac{1}{r}\right) + \frac{\partial}{\partial r}\left(\frac{1}{2}\frac{\partial \nu}{\partial r} + \frac{1}{r} + \frac{1}{r}\right)$$

$$R_{11} = \frac{1}{2}\frac{\partial^2 \nu}{\partial r^2} + \frac{1}{4}\left(\frac{\partial \nu}{\partial r}\right)^2 - \frac{1}{4}\frac{\partial \nu}{\partial r}\frac{\partial \lambda}{\partial r} - \frac{1}{r}\frac{\partial \lambda}{\partial r} \qquad (5.6.9)$$

Applying the same process we will get,

$$R_{22} = e^{-\lambda}\left(1 + \frac{1}{2}r\frac{\partial \nu}{\partial r} - \frac{1}{2}r\frac{\partial \lambda}{\partial r}\right) - 1 \qquad (5.6.10)$$

$$R_{33} = \left\{e^{-\lambda}\left(1 + \frac{1}{2}r\frac{\partial \nu}{\partial r} - \frac{1}{2}r\frac{\partial \lambda}{\partial r}\right) - 1\right\}sin^2\theta = R_{22}sin^2\theta \qquad (5.6.11)$$

Further, for the above line element all the off diagonal of $R_{ab}$are identically zero. Now we will consider the Einstein vacuum field equation for empty space, $R_{ab} = 0$ and to determine the unknown function ν and λ of the equation (5.5.4), we can write

$$R_{00} = -\frac{1}{2}e^{\nu-\lambda}\left(\frac{1}{2}\left(\frac{\partial \nu}{\partial r}\right)^2 - \frac{1}{2}\frac{\partial \nu}{\partial r}\frac{\partial \lambda}{\partial r} + \frac{2}{r}\frac{\partial \nu}{\partial r} + \frac{\partial^2 \nu}{\partial r^2}\right) = 0 \qquad (5.6.12a)$$

$$R_{11} = \frac{1}{2}\frac{\partial^2 \nu}{\partial r^2} + \frac{1}{4}\left(\frac{\partial \nu}{\partial r}\right)^2 - \frac{1}{4}\frac{\partial \nu}{\partial r}\frac{\partial \lambda}{\partial r} - \frac{1}{r}\frac{\partial \lambda}{\partial r} = 0 \qquad (5.6.12b)$$

$$R_{22} = e^{-\lambda}\left(1 + \frac{1}{2}r\frac{\partial \nu}{\partial r} - \frac{1}{2}r\frac{\partial \lambda}{\partial r}\right) - 1 = 0 \qquad (5.6.12c)$$

$$R_{33} = \left\{e^{-\lambda}\left(1 + \frac{1}{2}r\frac{\partial \nu}{\partial r} - \frac{1}{2}r\frac{\partial \lambda}{\partial r}\right) - 1\right\}sin^2\theta = R_{22}sin^2\theta = 0 \qquad (5.6.12d)$$

Thus , there are only there independent equation to solve, namely,

$$-\frac{1}{2}e^{\nu-\lambda}\left(\frac{1}{2}\left(\frac{\partial \nu}{\partial r}\right)^2 - \frac{1}{2}\frac{\partial \nu}{\partial r}\frac{\partial \lambda}{\partial r} + \frac{2}{r}\frac{\partial \nu}{\partial r} + \frac{\partial^2 \nu}{\partial r^2}\right) = 0 \qquad (5.6.13)$$

$$\frac{1}{2}\frac{\partial^2 \nu}{\partial r^2} + \frac{1}{4}\left(\frac{\partial \nu}{\partial r}\right)^2 - \frac{1}{4}\frac{\partial \nu}{\partial r}\frac{\partial \lambda}{\partial r} - \frac{1}{r}\frac{\partial \lambda}{\partial r} = 0 \qquad (5.6.14)$$

$$e^{-\lambda}\left(1 + \frac{1}{2}r\frac{\partial \nu}{\partial r} - \frac{1}{2}r\frac{\partial \lambda}{\partial r}\right) - 1 = 0 \qquad (5.6.15)$$

Dividing equation (5.6.13) by $e^{\nu-\lambda}$ and then substituting (5.6.14) from the resulting equation, we get,

$$\frac{1}{2}r\frac{\partial \nu}{\partial r} + \frac{1}{2}r\frac{\partial \lambda}{\partial r} = 0$$

$$\Rightarrow \quad \frac{\partial \nu}{\partial r} + \frac{\partial \lambda}{\partial r} = 0$$

Thus, integrating we get,

$$\nu + \lambda = A$$

Where, A is a constant of integration which may be set equal to zero without any loss of generality, since at $r \to \infty$, $\lambda = 0$and ν=0 hence,

$$\nu = -\lambda \qquad (5.6.16)$$

Substituting this in equation (5.6.15) we get,

$$e^{\lambda}\left(1 + \frac{1}{2}r\frac{\partial \nu}{\partial r} + \frac{1}{2}r\frac{\partial \nu}{\partial r}\right) = 1$$

$$\Rightarrow e^{\lambda}\left(1 + r\frac{\partial \nu}{\partial r}\right) = 1$$

$$\Rightarrow \frac{\partial}{\partial r}\left(re^{\lambda}\right) = 1$$

Therefore, integrating we get,

$$re^{\nu} = r + B$$

Where, b is another constant of integration, which has been chosen as $B = -2$ Min order to facilitate the physical interpretation of M as the mass of the gravitating particle. That is

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2M}{r} \qquad (5.6.17)$$

Hence the most general static, spherically symmetric and asymptotically flat solution of the vacuum field equation, which is the most famous Schwarzschild solution, is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \qquad (5.6.18)$$

Where, $d\Omega^2 = d\theta^2 + sin^2\theta d\varphi^2$

Thus, we can arrive at the conclusion that when the space time surrounding any object has spherical symmetry and is free of charge, mass and all field other than gravity, then one can introduce coordinates in which the metric is that of Schwarzschild. Conclusion restated in coordinate free language: ***the geometry of any spherical symmetric vacuum region of space time is a piece of the Schwarzschild geometry*** (Birkhoff's theorem).

5.7 PROPERTIES OF SCHWARZSCHILD METRIC

Consider a test particle moving in the Schwarzschild geometry, described by the line element (5.6.18). This expression for the geometry applies outside any spherically symmetric center of attraction of total mass energy M. It makes no difference, for the motion of the particle outside, what the geometry is inside, because the particle never gets there, before it can collides, it collides with the surface of the star if the center of attraction is a star that is to say, a fluid mass in hydrostatic equilibrium. At each point throughout such an equilibrium configuration, the Schwarzschild equilibrium exceeds the local value of the quantity 2m(r).

Therefore the Schwarzschild coordinate R of the surface exceeds 2M. Consequently, the above metric applies that one need not face the issue of the singularity $r = 2M$. The ideal limit is not a star in hydrostatic equilibrium. It is a star that has undergone complete gravitational collapse to a black hole.

(1) In equation (5.6.18) the coordinate t is time like and the other three coordinates r, θ, φ are space like. The radial coordinate r has the property that the two spheres given by $t = constant, r = constant$ has the metric given by

$$ds^2 = r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

It follows that the area of any such two spheres would be $4\pi r^2$.

(2) The coordinate r is restricted by the condition $r > 2M$ because the above metric has an apparent singularity at $r = 2M$. The coordinate t has the range $-\infty < t < +\infty$. The solution above is generated by solving the vacuum Einstein field equation for a spherically symmetric space time and the quantity M appears as the constant of integration. The value of the constant can be determined by considering the weak field Newtonian limit of general relativity. If a point mass M situation at the origin O in Newtonian theory gives rises to a potential $\varphi = -\frac{GM}{r}$, where G is the Newtonian constant gravity, then in non relativistic units

$$g_{00} = 1 + \frac{2\varphi}{c^2} = 1 - \frac{2GM}{rc^2} \qquad (5.7.1)$$

Where, c is velocity of light. This determines the constant of integration m is the Schwarzschild solution as $m = GM/r$.Thus the solution is interpreted as describing the gravitational field of a point particle with mass m. In relativistic units, $G = c = 1$ in which case m is measured in centimeters. So it is sometimes known as the **geometric mass**.

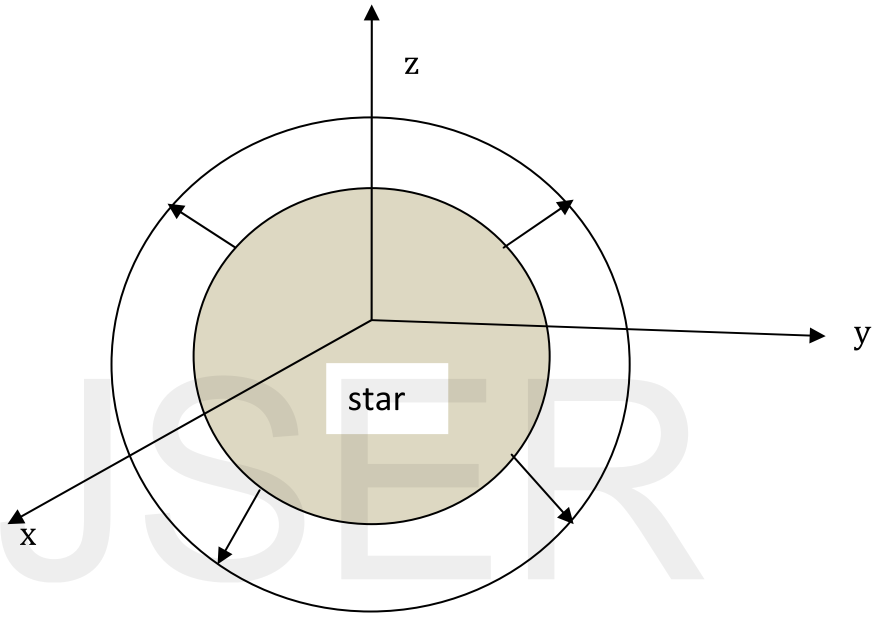

Fig: 5.6 A pulsating spherical star cannot emit gravitational waves

The Schwarzschild metric is static in the sense that the metric components $g_{ij}$are independent of the time. That is $g_{ij,t} = 0$. Further, there are no mixed terms in equation (5.6.18) between time and space. That is, $g_{\mu 0} = 0$ $(\mu = 1,2,3)$. This absence of terms like $dtd\varphi\left(g_{t\varphi} \equiv g_{03} = 0\right)$ indicates the source is not rotating and therefore there is no rotation inherent to the space time either. Space time's which exhibit these two properties are called '**static**'. Again the solution is stationary, since $\partial g_{ij}/\partial t = 0$. If a space time is stationary that does not involve in time. It is just that the time does not enter explicitly in the solution and there is no time evolution of the system, which is time symmetric about any origin of time. Time symmetric means that it is invariant under the time reflection $t \to t' = -t$.

Now we thus have proved some unexpected result by using a rigorous theorem known as the Birkhoff's theorem (1923), which states that any spherically symmetric vacuum solution of Einstein equation is necessarily the Schwarzschild solution that is static. This implies that if a spherically symmetric source like a star undergoes pulsation or changes its shape, while maintaining the spherical symmetry, it cannot radiate any disturbances in the exterior such as Schwarzschild exterior solution can be used to describe the outside metric for several situation such as spherically symmetric star which is either static or which undergoes radial spherically symmetric gravitational collapse

(a) The parameter m serves as the source of the gravitational field and $M = 0$ gives the flat Minkowskian space time. As pointed out above the comparison with Newtonian theory shows that m is to be treated as the gravitational mass of the body producing the field as measured from infinity.

(b) Again as $r \to \infty$, the metric becomes that of flat space time. That is

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

This is asymptotic flatness. As we go further from the isolated source mass M, the gravitational field progressively diminishes to zero.

The equation (5.5.18) is taken to represent the outside metric for a star with $r > a$, for some $a > 2M$, where a gives the boundary of the star. The metric inside $r > a$ is a different interior metric determined by the matter distribution T_{ij} inside the star and is matches at the boundary $r = a$ with equation (6.6.18). However in the case of a complete collapse, when all the mass collapses at $r = 0$, it necessary to consider the metric (5.5.18) as an empty space time solution for all the value of r. This has been found to be a true singularity. On the other hand, at $r = 2M, g_{00}$ becomes zero and g_{11} becomes infinity. Once upon a time this surface had been dubbed the Schwarzschild singularity. It was not taken seriously because the numerical value of 2M is quite small for any ordinary matter distribution and the surface lies within the matter, for instant, 2M is about a mere 3 km in the case of the sun. Within the matter distribution the matter is not the one given by equation (5.5.18) and $r = 2M$ has no special significance. As the star collapse unchecked, $r = 2M$ surface is exposed and the mass finally reaches the singularity $r = 0$ being progressively complicated in the process. No one can know the ultimate fate of the collapse matter, although quantum effects are expected to prevent the formation of the singularity. For our purpose however the important event is the collapse through the surface $r = 2M$ which can no longer be ignored as was done previously. The pathology exhibited by the metric components on this surface is only a coordinate effect similar to what happens in the case of polar coordinate poles coordinates can be found in terms of which no undesirable features are displayed at this surface. Such coordinates were discovered by **Kruskal** and **Szekers** .

On the other hand, strikingly interesting properties are exhibited by this surface and it is identified with the static or the non-rotating black hole. We shall now consider three basic properties of the Schwarzschild (or the static or the no rotating) black hole.

Consider the metric component $g_{00} = 1 - 2M/r$. For $r > 2M, g_{00} > 0$ and correspondingly t is a legitimate time coordinate. But for $r < 2M,\ g_{00} < 0$ and therefore t can no longer measure time. In this region a new time coordinate a mixture of t and r will have to be defined. Because of this the surface $r = 2M$ is called the static limit. A related consequences is as follows, outside the static limit we can defined static particles with $(r, \theta, \varphi) = constant$ with only time t changing. This is possible only up to the static limit within which t loses its character of being time. Objects will have both t and r coordinates changing within the static limit that is they have necessary to be in a static of full. We shall make these statements a little more precise and coordinate in independent as follows:

Consider the vector field $\xi^a = \delta_0^a = (1,0,0,0)$ defined at every point. This defines translation along time t which leaves the metric unchanged, since it is independent of t. Therefore, ξ^a is a vector defined a direction of symmetry, the motion along which leaves the space time geometry unaltered. Such a vector is called a killing vector. Consider then the four dimensional square ξ^2 of ξ^a, which is a scalar and therefore coordinate independent but coincides with g_{00} in the Schwarzschild coordinates, that is

$$\xi^2 = g_{ab}\xi^a\xi^b = g_{00} \tag{5.7.2}$$

We can speak of ξ^2 without reference to any coordinate system, it is convenient to refer to the Schwarzschild coordinates, to exact species information. We note

$$\xi^2 = \begin{cases} > 0; \xi^a time\ like\ for, r > 2M; g_{00} > 0 \\ = 0; \xi^a null\ like\ for, r = 2M; g_{00} = 0 \\ < 0; \xi^a space\ like\ for, r < 2M; g_{00} < 0 \end{cases}$$

Thus the static limit is the surface on which the time like killing vector becomes null. For $r > 2M$, we can define static particles (source, observer and so on) with four velocities following the killing direction

$$u^a = \frac{dx^a}{ds} = \frac{\xi^a}{(\xi^b\xi_b)^{1/2}} = (g_{00}) - \frac{1}{2}(1,0,0,0);\ u^a u_a = 1 \tag{5.7.3}$$

This is not possible on or within the static limit $\xi^2 \leq 0$.

The above discussion shows that the Schwarzschild black hole $r = 2M$ is the static limit on which the killing vector, which is asymptotically $(r \to \infty)$ time like, becomes null.

(1) Infinite red shift surface

Suppose we consider in any space time an observer with a four velocity u^a. Let him encounter a particle moving with four momentum p^a. The the energy of the particle as measured by the observer is given by

$$E = u^a p^a \tag{5.7.4}$$

For instance, in flat space time a static observer has $u^a = (1,0,0,0)$ and $p^a = (E, p)$ the above statement is true, the energy measured depends on the state of motion of the observer as given by u^a and change according to equation (5.7.4). This formula is true in the local elemental flat space time and hence in any coordinate system of an arbitrary space time since it is a scalar equation.

Now let us specialize to the static observers and sources following the killing vector direction for whom

$$u^a = \frac{\xi^a}{(\xi^b \xi_b)^{\frac{1}{2}}}$$

As we have seen. Let us assume that is the four momentum of a geodesic so that with proper parameterization

$$p_{ab} p^a = 0 \tag{5.7.5}$$

We have seen that defines the symmetry of space time. Whenever we have a symmetry there is a conserved quantity.

In mechanics corresponding to an ignorable coordinates $x^{(a)}$, the conjugate momentum is a constant. A similar situation exists here. Along the geodesic, the scalar $\xi^a p^a = constant$. Now we will consider the killing equations satisfied by ξ_a. These equations are

$$\xi_{a;b} + \xi_{b;a} = 0 \tag{5.7.6}$$

Taking the directional derivatives of $\xi^a p^a = constant$ along the geodesic tangent p^a is emitted at a point 1 by a four velocity u_1^a and is observer at point 2 by a static observer with four velocity u_2^a. Then the ratio of the energies measured at these two points (by static observer) is

$$\frac{E_1}{E_2} = \frac{(u^a p_a)_1}{(u^a p_a)_2}$$

$$= \frac{(\xi^a p_a)_1 (\xi^a \xi_b)_1^{-\frac{1}{2}}}{(\xi^a p_a)_2 (\xi^a \xi_b)_2^{-\frac{1}{2}}}$$

$$= \frac{(\xi^b \xi_b)_1^{-\frac{1}{2}}}{(\xi^b \xi_b)_1^{-\frac{1}{2}}} \qquad (5.7.8)$$

Since $\xi^a p_a = constant$, identifying $E = h\nu$, we get,

$$\frac{\nu_0}{\nu_s} = \frac{\left(1 - \frac{2M}{r_s}\right)^{\frac{1}{2}}}{\left(1 - \frac{2M}{r_0}\right)^{\frac{1}{2}}} \qquad (5.7.9)$$

Where o and s stand for observers and source respectively. This is the gravitational red shift formula for static sources and observers in the Schwarzschild space time. If r_0 is kept finite and larger then 2M, as smaller approaches 2M, we see that ν_0 goes to zero. In other words as the static sources approaches the black hole, the red shift tends to become infinite in the limit. The black hole is there an infinite red shift surface for static sources and observers.

(2)One way membrane

The proper of static limit showed the impossibility of defining static particles within the black hole. This as we found, is directly related to the idea of infinite red shift. We shall now discuss this defining property of the black hole.

Consider a surface given by the equation

$$\int (x^b) = constant \qquad (5.7.10)$$

Here f is any function of the space time coordinates represented by x^b. The normal to the surface na is given by the gradient of the function evaluated on the surface. So,

$$na = fa \qquad (5.7.11)$$

Then the square of the normal is given by

$$n^2 = n^a n_a = g^{ab} f_a f_b \qquad (5.7.12)$$

The normal is time like, space like or null according as $n^2$is greater than, less than or equal to zero. Let us concentrate on the case when $n^2 = 0$. The surface $f(x^b) = constant$ is then said to be null surface. Let us now see the significance of such a null surface.

Consider flat space with Cartesian coordinate(t, x, y, z). A wave front moving along x direction has the equation

$$f(x^b = t - x) = constant \qquad (5.7.13)$$

A normal to the wave front is given by,

$$n^a = f_a = (1, -1,0,0) \qquad (5.7.14)$$

That with the diagonal metric (1,-1,-1,-1), we find

$$n^2 = (1)^2 - (-1)^2 = 0 \qquad (5.7.15)$$

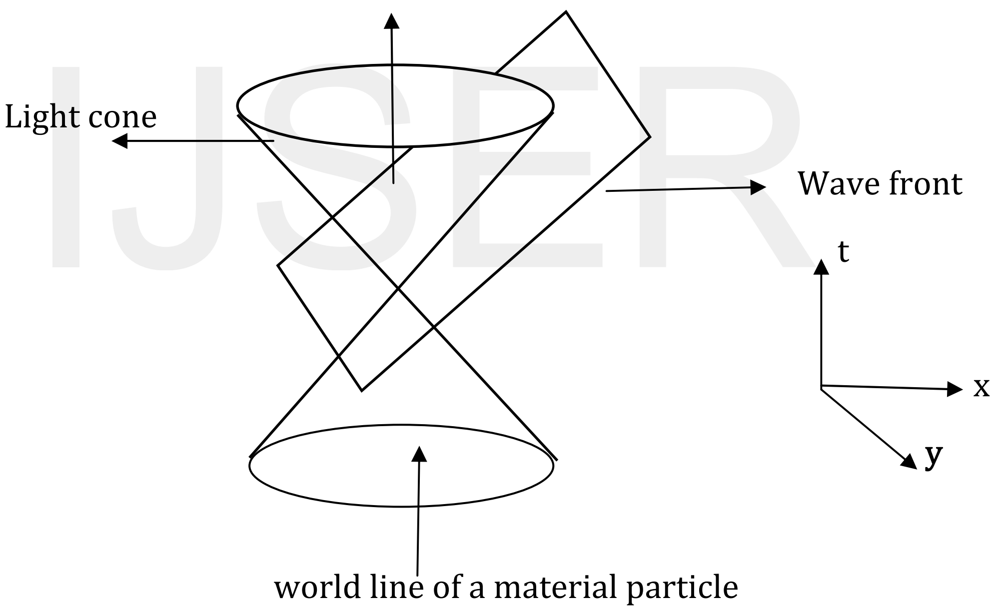

Fig: 5.7 Wave properties of particle

Therefore, the wave front is a null surface. It can be shown, that at every point on the wave front, the light cone is tangential to the surface. Any time like trajectory of a material particle confine to within the light cone can cross the wave front in only one direction. It cannot recross the wave front in the opposite direction. To do this the trajectory will have to turn around and go out the light cone. Physically, what this means is that once the wave front has crossed a material particle, the particle will have to travel faster than light in order to catch up with the wave front recross it in the other direction. Equivalently the particle; can cross a wave front in only one direction. Therefore the wave front is "**one way membrane**".

The above property is true for any null surface. The light cone is tangential to it and it behaves as a one way membrane. Material particles can cross it in one direction and cannot come out. In flat space time, only travelling wave front are example of null surfaces. When there is a gravitational field, the situation can be different, as in the case of the Schwarzschild space time.

Consider the family of surfaces given by

$$f(x^b) = \xi^a \xi^b = \left(1 - \frac{2M}{r}\right) = constant \tag{5.7.16}$$

These are two dimensional sphere $r = constant$. (provided we also take the section $t = constant$)Each of these surfaces has the normal

$$na = (\xi^b \xi_b)_a \left(0, \frac{2M}{r^2}, 0,0\right) \tag{5.7.17}$$

Then

$$n^2 = g^{ab} n_a n_b = -\left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^2}\right)^2 \tag{5.7.18}$$

When we set $r = 2M$. We see that the surface becomes null. It is like a spherical wave front frozen in space held in place by gravitation. Therefore a black hole, being a null surface is a one way membrane. Particles can go in but cannot come out (**fig: 5.7**). This is why; it is called a **black hole**.

"A luminous star of the same density as the Earth, and whose diameter should be two hundred and fifty times larger than that of the sun, would not, in consequence of its attraction, allow any of its rays to arrive at us: it is therefore possible that the largest luminous bodies in the Universe may, through the case, be invisible."

P.S.LAPLACE

CHAPTER 6

STELLAR EQUILIBRIUM AND COLLAPSE

6.1 INTRODUCTION

Gravitational fields are so weak that the practicing astrophysicist can usually ignore general relativity. This chapter deals with various sorts of object in which relativistic effects play an important, or in some cases a dominant, role. One of these is the neutron star, a "**cold**" star composed primarily of neutrons and supported against collapse by neutron degeneracy pressure. Another is the super-massive star, a giant object supported by radiation pressure, in which general relativistic effects can tip the balance between stability and instability. Most impressive of all is the black hole, a body caught in an inexorable gravitational collapse.

The existence of neutron stars and black holes was suggested in the 1930's on purely theoretical grounds, chiefly through the work of J. Robert Oppenheimer and his collaborators. However, these exotic objects remained a textbook curiosity until the 1960's, when the cooperative efforts of radio and optical astronomers began to reveal a great many strange new things in the sky.

A realistic discussion of quasi-stellar objects, galactic nuclei, pulsars, and so on, would require that we consider the effects of radiative energy transport, neutrino energy transport, turbulence, nuclear forces, magnetic fields and, above all, rotation. It would also require the discussion of massive calculations using automatic computers. In preparing this chapter, I have tried to restrict myself to the simplest calculations, which can be carried out analytically without too much trouble. These simple calculations are not very useful for a detailed understanding of astronomical observations, but they provide a valuable insight into the possible roles that general relativity can play in astrophysical phenomena.

6.2 STARS OF UNIFORM DENSITY

General relativity finds an interesting application to one other class of stable stars, those consisting of incompressible fluids, with equation of state

$$\rho = constant \tag{6.2.1}$$

These stars are of interest, not because they actually exist (they don't), but because they are simple enough to allow an exact solution of Einstein's equations, and because they set an upper limit to the gravitational red shift of spectral lines form the surface of any star.

With ρ constant, the fundamental equation $g^{ij}U^iU^j = -1$ may be written

$$\frac{-p'(r)}{[\rho + p(r)][(\rho/3) + p(r)]} = 4\pi G r \left[1 - \frac{8\pi G \rho r^2}{3}\right]^{-1} \tag{6.2.2}$$

The pressure must now be determined by integrating *inward* form the surface where $p = 0$, rather than outward, as for more realistic models. This gives

$$\frac{p(r) + \rho}{3p(r) + \rho} = \left[\frac{1 - 8\pi G \rho R^2/3}{1 - 8\pi G \rho r^2/3}\right]^{1/2}$$

Solving for $p(r)$, and expressing ρ in terms of the stellar mass,

$$\rho = \frac{3M}{4\pi R^3} \qquad for\, r < R \tag{6.2.3}$$

we find,

$$p(r) = \frac{3M}{4\pi R^3}\left\{\frac{[1 - (2MG/R)]^{1/2} - [1 - (2MGr^2/R^3)]^{1/2}}{[1 - (2MGr^2/R^3)]^{1/2} - 3[1 - (2MG/R)]^{1/2}}\right\} \tag{6.2.4}$$

The metric component $A(r)$ is immediately given by $A(r) = \left[1 - \frac{2G\mathcal{M}(r)}{r}\right]^{-1}$:

$$A(r) = \left[1 - \frac{2MGr^2}{R^3}\right]^{-1} \tag{6.2.5}$$

Whereas $B(r)$ can be calculated by using (6.2.4) in the integral

$$B(r) = exp\left\{-\int_r^\infty \frac{2G}{r^2}[\mathcal{M}(r') + 4\pi r'^3 p(r')]\left[1 - \frac{2G\mathcal{M}(r')}{r'}\right]^{-1} dr'\right\} \quad (6.2.5a):$$

$$B(r) = \frac{1}{4}\left[3\left(1 - \frac{2MG}{R}\right)^{1/2} - \left(1 - \frac{2MGr^2}{R^3}\right)^{1/2}\right]^2 \tag{6.2.6}$$

The most interesting feature of this solution is that it does not make sense for all values of M and R. The pressure given by equation (6.2.4) will become infinite at a point r_∞ where

$$r_\infty^2 = 9R^2 - \frac{4R^3}{MG} \tag{6.2.7}$$

(Also, the metric becomes singular at r_∞ because $B(r_\infty)$ vanishes.)but the pressure is a scalar, and so an infinity in $p(r)$ cannot be blamed on an injudicious choice of coordinate system. We must see to it that $p(r)$ is not singular for any real r, and the only way to accomplish this is to have r_∞^2 negative, or

$$\frac{MG}{R} < \frac{4}{9} \tag{6.2.8}$$

Note that the Schwarzschild radius $2MG$ is then less than $8/9$ the actual radius R, so there is no singularity in either the exterior solution

$$B(r) = A^{-1}(r) = \left[1 - \frac{2G\mathcal{M}(R)}{r}\right] \quad for\ \ r \geq R \tag{6.2.5b}$$

or the interior solution (6.2.5), (6.2.6).

This is not the first time that we have discovered an upper bound on the absolute value MG/R of the gravitational potential of a star. We learned that for a stable ideal-gas neutron star, MG/R is never greater than $0.36/3.2$, or 0.11. Is there than an absolute upper limit to MG/R imposed by the structure of the Einstein equations, irrespective of the equation of state?

To frame this question as a mathematical problem, we consider ρ as an arbitrary finite positive function, subject only to these general requirements:

(A) The radius R is fixed, with

$$\rho(r) = 0 \quad for\ \ r > R \tag{6.2.9}$$

(B) The mass M is fixed, with

$$\int_0^R 4\pi r^2 \rho(r) dr = M \tag{6.2.10}$$

(C) The metric coefficient $A(r)$ given by

$$A(r) = \left[1 - \frac{2G\mathcal{M}(R)}{r}\right]^{-1} \tag{6.2.10a}$$

must not be singular, so

$$\mathcal{M}(R) < \frac{r}{2G} \tag{6.2.11}$$

where

$$\mathcal{M}(R) \equiv \int_0^r 4\pi r'^2 \rho(r') dr'$$

(D) The density $\rho(r)$ must not increase outward :

$$\rho'^{(r)} \leq 0$$

(It is difficult to imagine that a fluid sphere with a larger density near the surface than near the center could be stable.) Given any function $\rho(r)$, satisfying these conditions, we can calculate $A(r)$ from Eq. (6.2.10a); we can determine $p(r)$ by integrating equation

$$-r^2 p^{'}(r) = G\mathcal{M}(r)\rho(r)\left[1+\frac{p(r)}{\rho(r)}\right]\left[1+\frac{4\pi r^3 p(r)}{\mathcal{M}(r)}\right]\left[1-\frac{2G\mathcal{M}(r)}{r}\right]^{-1} \quad (*)$$

inward from the surface (with the boundary condition that $p(R)=0$; and we can then calculate $B(r)$ from equation(6.2.5a)

$$B(r) = exp\left\{-\int_r^\infty \frac{2G}{r^2}[\mathcal{M}(r') + 4\pi r'^3 p(r')]\left[1-\frac{2G\mathcal{M}(r^{'})}{r^{'}}\right]^{-1} dr'\right\} \quad (6.2.5a)$$

Equation (6.1.11) guarantees that $A(r)$ is well behaved, and as long as $p(r)$ is finite, Eq. (*) will give $p(r) \geq 0$, and Eq. (6.2.5a) will give a finite positive-definite $B(r)$. Thus any absolute limitations on the input function $\rho(r)$(such as an upper bound on G/R) can only come from the condition that Eq. (*) must yield a finite solution for the pressure $p(r)$.

We shall exploit this condition rather indirectly, by concentrating on the metric coefficient $B(r)$ rather than on $p(r)$ itself. We first derive an equation that allows $B(r)$ to be calculated for a given density function $\rho(r)$, without having to solve for $p(r)$; from

$$R_{rr} = \frac{B''}{2B} - \frac{B'}{4B}\left(\frac{A'}{A}+\frac{B'}{B}\right) - \frac{A'}{rA} = -4\pi G(\rho - p)A$$

and

$$R_{tt} = -\frac{B''}{2A} + \frac{B'}{4A}\left(\frac{A'}{A}+\frac{B'}{B}\right) - \frac{B'}{rA} = -4\pi G(\rho + 3p)B$$

we have

$$3R_{rr}B + R_{tt}A = B'' - \frac{B}{2}\left(\frac{A'}{A}+\frac{B'}{B}\right) - \frac{3BA'}{rA} - \frac{B'}{r} = -16\pi G\rho AB$$

$$or, \qquad B'' - \frac{B'}{2}\left(\frac{A'}{A}+\frac{B'}{B}+\frac{2}{r}\right) = \frac{B}{rA}[3A^{'} - 16\pi G\rho r A^2]$$

This equation can be linearized by defining

$$B \equiv \zeta^2 \qquad (6.2.12)$$

Introducing Eq. (6.2.10a) for $A(r)$, and rearranging a bit, we find

$$\frac{d}{dr}\left[\frac{1}{r}\left(1-\frac{2G\mathcal{M}(r)}{r}\right)^{1/2}\frac{d\zeta(r)}{dr}\right]=G\left(1-\frac{2G\mathcal{M}(r)}{r}\right)^{-1/2}\left(\frac{\mathcal{M}(r)}{r^3}\right)'\zeta(r)$$

(6.2.13)

The initial conditions at $r = R$ can be determined directly from equation (6.2.5a), or from the condition that $B(r)$ fit smoothly to the exterior solution (6.2.5b); either way, we find that

$$\zeta(R)=\left[1-\frac{2MG}{R}\right]^{1/2} \tag{6.2.14}$$

$$\zeta'(R)=\frac{MG}{R^2}\left[1-\frac{2MG}{R}\right]^{-1/2} \tag{6.2.15}$$

The solution for $\zeta(r)$ must be *positive*, because $\zeta(r)$ can become negative only if it passes through the value zero, at which point B would vanish, and, according to Eq. (6.2.5a), B can vanish only if the pressure $p(r)$ has a singularity.

We next proceed to derive an upper bound for $\zeta(0)$. If ζ is positive, then the right-hand side of (6.2.13) is negative, because $3\mathcal{M}(r)/4\pi r^3$ is the mean density within the radius r, and the mean density cannot increase with r if the density does not. Thus (6.2.13) gives

$$\frac{d}{dr}\left[\frac{1}{r}\left(1-\frac{2G\mathcal{M}(r)}{r}\right)^{1/2}\frac{d\zeta(r)}{dr}\right]\leq 0$$

the equality being attained only for uniform density. Integrating this inequality from r to R and using (6.2.15), we have

$$\zeta'(R)\geq\frac{MGr}{R^3}\left[1-\frac{2G\mathcal{M}(r)}{r}\right]^{-1/2}$$

Integrating again from 0 to R and using (5.2.14), gives

$$\zeta(0)\leq\left[1-\frac{2MG}{R}\right]^{1/2}-\frac{MG}{R^3}\int_0^R\frac{rdr}{[1-(2G\mathcal{M}(r)/r)]^{1/2}}$$

The right-hand side is largest when $\mathcal{M}(r)$ is as small as possible. For a given mass M and radius R, the density distribution with $\rho'(r)\leq 0$ that gives an $\mathcal{M}(r)$ that is everywhere as small as possible has $\rho(r)$ constant, in which case

$$\mathcal{M}(r) = \frac{Mr^3}{R^3}$$

Using this in the integral, our inequality is

$$\zeta(0) \leq \frac{3}{2}\left[1 - \frac{2MG}{R}\right]^{1/2} - \frac{1}{2} \tag{6.2.16}$$

We have already noted that $\zeta(r)$ must be positive-definite; hence (6.2.16) implies that

$$\frac{MG}{R} < \frac{4}{9} \tag{6.2.17}$$

This is just the upper limit found earlier for stars of uniform density, but now we know that (6.2.17) holds for all stars, uniform or not.

It can also be proved that for a given mass and radius; the stable stars with smallest central pressure are those with uniform density. Hence the central pressure of any star is not less than the value obtained by setting $r = 0$ in Eq. (6.2.4), that is

$$p(0) \geq \frac{3M}{4\pi R^3}\left\{\frac{[1 - 2MG/R]^{1/2} - 1}{1 - 3[1 - 2MG/R]^{1/2}}\right\} \tag{6.2.18}$$

This again shows that MG/R can never equal the forbidden value $4/9$.

Our result can be immediately translated into a statement about the red shift of spectral lines from the surface of any star. According to Equations

$$\frac{v_2}{v_1} = \left(\frac{g_{00}(x_2)}{g_{00}(x_1)}\right)^{1/2},$$

$g_{rr} = A(r), g_{\theta\theta} = r^2, g_{\varphi\varphi} = r^2 sin^2\theta, g_{tt} = -B(r)$ and (6.2.5b), this is

$$z \equiv \frac{\Delta\lambda}{\lambda} = B^{-1/2}(R) - 1 = \left[1 - \frac{2MG}{R}\right]^{-1/2} - 1$$

Equations (6.2.17) imposes on z the upper bound

$$z < 2 \tag{6.2.19}$$

In fact, there seems to be a large concentration of quasi-stellar radio sources whose spectral lines show red shifts close to 1.95! However, we should not jump to the conclusion that these red shifts are necessarily due to strong gravitational fields, for red shifts near $z = 2$ require the star to be composed of a nearly incompressible fluid, with $\partial\rho/\partial p$ very small. This would seem unphysical, since we do not want the speed of sound $(\partial p/\partial\rho)^{1/2}$ to become larger than the speed of light! Bondi has shown that for a stable star with $\partial p/\partial\rho < 1$ and with $p/\rho \leq 1/3$ (as is the case for particles that interact only electromagnetically and/or in localized collisions) the red shift of spectral lines emitted from the surface is bounded by $z \leq 0.615$. In any case, there are quasi-stellar objects with red shifts $z > 2$, such as 4C25.5, with $z = 2.358$.

However, there is no theorem that limits the red shifts of light signals from the $interior$of static spherically symmetric bodies. For instance, a light signal from the center of a transparent uniform star would have a red shift given by equations

$\left\{ \frac{v_2}{v_1} = \left(\frac{g_{00}(x_2)}{g_{00}(x_1)}\right)^{1/2}, \quad g_{rr} = A(r), \quad g_{\theta\theta} = r^2, \quad g_{\varphi\varphi} = r^2 sin^2\theta, \quad g_{tt} = -Br \right.$ and (6.2.6)

$$1 + z = B^{-1/2}(0) = \frac{2}{3(1 - 2MG/R)^{1/2} - 1}$$

As MG/R approaches the maximum value $4/9$, this red shift becomes infinite. Hoyle and $Fowler$[29] have suggested that a quasi-stellar object can consist of a cluster of small dense stars, with the red shifts arising from emission and absorption in a hot cloud of gas trapped near the cluster center. It is not yet clear whether the red shifts of the QSO's arise internally, or from some other cause, such as the general cosmological recession of distant objects discussed.

6.3 TIME-DEPENDENT SPHERICALLY SYMMETRIC FIELDS

We now turn to the problems of stellar dynamics, and begin by writing down the metric and Ricci tensor for a spherically symmetric but time-dependent system. Spherical symmetry requires the proper time interval $d\tau^2$ to depend only on the rotational invariants

$$t, dt, r, x.dx = rdr, dx^2 = dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

so it can be written

$$d\tau^2 = C(r,t)dt^2 - D(r,t)dr^2 - 2E(r,t)drdt - F(r,t)r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

The function F can be removed by defining a new redial variable

$$r' \equiv rF^{1/2}(r,t)$$

The metric will then be of the same form, but with new functions C', D', E' in place of C, D, E, and of course with r' in place of r and no factor F. Dropping primes, we have then

$$d\tau^2 = C(r,t)dt^2 - D(r,t)dr^2 - 2E(r,t)drdt - F(r,t)r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

We next remove E by defining a new time

$$dt' = \eta(r,t)[C(r,t)dt - E(r,t)dr]$$

where η is an integrating factor defined to make the right-hand side a perfect differential, that is, so that

$$\frac{\partial}{\partial r}[\eta(r,t)C(r,t)] = -\frac{\partial}{\partial t}[\eta(r,t)E(r,t)]$$

(This equation can be solved by treating it as an initial value problem; given $\eta(r,t_0)$ for all r, we can solve for $\partial\eta(r,t)/\partial t$ at $t = t_0$ and determine $\eta(r, t_0 + dt)$ for all r.) The proper time is then

$$d\tau^2 = \eta^{-2}C^{-1}dt'^2 - (D + C^{-1}E^2)dr^2 - r^2\left(d\theta^2 + sin^2\theta d\varphi^2\right)$$

or, introducing new functions A and B in place $D + C^{-1}E^2$ and $\eta^{-2}C^{-1}$ and dropping the prime on t,

$$d\tau^2 = B(r,t)dt - A(r,t)dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2) \qquad (6.3.1)$$

Thus we can use the metric in its familiar "standard" form, the only new feature being that A and B now depend on t as well as r.

The non-vanishing elements of the metric tensor and its inverse are

$$g_{rr} = A \quad g_{\theta\theta} = r^2 \quad g_{\varphi\varphi} = r^2 sin^2\theta \quad g_{tt} = -B$$

$$g^{rr} = A^{-1} \quad g^{\theta\theta} = r^{-2} \quad g^{\varphi\varphi} = r^{-2}(sin\theta)^2 \quad g^{tt} = -B^{-1} \qquad (6.3.2)$$

It follows that the non-vanishing elements of the affine connection are

$$\Gamma^r_{rr} = \frac{A'}{2A} \qquad \Gamma^r_{\theta\theta} = -\frac{r}{A} \qquad \Gamma^r_{\varphi\varphi} = -\frac{rsin^2\theta}{A}$$

$$\Gamma^r_{tt} = \frac{B'}{2A} \qquad \Gamma^r_{rt} = \Gamma^r_{tr} = \frac{\dot{A}}{2A} \qquad \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r}$$

$$\Gamma^\theta_{\varphi\varphi} = -sin\theta cos\theta \qquad \Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} = \frac{1}{r} \qquad \Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = cot\theta$$

$$\Gamma^t_{rr} = +\frac{A}{2B} \qquad \Gamma^t_{tt} = \frac{\dot{B}}{2B} \qquad \Gamma^t_{tr} = \Gamma^t_{rt} = \frac{B'}{2B}$$

(A prime or a dot now denotes $\partial/\partial r$ or $\partial/\partial t$, respectively.) From

$$R^\lambda_{\mu\nu k} \equiv \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^k} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\nu} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta}$$

we obtain the independent nonzero components of the Ricci tensor :

$$R_{rr} = \frac{B''}{2B} - \frac{B'^2}{4B^2} - \frac{A'B'}{4AB} - \frac{A'}{Ar} + \frac{\ddot{A}}{2B} + \frac{\dot{A}\dot{B}}{4B^2} - \frac{\dot{A}^2}{4AB} \tag{6.3.3}$$

$$R_{\theta\theta} = -1 + \frac{1}{A} - \frac{rA'}{2A^2} + \frac{rB'}{2AB} \tag{6.3.4}$$

$$R_{tt} = -\frac{B''}{2A} + \frac{B'A'}{4A^2} - \frac{B'}{Ar} + \frac{B'^2}{4AB} + \frac{\ddot{A}}{2A} - \frac{\dot{A}^2}{4A^2} - \frac{\dot{B}\dot{A}}{4AB} \tag{6.3.5}$$

$$R_{tr} = -\frac{\dot{A}}{Ar} \tag{6.3.6}$$

Also it follows from the spherical symmetry of the metric that

$$R_{\varphi\varphi} = sin^2\theta R_{\theta\theta} \tag{6.3.7}$$

As a simple but important application of these result, let us consider a spherically symmetric but not necessarily static field in ***empty space***, where the field equations read $R_{ij} = 0$. According to (6.3.6) the field equation $R_{tr} = 0$ just tells us that A is time independent:

$$\dot{A} = 0$$

Inspection of (6.3.3) – (6.3.4) then shows that all time derivatives drop out of the field equations, and they become identical with the equations for a static isotropic gravitational field in empty space. The vanishing of R_{rr} and R_{tt} gives

$$(AB)' = 0$$

and the vanishing of $R_{\theta\theta}$ gives

$$\left(\frac{r}{A}\right)' = 1$$

Since A is time-independent, the general solution is

$$A = \left(1 - \frac{2MG}{r}\right)^{-1} \qquad B = f(t)\left(1 - \frac{2MG}{r}\right)$$

with GM a time-independent integration constant , and $f(t)$ an unknown function of t. The function $f(t)$ can be made to equal unity by defining a new time coordinate :

$$t^{'} = \int^{t} f^{1/2}(t)dt$$

The metric is now entirely time-independent, and agrees with the Schwarzschild solution

$$d\tau^2 = \left[1 - \frac{2MG}{r}\right] dt^2 - \left[1 - \frac{2MG}{r}\right]^{-1} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\varphi^2 \quad (6.3.8)$$

We have thus proved the $\boldsymbol{Birkhoff\ theorem}$, that a spherically symmetric gravitational field in empty space must be static, with a metric given by the Schwarzschild solution.

The Birkhoff theorem is analogous to the result proved by Newton in his theory of the lunar motion that the gravitational field outside a spherically symmetric body behaves as if the whole mass of the body were concentrated at the center. It is a little surprising that this result should apply in general relativity as well as in Newton's theory, for in general relativity a non-static body will usually radiate gravitational waves. The Birkhoff theorem tells us that, although a pulsating spherically symmetric body can of course produce non-static gravitational fields its mass, no gravitational radiation can escape into empty space. In this sense, the Birkhoff theorem is analogous to the well-known result of atomic theory, that a photon cannot be emitted in a quantum transition between two states of zero spin.

The Birkhoff theorem may be applied, not only to the gravitational field outside a body, but also to the field inside an empty spherical cavity at the center of a spherically symmetric (but not necessarily static) body. In this case the metric is again given by the Schwarzschild solution, but since the point $r = 0$ is here in empty space, there can be no singularity, so the integration constant MG must vanish. The Birkhoff theorem thus has the corollary that ***the metric inside an empty spherical cavity at the center of a spherically symmetric system must be equivalent to the flat-space Minkowski metric*** $\eta_{\mu\nu}$. This corollary is analogous to another famous result of Newtonian theory that the gravitational field of a spherical shell vanishes inside the shell. Stars do not usually have holes at their centers, so this corollary will not be of much use to as in this chapter. Its importance arises from the fact that the Birkhoff theorem is a local theorem, not depending on any conditions on the metric for $r \to \infty$ (aside from spherical symmetry), so that space must be flat in a spherical cavity at the center of a spherical symmetric system, even if the system is infinite even, in fact, if the system is the whole universe.

6.4 CO-MOVING COORDINATES

As a further preparation for our treatment of gravitational collapse, and also to lay a ground work for our discussion of cosmology, we know construct a very useful set of coordinates, ***the co-moving co-ordinate system*** which incorporates a more natural separation between space and time than that provided by the standard coordinates.

Imagine a finite region of space field with a dense cloud of freely falling particles. Each particle is assumed to carry along a little clock, and is given a fixed set of spatial coordinates, which can be defined as the coordinates x^i of the particle, in some arbitrary system, when its own clock reads $t = 0$.

(The rules for setting these different clocks are discussed below.) The space-time coordinates x, t of any event are defined by taking x as the spatial coordinate level of the particle that is just going by when and where the events and occurs, and by taking t as the time then shown on that particle's clock. We may think of the coordinates mesh as being dragged along by the cloud of particles, with time defined by clocks stuck on the mesh. This coordinate system will be useful throughout the region occupied by the particle cloud, for whatever interval of time in which particle trajectories do not cross.

The metric $g_{\mu\nu}$ in comoving coordinates is characterized by certain specially simple features. First, we note that the clocks are in free fall and therefore tell proper time, so the proper time interval between two points x, t and $x, t + dt$ on a given particle's trajectory is just dt, that is,

$$d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu = -g_{tt} dt^2$$

and therefore

$$g_{tt} = -1 \tag{6.4.1}$$

Also we note that the particle trajectory $x = constant,\ t = \tau$ satisfies the equation of free fall, so

$$0 = \frac{d^2x^i}{d\tau^2} + \Gamma^i_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \Gamma^i_{tt}$$

Using (6.4.1), this gives

$$0 = g^{ij}\frac{\partial g_{jt}}{\partial t}$$

or, since g^{ij} is generally a nonsingular matrix,

$$0 = \frac{\partial g_{jt}}{\partial t} \qquad (6.4.2)$$

We have kept open the option of setting the clocks attached to the different particles in an arbitrary fashion. Suppose that we rest these clocks by a transformation

$$t^{'} = t + f(x) \qquad x^{'} = x \qquad (6.4.3)$$

The new metric will have the elements

$$g^{'}_{tt} = -1 \qquad (6.4.4)$$

$$g^{'}_{ti} = g_{ti} + \frac{\partial f}{\partial x^i} \qquad (6.4.5)$$

$$g^{'}_{ij} = g_{ij} - g_{ti}\frac{\partial f}{\partial x^j} - g_{tj}\frac{\partial f}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} \qquad (6.4.6)$$

It would be great simplification if the function f could be chosen so that the two terms in Eq. (6.4.5) cancel, giving $g^{'}_{it} = 0$. There are two important cases where this is possible :

(A) Suppose that we can reset all clocks so that all particles are at rest at a time $t = 0$. This assumption can be given an absolute physical significance by interpreting it to mean that for each particle P at $t = 0$, it is possible to find a locally inertial coordinate system $\tilde{x}^\mu$ in which the separation between P and neighboring particles is purely spatial,

$$\left(\frac{\partial \tilde{x}^0}{\partial x^i}\right)_{t=0,x=x_p} = 0$$

and in which the movement of P in a time interval dt is purely temporal,

$$\left(\frac{\partial \tilde{x}^i}{\partial t}\right)_{t=0,\mathrm{x}=\mathrm{x_p}} = 0$$

The metric in this locally inertial system is the Minkowski metric η_{ij}, so the space-time components of the metric in the comoving system at $t = 0$ are

$$g_{t\mu}(x_p, 0) = \left[\eta_{ij}\frac{\partial \tilde{x}^i}{\partial x^\mu}\frac{\partial \tilde{x}^j}{\partial t}\right]_{t=0,x=x_p} = 0$$

With (5.4.2), it follows that g_{ti} vanishes everywhere, so the metric is given by

$$d\tau^2 = dt^2 - g_{ij}(x,t)dx^i dx^j \qquad (6.4.7)$$

(B) If the metric is manifestly spherically symmetric, then the line element must have the general form with which we started in the last section, that is,

$$d\tau^2 = C(r,t)dt^2 - D(r,t)dr^2 - 2E(r,t)drdt - F(r,t)r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

The only non-vanishing time-space component g_{tj} is $g_{tr} = 2E$, and (6.4.2) then tells us that E is time independent, so

$$g_{tr} = 2E(r)$$

$$g_{t\theta} = g_{t\varphi} = 0$$

We can therefore eliminate the components g_{tj} by resetting the clocks as in (6.4.3), with

$$f = -2\int^r E(r)dr$$

Using (6.4.4) and dropping primes, the metric is now of the form

$$d\tau^2 = dt^2 - U(r,t)dr^2 - V(r,t)(d\theta^2 + sin^2\theta d\varphi^2) \qquad (6.4.8)$$

with U and V new unknown functions that replace D and F .

it is of course possible to construct coordinate systems of this sort even if the cloud of freely falling particles is purely imaginary. In differential geometry, coordinate systems satisfying (6.4.1) and (6.4.2) are called ***Gaussian***, and if g_{ti} vanishes, so that the line element takes the form (6.4.7), then we call the coordinates ***Gaussian normal***. However, these coordinate systems find their most important applications to system that actually do consist of a freely falling fluid. In this case the fluid velocity four-vector by definition has zero space component,

$$U^i = 0 \tag{6.4.9}$$

and since U^i is normalized so that

$$g_{ij}U^iU^j = -1 \tag{6.4.10}$$

[see Eq.(5.2.5a)]the time component of U^i must be

$$U^t = (-g_{tt})^{-1/2} = 1 \tag{6.4.11}$$

We shall be working only with spherically symmetric comoving coordinate systems, with line element (6.4.8). The non-vanishing elements of the metric tensor are

$$\begin{gathered} g_{rr} = U \qquad g_{\theta\theta} = V \qquad g_{\varphi\varphi} = V sin^2\theta \qquad g_{tt} = -1 \\ g^{rr} = U^{-1} \qquad g^{\theta\theta} = V^{-1} \qquad g^{\varphi\varphi} = (V sin^2\theta)^{-1} \qquad g^{tt} = -1 \end{gathered} \tag{6.4.12}$$

The non-vanishing elements of the affine connection are readily calculated as

$$\left.\begin{aligned} &\Gamma^r_{rr} = \frac{U^{'}}{2U} \quad \Gamma^r_{\theta\theta} = -\frac{V^{'}}{2U} \quad \Gamma^r_{\varphi\varphi} = -\frac{V^{'}}{2U} sin^2\theta \quad \Gamma^r_{rt} = \Gamma^r_{tr} = \frac{\dot{U}}{2U} \\ &\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{V^{'}}{2V} \qquad \Gamma^\theta_{\theta t} = \Gamma^\theta_{t\theta} = \frac{\dot{V}}{2V} \qquad \Gamma^\theta_{\varphi\varphi} = -sin\theta cos\theta \\ &\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{V^{'}}{2V} \qquad \Gamma^\varphi_{\varphi t} = \Gamma^\varphi_{t\varphi} = \frac{\dot{V}}{2v} \qquad \Gamma^\varphi_{\theta\varphi} = cot\theta \\ &\Gamma^t_{rr} = \frac{\dot{U}}{2} \qquad \Gamma^t_{\theta\theta} = \frac{\dot{V}}{2} \qquad \Gamma^t_{\varphi\varphi} = \frac{\dot{V}}{2} sin^2\theta \end{aligned}\right\} \tag{6.4.13}$$

(A prime or dot denotes $\partial/\partial r$ or $\partial/\partial t$, respectively.) From

$R^\lambda_{\mu\nu\kappa} \equiv \frac{\partial\Gamma^\lambda_{\mu\nu}}{\partial x^k} - \frac{\partial\Gamma^\lambda_{\mu\kappa}}{\partial x^\nu} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\nu\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta}$ we obtain the independent nonzero components of the Ricci tensor :

$$R_{rr} = \frac{V''}{V} - \frac{V'^2}{2V^2} - \frac{U'V'}{2UV} - \frac{\ddot{U}}{2} + \frac{\dot{U}^2}{4U} - \frac{\dot{U}\dot{V}}{2V} \tag{6.4.14}$$

$$R_{\theta\theta} = -1 + \frac{V''}{2U} - \frac{V'U'}{4U^2} - \frac{\ddot{V}}{2} - \frac{\dot{V}\dot{U}}{4U} \tag{6.4.15}$$

$$R_{tt} = \frac{\ddot{U}}{2U} + \frac{\ddot{V}}{V} - \frac{\dot{U}^2}{4U^2} - \frac{\dot{V}^2}{2V^2} \qquad (6.4.16)$$

$$R_{tr} = \frac{\dot{V}'}{V} - \frac{V'\dot{V}}{2V^2} - \frac{\dot{U}V'}{2UV} \qquad (6.4.17)$$

Also, it again follows from the spherical symmetry of the metric that

$$R_{\varphi\varphi} = R_{\theta\theta} sin^2\theta \qquad (6.4.18)$$

$$R_{r\theta} = R_{r\varphi} = R_{\theta\varphi} = R_{\theta t} = R_{\varphi t} = 0 \qquad (6.4.19)$$

6.5 GRAVITATIONAL COLLAPSE

We saw that a cooling star of mass greater than a few solar masses cannot reach equilibrium as either a white dwarf or a neutron star. It may be that a massive star will always eject enough matter by the time it reaches the end of its thermonuclear evolution so that its mass drops below the **Chandrasekhar** or the **Oppenheimer-Volkoff** limits. If not, then it will collapse.

A proper treatment of gravitational collapse would be prohibitively complicated. In order to get some feeling for what can happen during collapse, we consider only the simplest *case*,[32] the spherically symmetric collapse of "**dust**" with negligible pressure. Since the dust particles are acted on by purely gravitational forces, they fall freely, and we can use them as the physical basis of a co-moving coordinate system of the sort discussed in the last section. The metric is then given by equation (5.4.8):

$$d\tau^2 = dt^2 - U(r,t)dr^2 - V(r,t)(d\theta^2 + sin^2\theta d\varphi^2) \qquad (6.5.1)$$

The energy-momentum tensor for a fluid of negligible pressure is given by

$$T^{ij} = \rho U^i U^j \qquad (6.5.2)$$

where $\rho(r,t)$ is the proper energy density and U^i is the velocity four-vector, given for a comoving coordinate system by Eqs. (6.4.9) and (6.4.11):

$$U^r = U^\theta = U^\varphi = 0, \qquad U^t = 1 \qquad (6.5.3)$$

The equations of momentum conservation $\left(T^i_\mu\right)_{;i} = 0$ are automatically satisfied, and the equation for energy conservation reads

$$0 = \left(T_{\mu}^{i}\right)_{;i} = -\frac{\partial\rho}{\partial t} - \rho\Gamma_{\lambda t}^{\lambda} = -\frac{\partial\rho}{\partial t} - \rho\left(\frac{\dot{U}}{2U} + \frac{\dot{V}}{V}\right)$$

or in other words

$$\frac{\partial}{\partial t}\left(\rho V\sqrt{U}\right) = 0 \qquad (6.5.4)$$

The Einstein field equations can be written

$$R_{ij} = -8\pi G S_{ij} \qquad (6.5.5)$$

where

$$S_{ij} = T_{ij} - \frac{1}{2}g_{ij}T_{\lambda}^{\lambda} = \rho\left[\frac{1}{2}g_{ij} + U_i U_j\right] \qquad (6.5.6)$$

This may be evaluated with the aid of Eqs. (6.5.1) and (6.5.3); we find that the only non-vanishing components of $S_{ij}$are

$$S_{rr} = \rho\frac{U}{2} \quad S_{\theta\theta} = \rho\frac{V}{2} \quad S_{\varphi\varphi} = S_{\theta\theta}sin^2\theta \quad S_{tt} = \frac{\rho}{2} \qquad (6.5.7)$$

In particular,

$$S_{tr} = 0 \qquad (6.5\ .8)$$

Using (6.5.7) – (6.5.8) and (6.4.14) – (6.4.17) in (6.5.5) yields four field equations :

$$\frac{1}{U}\left[\frac{V''}{V} - \frac{V'^2}{2v^2} - \frac{U'V'}{2UV}\right] - \frac{\ddot{U}}{2U} + \frac{\dot{U}^2}{4U^2} - \frac{\dot{U}\dot{V}}{2UV} = -4\pi G\rho \qquad (6.5.9)$$

$$-\frac{1}{V} + \frac{1}{U}\left[\frac{V''}{2V} - \frac{U'V'}{4UV}\right] - \frac{\ddot{V}}{2V} - \frac{\dot{V}\dot{U}}{4VU} = -4\pi G\rho \qquad (6.5.10)$$

$$\frac{\ddot{U}}{2U} + \frac{\ddot{V}}{V} - \frac{\dot{U}^2}{4U^2} - \frac{\dot{V}^2}{2V^2} = -4\pi G\rho \qquad (6.5.11)$$

$$\frac{\dot{V}'}{V} - \frac{V'\dot{V}}{2V^2} - \frac{\dot{U}V'}{2UV} = 0 \qquad (6.5.12)$$

Let us simplify our model even further, and assume that ρ is independent of position. We can now seek a separable solution, with

$$U = R^2(t)f(r) \qquad V = S^2(t)g(r)$$

Then (6.5.12) requires that $\dot{S}/S$ equal $\dot{R}/R$, so we can normalize f and g so that

$$S(t) = R(t)$$

Also, we are still free to redefine the redial coordinate as an arbitrary function $\tilde{r}$ of r, and in particular we can choose $\tilde{r} = \sqrt{g(r)}$, so f and g are replaced with $\tilde{f} = fg'^2/4g$ and $\tilde{g} = \tilde{r}^2$. Dropping the tildes, we have then

$$U = R^2(t)f(r) \qquad V = R^2(t)r^2 \tag{6.5.13}$$

Equations (5.5.9) and (5.5.10) then read

$$-\frac{f'(r)}{rf^2(r)} - \ddot{R}(t)R(t) - 2\dot{R}^2(t) = -4\pi G R^2(t)\rho(t) \tag{6.5.14}$$

$$\left[-\frac{1}{r^2} + \frac{1}{rf^2(r)} - \frac{f'(r)}{2rf^2(r)}\right] - \ddot{R}(t)R(t) - 2\dot{R}^2(t) = -4\pi G R^2(t)\rho(t)$$

(6.5.15)

The first terms in (6.5.14) and (6.5.15) must evidently be equal constants, which we shall call, $-2k$:

$$-\frac{f'(r)}{rf^2(r)} = -\frac{1}{r^2} + \frac{1}{rf^2(r)} - \frac{f'(r)}{2rf^2(r)} = -2k$$

The unique solution is

$$f(r) = [1 - kr^2]^{-1}$$

so the metric takes the form

$$d\tau^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2\right] \tag{6.5.16}$$

(Incidentally, the metric is spatially homogeneous as well as isotropic, and for this reason it will provide the kinematic framework for our treatment of relativistic cosmology)

Our remaining problem is to calculate the functions $\rho(t)$ and $R(t)$. Using (6.5.13) and (6.5.14) in the energy-conservation equation (6.5.4), we find that $\rho(t)R^3(t)$ is constant. We normalize the redial coordinate r so that

$$R(0) = 1 \tag{6.5.17}$$

and therefore

$$\rho(t) = \rho(0)R^{-3}(t) \tag{6.5.18}$$

The field equations (6.5.14) or (6.5.15) and (6.5.11) are now ordinary differential equations:

$$-2k - \ddot{R}(t)R(t) - 2\dot{R}^2(t) = -4\pi G\rho(0)R^{-1}(t) \tag{6.5.19}$$

$$\ddot{R}(t)R(t) = -\frac{4\pi G}{3}\rho(0)R^{-1}(t) \tag{6.5.20}$$

We can eliminate $\ddot{R}(t)$ by adding these two equations, and find

$$\dot{R}^2(t) = -k + \frac{8\pi G}{3}\,\rho(0)R^{-1}(t) \tag{6.5.21}$$

Equations (6.5.19) and (6.5.20) can be recovered from (6.5.21) and its time derivative, so we can forget about them and simply use (6.5.21) to calculate $R(t)$.

We shall now assume that the fluid is at rest (in standard coordinates) at $t = 0$, so

$$\dot{R}(0) = 0 \tag{6.5.22}$$

and therefore (6.5.21) and (6.5.17) give

$$k = \frac{8\pi G}{3}\rho(0) \tag{6.5.23}$$

Thus equation (6.5.21) can be written

$$\dot{R}^2(t) = k[R^{-1}(t) - 1] \tag{6.5.24}$$

The solution is given by the parametric equations of a ***cloid***:

$$t = \left(\frac{\psi + sin\psi}{2\sqrt{k}}\right)$$

$$R = \frac{1}{2}(1 + cos\psi) \tag{6.5.25}$$

Note that $R(t)$ vanishes when $\psi = \pi$, and hence when $t = T$, where

$$T = \frac{\pi}{2\sqrt{k}} = \frac{\pi}{2}\left(\frac{3}{8\pi G\rho(0)}\right)^{1/2} \tag{6.5.26}$$

Thus a fluid sphere of initial density $\boldsymbol{\rho(0)}$ and zero pressure will collapse from rest to a state of infinite proper energy density in the finite time $\boldsymbol{T}$.

Although the collapse is complete at a finite coordinate time $t = T$, any light signal coming to us from the sphere's surface will be delayed by its gravitational field, so we on earth will not see the star suddenly vanish. To make this more specific, we have to complete our calculation by finding the metric outside the star.

The Birkhoff theorem shows that it is always possible to find a "standard" coordinate system $\bar{r}, \bar{\theta}, \bar{\varphi}, \bar{t}$ in which the metric outside the sphere takes the form

$$d\tau^2 = \left(1 - \frac{2MG}{\bar{r}}\right) d\bar{t}^2 - \left(1 - \frac{2MG}{\bar{r}}\right)^{-1} d\bar{r}^2 - \bar{r}^2 d\bar{\theta}^2 - \bar{r}^2 sin^2\theta d\bar{\varphi}^2$$

But this metric is not in the Gaussian normal form (6.5.1), so in order to match solutions at the surface we either have to convert the interior solution (6.5.16) into standard coordinates, or the exterior solution (6.5.27) into Gaussian normal coordinates.

Inspection of Eq. (6.5.16) shows immediately that the standard spatial coordinate $\bar{r}, \bar{\theta}, \bar{\varphi}$ must be chosen as

$$\bar{r} = rR(t), \qquad \bar{\theta} = \theta, \qquad \bar{\varphi} = \varphi \tag{6.5.28}$$

In order to define a standard time coordinate such that $d\tau^2$ does not contain a cross-term $d\bar{r}\, d\bar{t}$, we employ the "integrating factor" technique which gives

$$\bar{t} = \left(\frac{1 - ka^2}{k}\right)^{1/2} \int_{S(r,t)}^{1} \frac{dR}{(1 - ka^2/R)} \left(\frac{R}{1 - R}\right)^{1/2} \tag{6.5.29}$$

where

$$S(r, t) = 1 - \left(\frac{1 - kr^2}{1 - ka^2}\right)^{1/2} (1 - R(t)) \tag{6.5.30}$$

The constant a arbitrary, but may conveniently be chosen as the radius of the sphere in comoving coordinates. It is straightforward to check that the metric in the coordinate system $\bar{r}, \bar{\theta}, \bar{\varphi}, \bar{t}$ takes the standard form

$$d\tau^2 = B(r, t) d\bar{t}^2 - A(r, t) d\bar{r}^2 - \bar{r}^2 (d\bar{\theta}^2 + sin^2\theta d\bar{\varphi}^2)$$

with

$$B = \frac{R}{S} \left(\frac{1 - kr^2}{1 - ka^2}\right)^{1/2} \frac{(1 - ka^2/S)^2}{(1 - kr^2/R)} \tag{6.5.31}$$

$$A = \left(1 - \frac{kr^2}{R}\right)^{-1} \tag{6.5.32}$$

it now being understood that S is a function of $\bar{t}$ defined by Eq. (6.5.29) and that r and $R(t)$ are function of $\bar{r}$ and S, $\bar{r}$ and $\bar{t}$, defined by solving Eqs. (6.5.28) and (6.5.30). This is a mess, but at the radius $r = a$ of the star (a constant, since r is a commoving coordinate) we have

$$\bar{r} = \bar{a}(t) \equiv aR(t) \tag{6.5.33}$$

$$\bar{t} = \left(\frac{1 - ka^2}{k}\right)^{1/2} \int_{R(t)}^{1} \frac{dR}{(1 - ka^2/R)} \left(\frac{R}{1 - R}\right)^{1/2} \tag{6.5.34}$$

$$B(\bar{a}, \bar{t}) = \left(1 - \frac{ka^2}{R(t)}\right) \tag{6.5.35}$$

$$A(\bar{a}, \bar{t}) = \left(1 - \frac{ka^2}{R(t)}\right)^{-1} \tag{6.5.36}$$

(Equation (6.5.34) could have been obtained by integrating the equations for free fall)Comparing with (6.5.27), we see that the interior and exterior solution fit continuously at $\bar{r} = aR(t)$ if

$$k = \frac{2MG}{a^3} \tag{6.5.37}$$

With (6.5.23), this just says that

$$M = \frac{4\pi}{3} \rho(0) a^3 \tag{6.5.38}$$

not a surprising result!

Now we return to the problem of calculating the behavior of light signals emitted from the surface of the collapsing sphere. A light signal emitted in a redial direction at a standard time $\bar{t}$ will have $d\bar{r}/d\bar{t}$ given by Eq. (6.5.27) and the condition $d\tau = 0$, so it will arrive at a distant point $\bar{r}$ at a time

$$\bar{t}' = \bar{t} + \int_{aR(t)}^{\bar{r}'} \left(1 - \frac{2MG}{r}\right)^{-1} dr$$

The most striking consequence of Eq. (6.5.39) and (6.5.34) is that both $\bar{t}$ and $\bar{t}'$ approach infinity when the radius (6.5.33) of the sphere approaches the Schwarzschild radius $2GM$, that is, when

$$R(t) \rightarrow \frac{2GM}{a} = ka^2 \tag{6.5.40}$$

The collapse to the Schwarzschild radius therefore appears to an outside observer to take an infinite time, and the collapse to $\boldsymbol{R = 0}$ is utterly unobservable from outside.

Although the collapsing sphere does not suddenly disappear, it does fade out of sight, because light from its surface is subject to an increasing red shift. The proper time for a light source on the sphere's surface is just the comoving time t, so the comoving time interval between emission of wave crests at the surface equals the natural wavelength λ_0 that would be emitted by the source in the absence of gravitation. The standard time interval $d\bar{t}'$ between arrivals of wave crests at $\bar{r}'$ is the observed wave length λ'; thus the functional change of wavelength is

$$z \equiv \frac{\lambda' - \lambda_0}{\lambda_0} = \frac{d\bar{t}'}{dt} - 1 = \frac{d\bar{t}}{dt} - a\dot{R}(t)\left(1 - \frac{2MG}{aR(t)}\right)^{-1} - 1$$

$$= -\dot{R}(t)\left(1 - \frac{ka^2}{R(t)}\right)^{-1}\left[\left(\frac{1 - ka^2}{k}\right)^{1/2}\left(\frac{R(t)}{1 - R(t)}\right)^{1/2} + a\right] - 1$$

Using (6.5.24) to determine $\dot{R}(t)$, this is

$$z = \left(1 - \frac{ka^2}{R(t)}\right)^{-1}\left[(1 - ka^2)^{1/2} + a\sqrt{k}\left(\frac{1 - R(t)}{R(t)}\right)^{1/2}\right] - 1 \tag{6.5.41}$$

In order to see how the red shift z varies with $\bar{t}'$, let us assume that the sphere is initially very much larger than its Schwarzschild radius

$$ka^2 = \frac{2GM}{a} \ll 1 \tag{6.5.42}$$

and distinguish two periods in the history of the collapse :

(A) Until t gets close to T, we have

$$\frac{ka^2}{R(t)} \ll 1 \tag{6.5.43}$$

Using (6.5.42) and (6.5.43) in (6.5.34), (6.5.39) and (6.5.41) gives (with $\bar{r}' \gg a$)

$$\bar{t} \simeq t$$

$$\bar{t} \simeq \bar{t} + \bar{r}' - aR(t) \simeq t + \bar{r}' - aR(t) \simeq t + \bar{r}'$$

$$z \simeq a\sqrt{k}\left(\frac{1-R(t)}{R(t)}\right)^{1/2} \simeq a\sqrt{k}\left(\frac{1-R(\bar{t}'-\bar{r}')}{R(\bar{t}'-\bar{r}')}\right)^{1/2} \qquad (6.5.44)$$

(B) Eventually we have

$$\frac{ka^2}{R(t)} \to 1$$

at a time t_1 given by (6.5.25) as

$$t_1 \simeq \frac{1}{2\sqrt{k}}\left[\pi - \frac{4}{3}(ka^2)^{3/2}\right] \qquad (6.5.45)$$

Now (6.5.34), (6.5.39), and (6.5.41) give

$$\bar{t} \simeq -ka^3 ln\left[1-\frac{ka^2}{R(t)}\right] + constant$$

$$\bar{t}' \simeq \bar{t} - ka^3 ln\left[1-\frac{ka^2}{R(t)}\right] + constant$$

$$\simeq -2ka^3 ln\left[1-\frac{ka^2}{R(t)}\right] + constant$$

$$z \simeq 2\left(1-\frac{ka^2}{R(t)}\right)^{-1} \propto exp\left(\frac{\bar{t}'}{2ka^3}\right) \qquad (6.5.46)$$

Putting (A) and (B) together, we conclude that the red shift z seen by an observer at $\bar{r}'$ is zero when the collapse is observed to begin, increases gradually but remains of order $a\sqrt{k} \ll 1$ until a time very close to $T = {}^{\pi}/_{2}\sqrt{k}$ has passed, and then grows exponentially with a rate ${}^{1}/_{2}\, ka^3$. For example, a

collapsing sphere with a mass $M = 10^8 M_{\odot}$and radius $a = 100$ light years will have a red shift z of order 10^{-3} for a period of order 10^5 years, after which the red shift suddenly begins growing exponentially with an e-folding time of order 1 min. for practical purposes, the collapsing sphere is suddenly and completely cut off from communication with the rest of the universe.

Completely cut off ? Even if a collapsing body does fade out of side, it still has a gravitational field, and the measurement of this field at great distances can be used to determine the energy, momentum, and angular momentum of the body. If the body has a net electric charge, then measurement of the electric field at great distances will, via Gauss's theorem, also tell as the charge. It is interesting to ask whether measurements of the gravitational and/or electromagnetic fields outside a collapsing body can yield any information about the body ***beyond*** the energy, momentum, angular momentum, and charge. In the case of a spherically symmetric electrically neutral body, which we have been considering in this chapter, the answer is provided by **Birkhoff's theorem**: ***The gravitational field outside a spherically symmetric body must be of the Schwarzschild form, so all we can ever learn about the body is its mass. (Spherical symmetry, of course, implies zero momentum and zero angular momentum.)***

Carter has shown that when the gravitational field of an ***axially symmetric*** collapsing body settles down to a stationary state, its exterior metric belongs to a ***two-parameter*** family of solutions, such as the Kerr metrics that are completely specified by the total mass and angular momentum. It is widely believed that the gravitational field of any electrically neutral collapsing body will eventually approach the Kerr form.

As mentioned in the introduction to this chapter, interest in the phenomenon of gravitational collapse was rekindled in the last decade by the discovery of quasi-stellar sources, which appears to require some powerful new source of energy. The maximum energy available from fusion of hydrogen into the most stable nuclei, say iron, is only 8 Mev per nucleon, or less than 1% of the rest-mass. Matter-antimatter annihilation could have 100% efficiency (apart from neutrino energy losses), but this process can be important only if there is some abundant natural source of anti-nucleons. Otherwise the only likely mechanism for conversion of mass into energy with high efficiency is gravitational collapse.

A cloud of dust that is collapsing as in the Oppenheimer-Snyder model will obviously release no energy to the outside world. To extract the growing kinetic energy of the falling particles, something must slow them on the way down−either a macroscopic "bounce" of the whole system, or particle-particle collisions that heat the collapsing gas. Detailed calculations reveal a discouragingly low efficiency for conversion of mass into available energy in the gravitational collapse of an ***isolated*** body. However, particles falling into a Kerr metric can reemerge with a higher energy, acquired at the expense of the rotational energy of the collapsing body. Whether or not gravitational collapse has anything to do with quasi-stellar sources, the question remains: What happens to a real cooling star whose mass is above the Chandrasekhar and Oppenheimer-Volkoff limits? In recent years topological methods have been used by Penrose and Hawking to prove a number of powerful theorems, to the effect that under reasonable conditions (validity of general relativity, positivity of energy, ubiquity of matter, causality) collapse becomes inevitable once a ***trapped surface*** forms. A trapped surface is a closed space-like two-dimensional surface for which both the outgoing and the ingoing families of future-directed null geodesics orthogonal to the surface are converging. (For the Schwarzschild metric, the spheres with are r and t constant are trapped surfaces for r within the Schwarzschild radius $2MG$.) However, it is not known whether a real massive star will actually develop a trapped surface, or merely explode into fragments with small enough mass to form stable neutron stars or white dwarfs. If gravitational collapse is indeed the inevitable fate of massive bodies, then we must expect that the universe is full of **black holes,** collapsing bodies whose presence is betrayed only through their gravitational fields or through the energy released when matter is drawn in. Our best hope of observing gravitational collapse would be to find a binary star, one member an ordinary visible star, and the other member a black hole.

“A singularity represents the ultimate unknown able in science”

PAUL DEVIES

CHAPTER 7

SINGULARITIES IN COSMOLOGY

7.1 INTRODUCTION

We saw that all the **Friedmann models** have **singularities** in the finite past, that is, at a finite time in the past, which we have called $\boldsymbol{t = 0}$; the **scale factor R(t) goes to zero** and correspondingly some **physical variables, such as the energy density, go to infinity**. Only exceptionally, such as in the de Sitter or the steady state models, is there no singularity in the finite past. But these latter models have some unphysical or unorthodox feature, such as the continuous creation of matter, which is not generally acceptable. The presence of singularities in the universe, where physical variables such as the mass-energy density or the pressure or the strength of the gravitational field go to infinity seems doubtful to many people, who therefore feel uneasy about this kind of prediction of the equations of general relativity. This was partly the motivation with which Einstein searched for a **'unified field theory'**. In this connection He says (1950):

The theory is based on a separation of the concepts of the gravitational field and matter. While this may be a valid approximation for weak fields, it may presumably be quite inadequate for very high densities of matter. One may not therefore assume the validity of the equations for very high densities and it is just possible that in a unified theory there would be no such singularity.

There was at one time the feeling that the singularities in the Friedmann models arise because of the highly symmetric and idealized form of the metric, and that, for example, if the metric were not spherically symmetric, the matter coming from different directions might '**miss**' each other and not gather at the centre of symmetry, as it does in the (spherically symmetric) Friedmann models. However, it was shown by Hawking and Penrose (1970) that spherical symmetry is not essential for the existence of a singularity. We shall consider this work later. There are in the main two possible approaches for dealing with the problem of singularities. **Firstly**, one can try to relax the symmetry conditions inherent in Robertson–Walker metrics and try to determine what the field equations predict in these more general cases. **Secondly**, one can try to derive some general results about singularities by using reasonable assumptions, say about the energy–momentum tensor, without considering the field equations in detail. The Penrose–Hawking results fall in the latter category. As regards the former approach, the simplest relaxation of the symmetries of the Robertson–Walker metrics (which are homogeneous and isotropic) is to drop the requirement of isotropy and consider metrics which are only homogeneous. We shall consider such metrics in some detail in the next section, partly with a view to explaining another approach to the question of singularities, pioneered by Lifshitz and

Khalatnikov (1963). There is an extensive literature on singularities and cosmological solutions, incorporating both the approaches mentioned above. This chapter is meant to be only a brief introduction to this work.

7.2 HOMOGENEOUS COSMOLOGIES

In this section we shall derive the metric and field equations for homogeneous (but not isotropic) cosmologies.

Consider the spatial part of the metric

$$ds^2 = c^2 dt^2 - h_{ij} dx^i dx^j \qquad (i, j = 1,2,3)$$

as follows:

$$dl^2 = h_{ij}(t, x^1, x^2, x^3) dx^i dx^j, \qquad (i, j = 1,2,3) \tag{7.2.1}$$

A metric is homogeneous if after a transformation of the spatial coordinates (x^1, x^2, x^3) to new coordinates x'^1, x'^2, x'^3 the metric (7.2.1) transforms to the following one:

$$dl^2 = h_{ij}\left(t, x'^1, x'^2, x'^3\right) dx'^i dx'^j, \qquad (i, j = 1,2,3) \tag{7.2.2}$$

with the same functional dependence as before of the h_{ij}on the new spatial coordinates. Further, this set of transformations must be able to carry any point to any other point. One way to characterize the invariance of the metric under spatial transformations is to consider a set of three differential forms $e_m^{(a)} dx^m$ (with $a = 1,2,3$) which are invariant under these transformations, as follows:

$$e_m^{(a)}(x) dx^m = e_m^{(a)}(x') dx'^m, \tag{7.2.3}$$

where we have written x for x^1, x^2, x^3, etc. in the arguments. With the use of these forms a metric invariant under spatial transformations can be constructed as follows (the η_{ab} are six functions of t):

$$dl^2 = \eta_{ab}\left(e_m^{(a)} dx^m\right)\left(e_n^{(b)} dx^m\right), \tag{7.2.4}$$

that is, the three-dimensional metric tensor h_{ij} of (7.2.2) is given as follows:

$$h_{ij} = \eta_{ab} e_i^{(a)} e_j^{(b)} \tag{7.2.5}$$

Note that in (7.2.3) the $e_m^{(b)}$ on the two sides of the equation are respectively the same functions of the old and new coordinates. We introduce the reciprocal triplet of vectors $e_{(a)}^m$ by the following relations:

$$e_{(a)}^m e_m^{(b)} = \delta_a^b . e_{(a)}^m e_n^{(a)} = \delta_n^m . \quad (7.2.6)$$

It can be shown after some manipulations [23], that (7.2.3) leads to the following equation for the reciprocal triplet $e_{(a)}^m$:

$$e_{(a)}^m \frac{\partial e_{(b)}^n}{\partial x^m} - e_{(a)}^m \frac{\partial e_{(a)}^n}{\partial x^m} = C_{ab}^c e_{(c)}^n , \quad (7.2.7)$$

where the C_{ab}^c are constants satisyfing $C_{ab}^c = -C_{ba}^c$. These are the so called structure constants of the groups of transformations. If we denote by x^a the following linear differential operator:

$$X_a = e_{(a)}^m \frac{\partial}{\partial x^m} , \quad (7.2.8)$$

then (7.2.7) can be written as follows:

$$[X_a, X_b] \equiv X_a X_b - X_b X_a = C_{ab}^c X_c \quad (7.2.9)$$

One can now use the Jacobi identity given by

$$\left[[X_a, X_b], X_c\right] + \left[[X_b, X_c], X_a\right] + \left[[X_c, X_a], X_b\right] = 0 \quad (7.2.10)$$

to derive the following relation for the structure constants:

$$C_{ab}^e C_{ec}^d + C_{bc}^e C_{ea}^d + C_{ca}^e C_{eb}^d = 0 \quad (7.2.11)$$

The different types of homogeneous spaces correspond to the different inequivalent solutions of (7.2.11) satisfying the anti-symmetry condition $C_{ab}^c = -C_{ba}^c$. Some solutions are equivalent to each other, reflecting the fact that the $e_{(a)}^m$ can still be subjected to a linear transformation with constant coefficients so that the operators $X_a$are not unique.

There are nine different types of homogeneous spaces that arise from the different inequivalent solutions of (7.2.11) with the required anti-symmetry condition. These are known as the **Bianchi types,** types I–IX. The Einstein equations for these spaces can be reduced to a system of ordinary differential equations for the η_{ab}, without the necessity of working out the frame vectors $e_{(a)}^m$, etc.

7.3 SOME RESULTS OF GENERAL RELATIVISTIC HYDRODYNAMICS

Before considering the results of Penrose and Hawking it is useful to have some idea of relativistic hydrodynamics. The fundamental quantity here is the four-velocity vector u^{μ} of a continuous distribution of matter in hydrodynamic motion. Thus $u^{\mu}$is a unit time-like vector. Some of the following formulae are valid for any arbitrary four-vector u^{μ}. With the use of the covariant derivative $u_{\mu;\nu}$ one can define the following quantities which are of **physical significance**:

(a) The scalar expansion $\theta = u^{\mu}_{;\mu}$, which gives the rate at which a volume element orthogonal to the vector u^{μ} expands or contracts.

(b) A measure of the departure of the velocity field from geodesic motion is given by the acceleration $\dot{u}_{\mu} = u_{\mu;\nu}u^{\nu}$. In the absence of non-gravitational forces, such as in the case of dust (pressure-less matter), the particles follow geodesics and the acceleration vanishes.

(c) The shear tensor is symmetric, trace-free and is orthogonal to the vector u_{μ}. It describes the manner in which a volume element orthogonal to u changes its shape, and is given as follows:

$$\sigma_{\mu\nu} = \frac{1}{2}\left(u_{\mu;\nu} + u_{\nu;\mu}\right) - \frac{1}{3}\left(g_{\mu;\nu} - u_{\mu}u_{\nu}\right)\dot{\theta} - \frac{1}{2}\left(\dot{u}_{\mu}u_{\nu} + \dot{u}_{\nu}u_{\mu}\right) \quad (7.3.1)$$

(d) A measure of the amount of rotational motion present in the matter is given by the vorticity tensor defined as follows:

$$w_{\mu\nu} = \frac{1}{2}\left(u_{\mu;\nu} - u_{\nu;\mu}\right) - \frac{1}{2}\left(\dot{u}_{\mu}u_{\nu} - \dot{u}_{\nu}u_{\mu}\right) \quad (7.3.2)$$

One can also define a vorticity vector w^{μ} as follows:

$$w^{\mu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}u_{\nu}u_{\rho;\sigma} \,, \quad (7.3.3)$$

where $\varepsilon^{\mu\nu\rho\sigma}$is the **Levi–Civita alternating tensor** which is antisymmetric in any pair of indices with $\varepsilon^{0123} = (-g)^{-1/2}, g$ being the determinant of the metric. If the vorticity vector or tensor vanishes, the vector u^{μ} is said to be

hypersurface orthogonal and this implies the absence of rotation in some invariant sense (rotation of the local rest frame relative to the compass of inertia; see, for example, Synge, 1937; Gödel, 1949).
Next we use $A_{\mu;\nu;\lambda} - A_{\mu;\lambda;\nu} = A_\sigma R^\sigma_{\mu\nu\lambda}$ with u_μ instead of A_μ and make slight changes in the indices to get the following equation:

$$u^\mu_{;\alpha;\beta} - u^\mu_{;\beta;\alpha} = R^\mu_{\nu\beta\alpha} u^\nu \tag{7.3.4}$$

In this equation we set μ equal to β and multiply the resulting equation with u^α as follows:

$$u^\sigma \left(u^\mu_{;\alpha;\mu} - u^\mu_{;\beta;\alpha} \right) = R_{\nu\alpha} u^\nu u^\alpha \tag{7.3.5}$$

where we have used $R_{\mu\nu} = g^{\lambda\sigma} R_{\lambda\mu\sigma\nu} = R^\sigma_{\mu\sigma\nu}$.

From the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \left(8\pi G/c^4\right) T_{\mu\nu} \tag{7.3.6a}$$

with

$$T^{\mu\nu} = (\varepsilon + p) u^\mu u^\nu - p g^{\mu\nu} \tag{7.3.6b}$$

we readily get

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \left[(\varepsilon + p) u^\mu u^\nu + \frac{1}{2} (p - \varepsilon) g_{\mu\nu} \right] \tag{7.3.6c}$$

whence it follows:

$$R_{\mu\nu} u^\mu u^\nu = \frac{4\pi G}{c^4} (\varepsilon + 3p) \tag{7.3.7}$$

One can use the definitions of expansion, shear, vorticity and acceleration given above to write (7.3.5) as follows:

$$\theta_{,\alpha} u^\alpha + \frac{1}{3} \theta^2 - \dot{u}^\alpha_{;\alpha} + 2(\sigma^2 - w^2) = -R_{\mu\nu} u^\mu u^\nu \tag{7.3.8}$$

In deriving this relation the following equations have been used (the first one follows by taking the dot-derivative of $u^\mu u_\mu = 1$);

$$\dot{u}_\mu u^\mu = 0, \tag{7.3.9a}$$
$$\sigma_{\mu\nu} u^\nu = w_{\mu\nu} u^\nu = 0, \tag{7.3.9b}$$
$$\sigma^2 \equiv \frac{1}{2} \sigma_{\mu\nu} \sigma^{\mu\nu}, \tag{7.3.9c}$$
$$w^2 \equiv \frac{1}{2} w_{\mu\nu} w^{\mu\nu}. \tag{7.3.9d}$$

Equation (7.3.8) holds for an arbitrary four-vector u^μ. We now let u^μ be the four-velocity of matter, so that (7.3.7) can be used in (7.3.8). We then get the following important equation, known as the **Raychaudhuri equation** (Raychaudhuri, 1955, 1979):

$$\theta_{,\alpha} u^\alpha + \frac{1}{3}\theta^2 - \dot{u}^\alpha_{;\alpha} + 2(\sigma^2 - w^2) + 4\pi(\varepsilon + 3p)Gc^{-4} = 0 \tag{7.3.10}$$

The importance of this equation derives from the fact that in one form or another it is used in most if not all singularity theorems of general relativity. To see the relevance of this equation to the question of singularities we consider a simple and somewhat crude analysis. Consider a set of time-like geodesics described by the four-vector u^μ. Let these geodesics be irrotational. Thus we have $\dot{u}^\mu = w = 0$. Let λ be a parameter along a typical geodesic so that $u^\mu = dx^\mu/d\lambda$. Then

$$\theta_{,\alpha} u^\alpha = \frac{\partial\theta}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda} = \frac{d\theta}{d\lambda} = -\frac{1}{3}\theta^2 - 2\sigma^2 - 4\pi(\varepsilon + 3p)Gc^{-4} \tag{7.3.11}$$

Now make the assumption that $2\sigma^2 + 4\pi(\varepsilon + 3p)Gc^{-4}$ is greater than a positive constant $\frac{1}{3}\xi^2$. Then the behavior of θ is governed by the following differential equation:

$$d\theta/d\lambda = -\frac{1}{3}(\theta^2 + \xi^2), \tag{7.3.12}$$

which has the solution

$$\theta = \theta_0 - \xi \tan[(\xi/3)(\lambda - \lambda_0)] \tag{7.3.13}$$

θ_0 being the value of θ at $\lambda = \lambda_0$. From this equation it is clear that θ becomes infinite as λ is decreased from the value λ_0 to $\lambda_0 - 3\pi/2\xi$. If, for example, λ denotes the proper time along the geodesic, then this shows that at a finite time in the past the expansion θ becomes infinite. An infinite value of θ indicates that at that point geodesics cross each other and there is a sort of '**explosion**' like the big bang. In the Friedmann models u^μ is given by the vector $(1, 0, 0, 0)$ and it is readily verified that θ, which is the covariant divergence of this vector, is given by $3\dot{R}/R$. In the case $k = 0$, for example, from

$$\dot{R}^2 = (8\pi G/3)\varepsilon\, R^2/c^2$$

we see that this is proportional to $\varepsilon^{1/2}$. We know that this tends to infinity as the big bang $t = 0$ is approached. Thus the expansion θ tends to infinity at a finite time in the past. The assumption $2\sigma^2 + 4\pi(\varepsilon + 3p)Gc^{-4} = \frac{1}{3}\xi^2$ is a limiting case. If $2\sigma^2 + 4\pi(\varepsilon + 3p)Gc^{-4} > \frac{1}{3}\xi^2$ the infinity in θ occurs at a shorter distance away from $\lambda = \lambda_0$.

The above somewhat crude analysis can be made more precise, and this is essentially what is done in the singularity theorems. These theorems are very technical and need a great deal of preliminary apparatus. We shall here give only the statement of one of these theorems, but we need some familiarity with singularities.

7.4 DEFINITION OF SINGULARITIES

The question of a definition of singularities in general relativity is a highly complex one and we can only consider a bare outline of the extensive literature on the subject. An excellent account of this topic is given in Hawking and Ellis (1973).

We have encountered a simple case of a singularity in the Friedmann models, where at $t = 0$ the mass-energy density goes to infinity. The mass energy density is a simple example of the so-called '**curvature scalars**' or '**curvature invariants**' whose values do not change under a coordinate transformation, so that if they are infinite at a certain point in one coordinate system, they will be infinite at that point in every coordinate system. Another example of a curvature scalar is the **Ricci scalar** defined by $R = g^{\mu\nu}R_{\mu\nu}$. It is well known that in empty space (where the Ricci tensor vanishes), there are four curvature invariants, one of these being $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ (see for example, Weinberg (1972). If one of the curvature scalars goes to infinity at a point, that point is a **space-**

time singularity, and cannot be considered as a part of the space-time manifold, whose points are defined to be such that one can introduce a coordinate system so that the metric and its derivatives to second order are well behaved. Such points may be called '**regular**' points.

However, all the curvature scalars remaining finite at a point does not necessarily imply the point is regular. The usual example of this that is cited is that of the two-dimensional surface of an ordinary cone in three dimensions. The curvature scalars of this surface remain finite as one approaches the apex of the cone, but the latter is not a regular point as it is not possible to introduce any coordinate system that is well behaved at that point. On the other hand, the metric behaving badly at a point does not necessarily mean that the point is singular, because the bad behaviour may be simply due to the unsuitable nature of the coordinate system. These matters are illustrated well by the Schwarzschild metric.
The Schwarzschild solution is given as follows:

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \qquad (7.4.1)$$

Here the coefficient of dt^2 goes to infinity at $r = 0$ and that of dr^2 goes to infinity at $r = 2m$. The curvature invariants are well behaved at $r = 2m$, but some of them go to infinity at $r = 0$. Thus the bad behaviour of the metric cannot be removed at $r = 0$, so the latter is a singularity. However, as mentioned earlier, the fact that the curvature invariants are regular at $r = 2m$ does not necessarily mean that the latter is not a singularity. To prove this one would have to find a coordinate system which is well behaved at the point. For a long time after the Schwarzschild solution was discovered, in 1916, such a coordinate system could not be found. It was observed that the radial time-like and null geodesics displayed no unusual behaviour at $r = 2m$. Finally, in 1960 **Kruskal** found the following transformation from (r, t) to new coordinates (u, v) which shows that, the point $r = 2m$ is regular:

$$u^2 - v^2 = (2m)^{-1}(r - 2m)exp(r/2m), v = u\ tanh(ct/4m) \qquad (7.4.2)$$

with the metric (7.4.1) given as follows:

$$ds^2 = r^{-1}(32m^3)exp(-r/2m)(du^2 - dv^2) - r^2(d\theta^2 + sin^2\theta d\phi^2), \qquad (7.4.3)$$

where r is to be interpreted as a function of u and given implicitly by the first equation in (7.4.2).

Another aspect of the question of singularities can be illustrated with the Schwarzschild metric, as follows (Raychaudhuri, 1979, p. 146). Transform the coordinate r in (7.4.1) to a new coordinate r'given by

$$r - 2m = r'^2, \quad (7.4.4)$$

this changes (7.4.1) to the following form:

$$ds^2 = c^2 r'^2 / \left(r'^2 + 2m\right) dt^2 - \left(r'^2 + 2m\right)(d\theta^2 + sin^2\theta d\phi^2) - 4\left(r'^2 + 2m\right) dr'^2 \quad (7.4.5)$$

Clearly this metric is regular for all values of r'in$0 \leq r \leq \infty$. But this is only a part of the space represented by (7.4.1) with $0 \leq r \leq \infty$. In (7.4.5) there would be no singularities of the curvature scalars such as $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ for any values of r'. It is thus not always satisfactory simply to see if the metric components are regular. One way to demand regularity which is physically meaningful is to require that all time-like and null geodesics should be complete in the sense that they can be extended to arbitrary values of their affine parameters. Since time-like and null geodesics give respectively the paths of freely falling (that is, in motion under purely gravitational forces) massive and massless particles, this requirement means that the space-time must contain complete histories of such freely falling particles, and that these geodesics should not suddenly come to an end at any point. In fact even this may not be satisfactory as the definition of a regular space-time, as Geroch (1967) has provided an example of a space time that is geodesically complete (that is, the geodesics can be extended arbitrarily) but one that has a non-geodetic time-like curve (for example an observer propelled by a space-ship, that is, non-gravitational forces) with bounded acceleration which has a finite length. To get over these kinds of difficulties a modified definition of completeness, called b-completeness, has been given by Schmidt (1973).

7.5 AN EXAMPLE OF A SINGULARITY THEOREM

As indicated earlier, there are various forms of singularity theorems, mostly due to Penrose, Hawking and Geroch (see Hawking and Ellis, 1973), which involve elaborate conditions, some of which are quite technical.

Roughly speaking, these theorems show that quite reasonable assumptions lead to at least one consequence which is physically unacceptable. We will give here the statement of one of these theorems, due to Hawking and

Penrose (1970), which is as follows:
Space-time is not time-like and null geodesically complete if:

(a) $R_{\mu\nu}K^{\mu}K^{\nu} \geq 0$for every non-space-like vector K^{μ}. If the Einstein equations (7.3.6a) are valid, and if $K^{\mu}$is taken to be a unit timelike vector, this condition is readily seen to imply $T_{\mu\nu}K^{\mu}K^{\nu} \geq \frac{1}{2}T$. If, in addition, $T_{\mu\nu}$ is that for a perfect fluid given by (7.3.6b) and K^{μ} is taken to be the four-velocity u^{μ}, then this condition implies $\varepsilon + 3p \geq 0$. For this reason this is sometimes referred to as the energy condition. Physically it is very reasonable.

(b) Every non-space-like geodesic contains a point at which

$K_{[\mu}R_{\nu]\lambda\sigma[\rho}K_{\tau]}K^{\lambda}K^{\sigma} \neq 0$ [ ] implies anti symmetrization,

where $K_{\mu}$is the tangent vector to the geodesics. This is one of the rather technical conditions and it appears that this is true for any general solution of Einstein's equations.

(c) There are no closed time-like curves. Physically this means that no observer can go to his past.

(d) There exists a point p such that the future or past null geodesics from p are focussed by the matter or curvature and start to reconverge. Penrose and Hawking show that observations on the microwave background radiation indicate that this condition is satisfied.

There are actually two alternatives to the condition (d) which are more technical. We thus see that assumptions which are quite reasonable lead to consequences which are physically very strange, such as a particle's world line suddenly coming to an end, or an observer meeting his past.

7.6 AN ANISOTROPIC MODEL

To see an example of singularities which is different from the simple Friedmann cases and yet not too complicated, we will consider in this section a model that is homogeneous but anisotropic. It is, in fact, the metric

$$ds^2 = A(t)dt^2 - B(t)dx^2 - C(t)dy^2 - D(t)dz^2$$

with $A = 1$, and we use X^2, Y^2, Z^2instead of B, C, D in that equation, so that our metric is as follows:

$$ds^2 = c^2dt^2 - X^2(t)dx^2 - Y^2(t)dy^2 - Z^2(t)dz^2 \quad (7.6.1)$$

Such models have been studied by Raychaudhuri (1958), Schücking and Heckmann (1958) and others. The case $X = Y$ with dust was considered by Thorne (1967). An account of this model is given in Hawking and Ellis (1973, p. 142).The fact that the metric (7.6.1) is homogeneous. It is anisotropic because not all directions from a point are equivalent. There are several reasons for studying anisotropic universes. We have mentioned earlier that the universe displays a high degree of isotropy in the present epoch. However, in earlier epochs, perhaps very early ones, there may have been a significant amount of anisotropy. Also, in a realistic situation the singularity in the universe is unlikely to possess the high degree of symmetry that the Friedmann models have. The observed isotropy of the universe needs to be explained and, in the process of seeking this explanation, one must consider more general models of the universe than the Friedmann ones. We will consider solutions of Einstein's equations for the metric (7.6.1) for a perfect fluid with zero pressure, that is, dust. We set $G = 1$ and $c = 1$ for this section and the next, and define a function $S(t)$ by $S^3 = XYZ$. A solution of Einstein's equation is given as follows (M, a, b are constants):

$$\left.\begin{array}{l} \epsilon = 3M/(4\pi S^3), \quad X = S\left(t^{2/3}/S\right)^{2sina}, \quad Y = S\left(t^{2/3}/S\right)^{2sin(a+2\pi/3)}, \\ Z = S\left(t^{2/3}/S\right)^{2sin(a+4\pi/3)}, \quad S^3 = \dfrac{9}{2}Mt(t+b). \end{array}\right\} (7.6.2)$$

The constant b determines the amount of anisotropy, the value $b = 0$ giving the isotropic Einstein–de Sitter universe. The constant 'a' determines the direction of most rapid expansion, the domain of 'a' being $-\pi/6 < a < \pi/2$. We have

$$\dot{S}/S = (2/3t)\left(t + \frac{1}{2}b\right)(t+b), \dot{X}/X = (2/3t)\left[t + \frac{1}{2}b(1 + 2sina)\right]/(t+b) \quad (7.6.2)$$

the expressions for $\dot{Y}/Y$ and $\dot{Z}/Z$ being obtained by replacing a in $\dot{X}/X$ by $a + 2\pi/3$ and $a + 4\pi/3$ respectively. This universe has a highly anisotropic singular state at $t = 0$. For large t it tends to isotropy, in fact to the Einstein–de Sitter universe.

Suppose we follow the time t backwards to the initial singularity. At first there is isotropic contraction. Let $a \neq \frac{\pi}{2}$. Then $1 + 2sin(a + 4\pi/3)$ is negative. Thus the collapse in the z-direction halts and is replaced by expansion, the rate of which becomes infinite as t tends to zero. The collapse is monotonic in the x- and y-directions. Consider now the situation forwards from $t = 0$. The matter collapses from infinity in the z-direction, then halts and expands. In the x- and y-directions it expands monotonically. Thus we have here a **cigar-shaped singularity**. If one could observe the matter far back in time, one would see a maximum red-shift in the z-direction, then the red-shift would decrease to zero (corresponding to the halt), then one would get indefinitely large blue-shifts, the latter occurring in light given off by the matter near $t = 0$.

The case $a = \frac{\pi}{2}$ is somewhat different. Here we have

$$\dot{X}/X = (2/3t)\left(t + \frac{3}{2}b\right)/(t + b), \dot{Y}/Y = \dot{Z}/Z = (2/3)(t + b)^{-1} \qquad (7.6.4)$$

Following time backwards again, the initially isotropic contraction slows down to zero in the y- and z-directions but the collapse is monotonic in the x-direction. Going forwards in time, the rate of expansion of the universe in the y- and z-directions starts from a finite value but the expansion rate in the x-direction is infinite. This is thus a '**pancake**' singularity. There are limiting red-shifts in the y- and z-directions, but no limit to the red-shifts in the x-direction.

7.7 THE OSCILLATORY APPROACH TO SINGULARITIES

In this section we consider an interesting approach to singularities developed by Lifshitz and Khalatnikov (1963) and by Belinskii, Khalatnikov and Lifshitz (1970). We study one of the homogeneous spaces that were introduced in Section 7.2, namely, Bianchi type IX, whose structure constants are as follows (see (7.2.11)):

$$C^1_{11} = C^2_{31} = C^3_{12} = 1 \qquad (7.7.1)$$

Denoting $(x^1, x,^2, x^3)$ by (θ, ϕ, ψ), the three vectors $e^{(a)}_m$ (see (7.2.3) and (7.2.4)) can be taken as follows:

$$e_m^{(1)} = (sin\psi, -cos\psi sin\theta, 0), e_m^{(2)} = (cos\psi, sin\psi sin\theta, 0), e_m^{(3)} = (0, cos\theta, 1) \quad (7.7.2)$$

The metric (7.2.4) is given as follows, where we have taken $\eta_{ab}(t)$ to be diagonal and set $\eta_{11} = a^2, \eta_{22} = b^2$, and $\eta_{33} = c^2$.

$$ds^2 = dt^2 - a^2(sin\psi d\theta - cos\psi sin\theta d\phi)^2 - b^2(cos\psi d\theta + sin\psi sin\theta d\phi)^2 - c^2(cos\theta d\phi + d\psi)^2 \quad (7.7.3)$$

In the isotropic models, near the singularity the spatial curvature term behaves as R^{-2}whereas the mass-energy density behaves as R^{-3}(for zero pressure) and as R^{-4} (for radiation). Thus in the Friedmann models the curvature terms go to infinity slower than the terms arising from T and the derivatives with respect to time of the metric (that is, R terms). This kind of singularity is referred to as a **velocity-dominated singularity** (Eardley, Liang and Sachs, 1972). In the anisotropic models which are our concern in this section the behaviour near the singularity is dominated by curvature terms as observed by Belinskii and his coworkers and by Misner (1969) and is called the **mixmaster singularity**.

Thus if we are interested in the behaviour near the initial singularity for the anisotropic metric (7.7.3), it is sufficient to consider the empty space or vacuum Einstein equations where $T_{\mu\nu} = 0$, for the terms arising from $T_{\mu\nu}$ are negligible in comparison to the other terms. The empty space Einstein equations can be written as follows:

$$(ab\dot{c})^{\cdot}/(abc) = (2a^2b^2c^2)^{-1}[(a^2 - b^2)^2 - c^4], \quad (7.7.4a)$$
$$(\dot{a}bc)^{\cdot}/(abc) = (2a^2b^2c^2)^{-1}[(b^2 - c)^2 - a^4], \quad (7.7.4b)$$
$$(a\dot{b}c)^{\cdot}/(abc) = (2a^2b^2c^2)^{-1}[(c^2 - a^2)^2 - b^4], \quad (7.7.4c)$$
$$\ddot{a}/a + \ddot{b}/b + \ddot{c}/c = 0 \quad (7.7.4d)$$

Here a dot represents differentiation with respect to t. If the right hand sides in (7.7.4a)–(7.7.4c) were absent, we would get the following well-known **Kasner solution** (1921) (of Bianchi type I):

$$a = t^q, b = t^r, c = t^p, \quad (7.7.5)$$

where p, q, r are constants satisfying

$$p + q + r = p^2 + q^2 + r^2 = 1 \quad (7.7.6)$$

Suppose now that even when the terms on the right hand sides of $(7.7.4a) -$

$(7.7.4c)$ are present, there exist certain ranges of values of t for which the metric is given approximately by (7.7.5):

$$a{\sim}t^q, b{\sim}t^r, c{\sim}t^p, \qquad (7.7.7)$$

Then from (7.7.4d) we get

$$p^2 + q^2 + r^2 = p + q + r \qquad (7.7.8)$$

It is readily verified that not all the three expressions on the right hand sides of $(7.7.4a)$– $(7.7.4c)$ can be positive, that is, one of these at least must be negative. From this it follows, substituting (7.7.7) into the left hand sides of $(7.7.4a)$– $(7.7.4c)$, that at least one of the expressions $p(p + q + r - 1)$, $q(p + q + r - 1), r(p + q + r - 1)$ must be negative. The possibility that, p, q, r are all positive with $(p + q + r - 1)$negative is inadmissible because it contradicts (7.41) (for in this case we must have $0 < p < 1, 0 < q < 1$, $0 < r < 1$, so that $p^2 < p$, $q^2 < q$, $r^2 < r$, and (7.7.8) becomes impossible). Thus at least one of the indices p, q, r is negative. This implies that the length along at least one direction shrinks while (since $p + q + r > 0$ from (7.7.8)) the spatial volume, which is determined by the product $(abc)^2$ expands. In fact $(7.7.4a)$– $(7.7.4c)$ do not allow two of the exponents p, q, r to be negative at the same time.

We suppose that p is negative and $q < r$. Then (7.7.7) implies that for small t, a and b can be neglected in comparison with c. We now define new dependent variables α, β, γ and a new independent variable τ by the following relations:

$$a = exp(\alpha), b = exp(\beta), c = exp(\gamma); dt/d\tau = abc \qquad (7.7.9)$$

These transformations, together with the approximations introduced above, enable us to write $(7.7.4a)$– $(7.7.4c)$ as follows:

$$\gamma'' = -\frac{1}{2}exp(4\gamma), \qquad (7.7.10a)$$

$$\alpha'' = \beta'' = \frac{1}{2}exp(4\gamma) \qquad (7.7.10b)$$

where a prime denotes differentiation with respect to τ. Equation $(7.7.10a)$ is in the form of the equation of motion of a particle which is moving in a potential well which is exponential. The 'velocity' thus changes sign corresponding to a change from a region where c is decreasing to one where

c is increasing. Belinskii et al. assume that the right hand sides of $(7.7.4a)$– $(7.7.4c)$ are small enough at a certain epoch such that $p\ +q\ +r$ is nearly unity and one has the Kasner solution with

$$abc = wt, \tau = w^{-1} logt + constant, \quad (7.7.11)$$

where w is a constant. Equations (6.43a) and (6.43b) can then be integrated as follows:

$$a^2 = a_0^2[1 + exp(4pw\tau)]exp(2qw\tau), \quad (7.7.12a)$$
$$b^2 = b_0^2[1 + exp(4pw\tau)]exp(2rw\tau), \quad (7.7.12b)$$
$$c^2 = 2|p|[cosh(2wp\tau)]^{-1}, \quad (7.7.12c)$$

where we have chosen the integration constants so that as tends to infinity, a, b, c go to the assumed Kasner solution with a negative p. We get the following asymptotic values of a, b, c as tends to infinity and minus infinity respectively:

As $\tau \to \infty, a{\sim}exp(qw\tau), b{\sim}exp(rw\tau), c{\sim}exp(pw\tau),$ $(7.7.13a)$

As $\tau \to -\infty, a{\sim}exp[w(q + 2p)\tau], b{\sim}exp[w(q + 2p)\tau], c{\sim}exp(-pw\tau),$ $(7.7.13b)$

In $(7.7.13a)$ we have $w\tau{\sim}log\ t$ while in $(7.7.13b)$, $w(1 + 2p){\sim}t$. In the second of these limits, that is in $(7.7.13b)$, transforming back to t from τ (with $w(1 + 2p) = t$), we get

$$a{\sim}t^{q'}, b{\sim}t^{r'}, c{\sim}t^{p'}, \quad (7.7.14)$$

Where

$$p' = -p/(1 + 2p) > 0, \quad (7.7.15a)$$

$$q' = (2p + q)/(1 + 2p) < 0 \quad (7.7.15b)$$

$$r' = (r + 2p)/(1 + 2p) > 0 \quad (7.7.15c)$$

This behaviour is different from that existing in the limit $\tau \to \infty$ which is given by (7.7.7), in the sense that the exponent in c has changed from negative to positive, while that of a has become negative (that is, q is positive but q'negative). Thus the a- and c-axes have interchanged their expanding and contracting behaviours. This indicates that, as we move towards the singularity, distances along two of the axes oscillate while that along the third axis decreases monotonically. This happens in successive periods which are called 'eras'. On going from one era to the next, the axis along which distances decrease monotonically changes to another one. Asymptotically the order in which this change occurs becomes a random process [23]. One has a particularly long era if (p, q, r) corresponds to the triplet$(1, 0, 0)$. In this case there are no particle horizons in the direction for which the index is unity, since $\int_0 \; t^{-1}\, dt$ diverges. In the course of evolution this particular direction also changes and this phenomenon may lead to effective abolition of all particle horizons. This was one of the motivations of the mixmaster model of Misner which was thought to provide the solution to the '**horizon**' problem, that is, to explain why the universe is so isotropic and homogeneous. But this model did not provide a solution to the problem, although some interesting insights were gained. This completes our brief exposition of singularities in cosmology.

7.8 A SINGULARITY-FREE UNIVERSE ?

A new class of inhomogeneous cosmological solutions has been found by Senovilla (1990) which does not seem to possess any singularities in the past, with the curvature and matter invariants regular and smooth everywhere. The source is a perfect fluid with equation of state $\varepsilon = 3p$. The metric is as follows (with signature +2):

$$ds^2 = e^2(-dt^2 + dx^2) + K(qdy^2 + q^{-1}dz^2) \qquad (7.8.1)$$

where the functions f, K and q depend on t and x only and are given explicitly as follows:

$$\begin{aligned} e^f &= [Acosh(at) + Bsinh(at)]^2 cosh(3ax) \\ K &= [Acosh(at) + Bsinh(at)]^2 sinh(3ax)[cosh(3ax)]^{-3/2} \\ q &= [Acosh(at) + Bsinh(at)]^2 sinh(3ax) \end{aligned} \qquad (7.8.2)$$

where a, A, B are arbitrary constants. The pressure and energy density are given as follows:

$$p = \frac{1}{3}\varepsilon = 5\chi^{-1}a^2[Acosh(at) + Bsinh(at)]^{-2} + [cosh(3ax)]^{-4} \qquad (7.8.3)$$

where χ is the gravitational constant in suitable units.

In two important papers, Raychaudhuri (1998, 1999) evaluates the new Senovilla solution and re-examines the singularlity theorems, and offers an additional theorem. To recapitulate, there are essentially **four conditions**:

(1) the causality condition forbidding closed time-like lines,

(2) the strong energy condition $\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) u^{\mu} u^{\nu} \geq 0$,

(3) a condition on the Riemann-Christoffel tensor, and

(4) existence of a trapped surface.

Raychaudhuri quotes from Misner, Thorne and Wheeler (1973): '**All the conditions except the trapped surface seem eminently reasonable for any physically realistic space time**' (p. 935). Raychaudhuri also discusses the further solutions found by Ruiz and Senovilla (1992). One of the important points to notice is that it is the last condition that is violated by the new singularity-free solution.

However, as Raychaudhuri shows, the average of the physical and kinematic scalars taken over the entire space-time vanishes. In the new solution the space-time is open in all directions, which means, according to Raychaudhuri, that the space-time has topology $R^3 \times R$. Raychaudhuri goes on to enunciate and prove an interesting new theorem: '**In any singularity free non-rotating universe, open in all directions, the space-time average of all stress energy invariants including the energy density vanishes**.' Here '**non-rotating**' means all matter has world lines forming a normal congruence, that is, one that is hyper surface orthogonal. This means essentially that the tangent four-vectors to the world lines are orthogonal to the space-like three-surface on which the matter lies at any instant. The proof is based on Raychaudhuri's earlier equation (7.3.10).

"Evolutionary cosmology formulates theories in which a universe is capable of giving rise to and generating future universes out of itself within black holes or whatever."

ROBERT NOZICK

CHAPTER 8

THE NATURE OF SINGULARITIES IN SYMMETRIC SCALAR FIELD COSMOLOGIES

8.1 INTRODUCTION

The nature of singularities in general solution of the Einstein equations is a subject about which mach remains to be learned. Various classes of singularities have been defined which represent possible models for general behavior. Examples are curvature singularities, crushing singularities[1], velocity dominated singularities[2] and isotropic singularities[3]. In this paper spacetimes belonging to one of the simplest classes of inhohomogeneous cosmologies will be examined in order to get as much information as possible about their singularities and to test the applicability of the models just mentioned.

The space-times considered in the following are solutions of the Einstein equations coupled to a massless scalar field in the standard way.

Thus, if ϕ denotes the scalar field they are solutions of

$$G_{\alpha\beta} = 8\pi \left[\nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2}(\nabla^\gamma \phi \nabla_\gamma \phi) g_{\alpha\beta}\right] \qquad (8.1.1)$$

The Bianchi identities imply that ϕ satisfies the wave equation. These space-times are further assumed to be plane symmetric. Plane symmetric solutions of the Einstein equations with a scalar field as matter source have been discussed by Tabensky and Taub [4]. In fact their paper is on stiff fluids but, as they show, it is possible to transform between these two matter models under rather general circumstances. They write the field equations in a particularly simple form. If the gradient of the area of the orbits is everywhere time-like then these equations can be simplified further. This condition will be assumed in the following. It has been shown elsewhere that for appropriate boundary conditions it is automatically fulfilled unless the space-time is flat [5]. Tabensky and Taub show that the only non-trivial equation to be solved is the linear hyperbolic equation

$$\phi_{tt} + t^{-1}\phi_t = \phi_{xx} \qquad (8.1.2)$$

When this has been done a quality Ω is obtained by integrating the ordinary differential equation

$$\Omega_t = t(\phi_t^2 + \phi_x^2) \qquad (8.1.3)$$

This can be done starting on an initial hypersurface of constant t. In order that all Einstein equations should be satisfied the constraint equation

$$\Omega_x = 2t\phi_t\phi_x \qquad (8.1.4)$$

should hold on the initial hypersurface. The space-time metric is

$$ds^2 = t^{-1/2}e^{\Omega}(-dt^2 + dx^2) + t(dy^2 + dz^2) \qquad (8.1.5)$$

Here t belongs to the interval $(0, \infty)$. To avoid spurious singularities it is assumed that the space-time is spatially compact. This can be arranged by demanding that the coordinates x, y and z be periodic. The periodicity of y and z plays no significant role in the following but the periodicity of x means that ϕ and Ω (which only depend on t and x) are required to be periodic in x. In order to say anything about the nature of singularities in some general class of space-times it seems unavoidable to demand some kind of spatial boundary conditions since otherwise anything could happen. Spatial compactness is the simplest possibility of specifying boundary conditions for cosmological space-times.

The initial value problem for data given on the hypersurface $t = t_0 > 0$ can be solved as follows. An initial data set consists of periodic functions ϕ_0, ϕ_1 and Ω_0 which satisfy the equation

$$(\Omega_0)_x = 2t_0\phi_1(\phi_0)_x \qquad (8.1.6)$$

For simplicity they will be assumed to be C^∞ although the arguments which follow can also be carried through when these functions have an appropriate finite degree of differentiability. A solution is sought with $\phi(t_0, x) = \phi_0(x)$, $\phi_t(t_0, x) = \phi_1(x)$ and $\Omega(t_0, x) = \Omega_0(x)$. Under these conditions (8.6) is just the constraint equation (8.1.4) on the hypersurface $t = t_0$. To construct the solution first solve the linear hyperbolic equation (8.1.2) on the time interval $(0, \infty)$ with initial data ϕ_0 and ϕ_1. Standard theory ensures the existence of a unique C^∞ solution ϕ. Then Ω may be determined by integrating (8.1.3) with initial value Ω_0 for each fixed value of x.

The solutions of the initial value problem have an apparent singularity at $t = 0$. The aim of the following is to show that this is a true singularity (i.e. that space-time cannot be extended through it) and to obtain more detailed information about its nature. In Section 2 it is shown that $t = 0$ is always a curvature singularity and that the Kretschmann scalar $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ blows up uniformly as $t \to 0$. The consequences for strong cosmic censorship are discussed. In Section 3 the singularity is shown to be crushing and it is concluded that a neighbourhood of the singularity can be foliated by constant mean curvature hypersurfaces. An asymptotic expansion for the solution in a neighbourhood of $t = 0$ is obtained in Section 4 which shows in particular that the singularity is velocity dominated. In the final section a sufficient condition is given for the singularity to be isotropic.

Note that the simplification of the field equations which allows the analysis which follows to be carried out depends very much on the symmetry and the fact that the matter content of space-time is described by a massless scalar field. If plane symmetry is replaced by spherical symmetry or if the massless scalar field is replaced by almost any other kind of matter, then the equations for the matter fields, the equation for Ω and the equation for the area of the orbits are all coupled. The property of the matter which is needed for decoupling is that the trace of the projection of the energy-momentum tensor to the orthogonal complement of the orbits should vanish.

8.2 CURVATURE SINGULARITIES

The curvature of a general plane-symmetric space-time will now be computed. It is always possible to introduce local coordinates so that the metric takes the form

$$ds^2 = g_{ab}dx^a dx^b + r^2\delta_{AB}dy^A dy^b. \tag{8.2.1}$$

Here lower and upper case indices take the values 0,1 and 2,3 respectively. Let K denote the Gaussian curvature of the two-dimensional metric g_{ab} and let ∇_a denote the covariant derivative associated to that metric. Then the curvature components are

$$R^a_{bcd} = K(\delta^a_c g_{bd} - \delta^a_d g_{bc}) \tag{8.2.2}$$

$$R^A_{BCD} = -\nabla^a r\nabla_a r(\delta^A_C\delta_{BD} - \delta^A_D\delta_{BC}) \tag{8.2.3}$$

$$R^a_{BcD} = -r\nabla^a\nabla_c r\delta_{BD}. \tag{8.2.4}$$

Hence

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = 4K^2 + 4r^{-4}(\nabla^a r\nabla_a r)^2 + 8r^{-2}\nabla_a\nabla_b r\nabla^a\nabla^b r \tag{8.2.5}$$

When the curvature components have been computed the Einstein equations can easily be obtained. One combination of the field equations gives

$$\nabla_a\nabla_b r = -\frac{1}{2r}\nabla^c r\nabla_c r g_{ab} - 4\pi r(T_{ab} - trT\,g_{ab}) \tag{8.2.6}$$

where $tr\,T = g^{ab}T_{ab}$. Combining (8.2.5) and (8.2.6),

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = 4K^2 + 4r^{-4}(\nabla^a r\nabla_a r)^2 + r^{-4}\left(\frac{1}{2r}\nabla^c r\nabla_c r - 2\pi r trT\right)^2 + 16\pi^2 r^{-2}\left(T_{ab} - \frac{1}{2}trTg_{ab}\right)\left(T^{ab} - \frac{1}{2}trTg^{ab}\right) \quad (8.2.7)$$

The first three terms on the light hand side of (8.2.7) are obviously non-negative and when the matter content of space-time is described by a massless scalar field the fourth term is non-negative. (This condition also holds for many other physically reasonable matter fields but that fact is not relevant for this paper.) It follows that if $m = -r\nabla_a r\nabla^a r/2$ then

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} \geq \frac{16m^2}{r^6} \quad (8.2.8)$$

Returning from these general considerations to the particular class of space-time considered here, it turns out that in that case $m = \frac{1}{8}e^{-\Omega}$ and r is a constant times $t^{1/2}$. It follows from (8.1.3) that Ω is non-decreasing. Hence the curvature invariant $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ blows up at least as fast as t^{-3}as $t = 0$ is approached.

It has now been shown that $t = 0$ is a curvature singularity and thus the spacetime cannot be extended further. This gives a statement which might be called '***strong cosmic censorship in the past***' for the class of space times considered here. (This terminology assumed that a time orientation of space-time has been chosen so that t increases towards the future.) It says that if a spacetime of this type is the maximal globally hyperbolic development of initial data on some hypersurface then no extension of that space-time contains a point to the past of the initial hypersurface which does not belong to the original space-time.

8.3 CRUSHING SINGULARITIES

The mean curvature of the hypersurface of constant t is given by

$$H = -\frac{1}{2}t^{1/4}e^{-\Omega/2}\left(\Omega_t + \frac{3}{2t}\right) \quad (8.3.1)$$

Equation (8.1.3) shows that $\Omega_t \geq 0$ and so (8.3.1) implies that $|H| \geq \frac{3}{4}t^{-3/4}e^{-\Omega/2} \geq Ct^{-3/4}$.

Thus it can be seen that H tends uniformly to $-\infty$ as the singularity is approached. This means that this singularity is a **crushing singularity**[1]. A crushing singularity in a spatially compact space-time always has a neighbourhood which can be foliated by hypersurface of constant mean curvature. The proof of this will now be recalled. Note first that a space-time which has a compact Cauchy hypersurface can contain at most one compact space-like hypersurface with a given non-zero constant mean curvature. The fact that $|H|$ tends uniformly to infinity shows that given any real number H_1 which is sufficiently large and negative there exist $t_1, t_2 > 0$ such that the hypersurface $t = t_1$ has mean curvature less than H_1 and the hypersurface $t = t_2$ has mean curvature greater than H_1. These hypersurfaces provide barriers which ensure that there exist a hypersurface of constant mean curvature H_1 between the hypersurfaces, of $t = t_1$ and $t = t_2$. Thus there is an interval $(-\infty, H_0)$ such that the spacetime contains exactly one compact hypersurface of constant mean curvature H_1 for each H_1 in this interval. It remains to show that these hypersurfaces cover a neighbourhood of the singularity. A standard result implies that if $H_2 > H_1$ the hypersurface of mean curvature H_2 lies strictly to the past of that with mean curvature H_1. By construction the hypersurfaces tend to the singularity as $H_1 \to -\infty$. In other words there is no point which lies to the past of all these hypersurfaces. It remains to show that there are no gaps, i.e. that there is no point which lies to the past of the hypersurface with mean curvature H_1 but to the future of the hypersurfaces with mean curvature H_2 for all $H_2 > H_1$.

Suppose that a point p with this property existed. Then there would be an open neighbourhood U of the hypersurface with mean curvature H_1 disjoint from the future of p. In U there exist hypersurfaces of constant mean curvature H_2 for all H_2 in some interval $[H_1 - \epsilon, H_1 + \epsilon]$ with $\epsilon > 0$. Hence there is a point of the hypersurface with mean curvature $H_1 + \epsilon$ which lies to the past to the hypersurface with $H_1 - \epsilon/2$, contradicting a statement made earlier. It follows that no point p with the above property can exist.

8.4 VELOCITY DOMINATED SINGULARITIES

The central problem in analyzing the singularities in the class of space-times considered here is to determine the behavior of a general spatially periodic solution of eq. (8.1.2) as $t \to 0$. Letelier and Tabensky have written down an integral formula for solutions of this equation but they give an explicit example of a solution which is not of that form. They conjecture that all solutions can be obtained as limits of solutions given by the integral formula. Without a proof of this conjecture their analysis is incomplete. This problem can be circumvented by the direct use of energy estimates, as has been shown by Isenberg and Moncrief in the course of a study of polarised Gowdy spacetimes. A sketch of the argument will now be. It will be supposed for simplicity that the solution ϕ being considered is C^∞. A computation gives the inequality

$$\frac{d}{dt}\int t^2(\phi_t^2 + \phi_z^2) \geq 0 \tag{8.4.1}$$

for $t > 0$ when ϕ is a solution of (8.1.2). Since the coefficients in the equation do not depend explicitly on the spatial coordinate the derivatives of ϕ of any order with respect to x satisfies the same equation as ϕ itself. Hence all spatial derivatives of ϕ satisfy inequalities analogous to (8.4.1). The Sobolev embedding theorem then implies that $t\phi_t, t\phi_z$ and the derivatives of these quantities with respect to x of any order are bounded in a neghibourhood of $t = 0$. Equation (8.1.2) can be rewritten as

$$(t\phi_t)_t = t\phi_{xx}. \tag{8.4.2}$$

Knowing that $t\phi_{xx}$ is bounded allows us to conclude that $t\phi_t$ has a continues extension to $t = 0$. Integrating twice in time gives the asymptotic expansion

$$\phi(t,x) = \pi(x)\log t + \omega(x) + O(t) \tag{8.4.3}$$

for some smooth functions $\pi(x)$ and $\omega(x)$ as $t = 0$. The expression obtained by formally differentiating (8.4.3) once with respect to t and as many times as desired with respect to x is also a valid asymptotic expansion. Substituting these asymptotic expansions into the evolution equation for Ω gives

$$\Omega_t = \pi^2(x)t^{-1} + O(1) \tag{8.4.4}$$

Integrating this with respect to t gives

$$\Omega(t,x) = \pi^2(x)\log t + \alpha(x) + O(t) \tag{8.4.5}$$

for some α. Let the parts of the right hand sides of (8.4.4) and (8.4.5) explicitly written out be denoted by $\tilde{\phi}$ and $\tilde{\Omega}$ respectively so that $\phi = \tilde{\phi} + O(t)$ and $\Omega = \tilde{\Omega} + O(t)$. The quantities $\tilde{\phi}$ and $\tilde{\Omega}$ are solutions of the equations obtained from the Einstein evolution equations by dropping all spatial derivatives. This is what Isenberg and Moncrief [13] call the velocity dominated system. Thus the solution of the full Einstein equations are approximated asymptotically near the singularity by solutions of the velocity dominated system and these space-times have what Isenberg and Mincrief call the AVTDS property (asymptotically velocity-term dominated near the singularity). This is not literally the same as the original definition of velocity dominated singularities which were given by Eardley, Liang and Sachs [2] but the spirit is the same and so for brevity this property is described here as the property that the singularity is velocity dominated.

The definition of a velocity dominated singularity makes use of a preferred foliation by space-like hypersurfaces. A singularity which has the velocity dominated property with respect to one foliation will in general not have it with respect to a different foliation. In the present case it has been shown that the property holds with respect to the foliation defined by the time coordinate t and this could be interpreted as saying that this foliation is in same sense well-behaved near the singularity. Taking this view it is natural to ask whether the foliation by hypersurfaces of constant mean curvature, whose existence was shown in Section 3, is also well-behaved in this sense.

Despite the excellent control over the space-time which is available, this question appears difficult to decide. It would be interesting to know the answer for the following reason. The time coordinate t is defined in terms of the symmetry of the solution and so has no obvious analogue in general space-times with less symmetry. On the other hand the constant mean curvature condition makes sense in any space-time and it seems reasonable to hope that foliation of constant mean curvature exist in a wide class of space-times.

8.5 ISOTROPIC SINGULARITIES

In the literature there has been some discussion of isotropic singularities, a class of singularities which is of relevance to Penrose's Weyl curvature hypothesis. A singularity of this kind can be defined [3,14] by the condition that it should be possible to conformally rescale the given metric so that the rescaled metric extends regularly through the singularity. In general the asymptotic form of the space-time metric near the singularity in the class of space-times considered here is

$$t\left[\, t^{\pi^2-3/2}e^{\alpha}(-dt^2+dx^2)+(dy^2+dz^2)\right] \tag{8.5.1}$$

This shows that if $\pi^2=3/2$ everywhere the conformal class of the metric extends continuously to $t=0$. Thus the singularity is isotropic in this case.

In fact it is desirable to require a little more of an isotropic singularity than what has just been demonstrated. The conformally rescaled metric should extend not just in a continuous non-degenerate manner to the singularity. It should also have some degree of differentiability there. This question of the differentiability of the rescaled metric (or more precisely one question of the simultaneous differentiability of the rescaled metric and the conformal factor) is in general somewhat subtle [14]. However it turns out that in present case everything can be made C^∞. To show this it is necessary to extend asymptotic expansions (8.4.2) and (8.4.4) to all powers of t. An asymptotic expansion of this type for the solution of (8.4.2) has been given in [15]. This in an expansion in integral powers of t and $\log t$. However, if π is constant, the only term containing a logarithm is that written out explicitly in equation (8.4.2). It follows immediately that in the case $\pi =$ const. (8.4.4) can be extended to an asymptotic expansion to all orders which except for the first term is an expansion in integral powers of t this shows that the rescaled metric is C^∞ in the case identified as being isotropic above.

"Go, wond'rous creature, mount where, Science guides, Go, measure earth, weigh air, and state the tides; Correct old time, and regulate the Sun."

ALEXANDER POPE

CHAPTER 9

SCHWARZSCHILD METRIC WITH COSMOLOGICAL CONSTANT

9.1 INTRODUCTION

In this chapter we have tried to modify the Schwarzschild metric as follows by the Λ-term (here r has definition of length):

$$ds^2 = \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right) c^2 dt^2 - \left[1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right]^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

Like the equation (12), where M is the mass of the sun, multiplied by $\frac{G}{C^2}$. It is well-known that the usual Schwarzschild solution implies a perihelion shift of Mercury of about 43'' per century. This shift is known with an accuracy of about half a percent. Again vanishing the Λ-term, we get a final solution of $r > \frac{2GM}{c^2}$ or $r < \frac{2GM}{c^2}$.

9.2 SCHWARZSCHILD-LIKE SOLUTION OF NON-CONSERVATIVE GRAVITATIONAL EQUATIONS

In this section, we shall discuss the Schwarzschild-like solution of the non-conservative gravitational equations

$$D_a \psi_i^a = R\lambda_i \quad (1)$$

Since, it is very well-known, at the very basis of the main experimental tests of general relativity.

In the absence of matter (empty space), i.e. for $T_{ij} = 0$, equation(1) become

$$g^{ha} D_a R_{ih} = \lambda_i R \quad (2)$$

Equations (2) represent, in the new theory of gravitation based on equations (1), the generalization of the case leading, for a central, symmetric gravitational field, to the well-known Schwarzschild metric in general relativity. Therefore, in order to study, the possible new implifications of equations

$$D_i V_a = F_i(V_a) \quad (3)$$

Let us solve equation (2) by assuming a spherical symmetric field.

As is well-known, the general static isotropic metric in this case can be written in the form

$$ds^2 = e^{\nu(r)}c^2dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2) \tag{4}$$

Therefore by assuming that x^0, x^1, x^2, x^3 represent the co-ordinates ct, r, θ, φ respectively the only non-vanishing contravariant components of the metric tensor are given by

$$g_{00} = e^{\nu(r)}, g_{11} = -e^{\lambda(r)}, g_{22} = -r^2, g_{33} = -r^2 sin^2\theta$$

$$g^{00} = e^{-\nu(r)}, g^{11} = -e^{-\lambda(r)}, g^{22} = -\frac{1}{r^2}, g^{33} = -\frac{1}{r^2 sin^2\theta}$$

The christoffel symbols can be calculated from the equation,

$$\Gamma^s_{ij} = \frac{1}{2}g^{sk}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) \tag{5a}$$

Then we have calculated the non-zero affine connections are,

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}g^{00}\left(\frac{\partial g_{10}}{\partial x^0} + \frac{\partial g_{00}}{\partial x^1} - \frac{\partial g_{10}}{\partial x^0}\right)$$

$$= \frac{1}{2}g^{00}\left(\frac{\partial g_{00}}{\partial x^1}\right)$$

$$= \frac{1}{2}e^{-\nu(r)}\frac{\partial}{\partial r}\left(e^{\nu(r)}\right)$$

$$= \frac{1}{2}\nu'$$

$$\Gamma^1_{00} = \frac{1}{2}g^{11}\left(\frac{\partial g_{10}}{\partial x^0} + \frac{\partial g_{01}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^1}\right)$$

$$= \frac{1}{2}g^{11}\left(-\frac{\partial g_{00}}{\partial x^1}\right)$$

$$= \frac{1}{2}\left(-e^{-\lambda(r)}\right)\left\{-\frac{\partial}{\partial r}\left(e^{\nu(r)}\right)\right\}$$

$$= \frac{1}{2}\nu' e^{\nu-\lambda}$$

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1}\right)$$

$$= \frac{1}{2}g^{11}\left(\frac{\partial g_{11}}{\partial x^1}\right)$$

$$= \frac{1}{2}\left(-e^{-\lambda(r)}\right)\left\{\frac{\partial}{\partial r}\left(-e^{\lambda(r)}\right)\right\}$$

$$= \frac{1}{2}\lambda'$$

$$\Gamma^1_{22} = \frac{1}{2}g^{11}\left(\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right)$$

$$= \frac{1}{2}g^{11}\left(-\frac{\partial g_{22}}{\partial x^1}\right)$$

$$= \frac{1}{2}\left(-e^{-\lambda(r)}\right)\left\{-\frac{\partial}{\partial r}(-r^2)\right\}$$

$$= -re^{-\lambda}$$

$$\Gamma^1_{33} = \frac{1}{2}g^{11}\left(\frac{\partial g_{31}}{\partial x^3} + \frac{\partial g_{13}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^1}\right)$$

$$= \frac{1}{2}g^{11}\left(-\frac{\partial g_{33}}{\partial x^1}\right)$$

$$= \frac{1}{2}\left(-e^{-\lambda(r)}\right)\left\{-\frac{\partial}{\partial r}(-r^2 sin^2\theta)\right\}$$

$$= -re^{-\lambda} sin^2\theta$$

$$\Gamma^2_{33} = \frac{1}{2}g^{22}\left(\frac{\partial g_{32}}{\partial x^3} + \frac{\partial g_{23}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^2}\right)$$

$$= \frac{1}{2}g^{22}\left(-\frac{\partial g_{33}}{\partial x^2}\right)$$

$$= \frac{1}{2}\left(-\frac{1}{r^2}\right)\left\{-\frac{\partial}{\partial \theta}(-r^2 sin^2\theta)\right\}$$

$$= -sin\theta cos\theta$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}g^{22}\left(\frac{\partial g_{12}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^2}\right)$$

$$= \frac{1}{2}g^{22}\left(\frac{\partial g_{22}}{\partial x^1}\right)$$

$$= \frac{1}{2}\left(-\frac{1}{r^2}\right)\left\{\frac{\partial}{\partial r}(-r^2)\right\} = \frac{1}{r}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2}g^{33}\left(\frac{\partial g_{13}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^1} - \frac{\partial g_{13}}{\partial x^3}\right)$$

$$= \frac{1}{2}g^{33}\left(\frac{\partial g_{33}}{\partial x^1}\right)$$

$$= \frac{1}{2}\left(-\frac{1}{r^2 sin^2\theta}\right)\left\{\frac{\partial}{\partial r}(-r^2 sin^2\theta)\right\}$$

$$= \frac{1}{r}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2}g^{33}\left(\frac{\partial g_{23}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^2} - \frac{\partial g_{23}}{\partial x^3}\right)$$

$$= \frac{1}{2}g^{33}\left(\frac{\partial g_{33}}{\partial x^2}\right)$$

$$= \frac{1}{2}\left(-\frac{1}{r^2 sin^2\theta}\right)\left\{\frac{\partial}{\partial \theta}(-r^2 sin^2\theta)\right\}$$

$$= cot\theta$$

This calculation leads the following expressions:

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}\nu', \Gamma^1_{00} = \frac{1}{2}\nu' e^{\nu-\lambda}, \Gamma^1_{11} = \frac{1}{2}\lambda', \Gamma^1_{22} = -re^{-\lambda}, \Gamma^1_{33} = -re^{-\lambda}sin^2\theta$$

$$\Gamma^2_{33} = -sin\theta cos\theta, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{23} = \Gamma^3_{32} = cot\theta$$

The Ricci tensors can be calculated from the equation

$$R_{jk} = \Gamma^i_{rk}\Gamma^r_{ji} + \frac{\partial \Gamma^i_{ji}}{\partial x^k} - \Gamma^i_{ri}\Gamma^r_{jk} - \frac{\partial \Gamma^i_{jk}}{\partial x^i} \qquad (5b)$$

$$\therefore R_{00} = \Gamma^i_{r0}\Gamma^r_{0i} + \frac{\partial \Gamma^i_{0i}}{\partial x^0} - \Gamma^i_{ri}\Gamma^r_{00} - \frac{\partial \Gamma^i_{00}}{\partial x^i}$$

Now,

$$\Gamma^i_{r0}\Gamma^r_{0i} = \Gamma^0_{r0}\Gamma^r_{00} + \Gamma^1_{r0}\Gamma^r_{01} + \Gamma^2_{r0}\Gamma^r_{02} + \Gamma^3_{r0}\Gamma^r_{03}$$

$$= \Gamma^0_{10}\Gamma^1_{00} + \Gamma^1_{00}\Gamma^0_{01} + 0 + 0$$

$$= \frac{1}{2}\nu' . \frac{1}{2}\nu' e^{\nu-\lambda} + \frac{1}{2}\nu' e^{\nu-\lambda} . \frac{1}{2}\nu' \quad = \frac{1}{2}\nu'^2 e^{\nu-\lambda}$$

$$\frac{\partial \Gamma^i_{0i}}{\partial x^0} = \frac{\partial}{\partial t}\left(\Gamma^i_{0i}\right) = 0$$

$$\Gamma^i_{ri}\Gamma^r_{00} = \Gamma^0_{r0}\Gamma^r_{00} + \Gamma^1_{r1}\Gamma^r_{00} + \Gamma^2_{r2}\Gamma^r_{00} + \Gamma^3_{r3}\Gamma^r_{00}$$

$$= \Gamma^0_{10}\Gamma^1_{00} + \Gamma^1_{11}\Gamma^1_{00} + \Gamma^2_{12}\Gamma^1_{00} + \Gamma^3_{13}\Gamma^1_{00}$$

$$= \frac{1}{2}\nu'.\frac{1}{2}\,\nu' e^{\nu-\lambda} + \frac{1}{2}\lambda'.\frac{1}{2}\,\nu' e^{\nu-\lambda} + \frac{1}{r}.\frac{1}{2}\,\nu' e^{\nu-\lambda} + \frac{1}{r}.\frac{1}{2}\,\nu' e^{\nu-\lambda}$$

$$= \frac{1}{4}\nu'^2 e^{\nu-\lambda} + \frac{1}{4}\lambda'\nu' e^{\nu-\lambda} + \frac{1}{r}\nu' e^{\nu-\lambda}$$

$$\frac{\partial \Gamma^i_{00}}{\partial x^i} = \frac{\partial \Gamma^1_{00}}{\partial x^1} = \frac{\partial}{\partial r}\left(\frac{1}{2}\,\nu' e^{\nu-\lambda}\right) = \frac{1}{2}\,\nu'' e^{\nu-\lambda} + \frac{1}{2}\,(\nu' - \lambda')\nu' e^{\nu-\lambda}$$

so, $$R_{00} = \Gamma^i_{r0}\Gamma^r_{0i} + \frac{\partial \Gamma^i_{0i}}{\partial x^0} - \Gamma^i_{ri}\Gamma^r_{00} - \frac{\partial \Gamma^i_{00}}{\partial x^i}$$

$$= \frac{1}{2}\nu'^2 e^{\nu-\lambda} + 0 - \frac{1}{4}\nu'^2 e^{\nu-\lambda} - \frac{1}{4}\lambda'\nu' e^{\nu-\lambda} - \frac{1}{r}\nu' e^{\nu-\lambda} - \frac{1}{2}\,\nu'' e^{\nu-\lambda} - \frac{1}{2}\,(\nu' - \lambda')\nu' e^{\nu-\lambda}$$

$$= e^{\nu-\lambda}\left(-\frac{1}{2}\,\nu'' - \frac{1}{4}\nu'^2 + \frac{1}{4}\lambda'\nu' - \frac{1}{r}\nu'\right)$$

$$\therefore R_{11} = \Gamma^i_{r1}\Gamma^r_{1i} + \frac{\partial \Gamma^i_{1i}}{\partial x^1} - \Gamma^i_{ri}\Gamma^r_{11} - \frac{\partial \Gamma^i_{11}}{\partial x^i}$$

Now

$$\Gamma^i_{r1}\Gamma^r_{1i} = \Gamma^0_{r1}\Gamma^r_{10} + \Gamma^1_{r1}\Gamma^r_{11} + \Gamma^2_{r1}\Gamma^r_{12} + \Gamma^3_{r1}\Gamma^r_{13}$$

$$= \Gamma^0_{01}\Gamma^0_{10} + \Gamma^1_{11}\Gamma^1_{11} + \Gamma^2_{21}\Gamma^2_{12} + \Gamma^3_{31}\Gamma^3_{13}$$

$$= \frac{1}{2}\nu'.\frac{1}{2}\nu' + \frac{1}{2}\lambda'.\frac{1}{2}\lambda' + \frac{1}{r}.\frac{1}{r} + \frac{1}{r}.\frac{1}{r}$$

$$= \frac{1}{4}\nu'^2 + \frac{1}{4}\lambda'^2 + \frac{2}{r^2}$$

$$\frac{\partial \Gamma_{1i}^{i}}{\partial x^{1}} = \frac{\partial}{\partial r}\left(\Gamma_{10}^{0} + \Gamma_{11}^{1} + \Gamma_{12}^{2} + \Gamma_{13}^{3}\right)$$

$$= \frac{\partial}{\partial r}\left(\frac{1}{2}\nu' + \frac{1}{2}\lambda' + \frac{1}{r} + \frac{1}{r}\right)$$

$$= \frac{1}{2}\nu'' + \frac{1}{2}\lambda'' - \frac{2}{r^{2}}$$

$$\Gamma_{ri}^{i}\Gamma_{11}^{r} = \Gamma_{r0}^{0}\Gamma_{11}^{r} + \Gamma_{r1}^{1}\Gamma_{11}^{r} + \Gamma_{r2}^{2}\Gamma_{11}^{r} + \Gamma_{r3}^{3}\Gamma_{11}^{r}$$

$$= \Gamma_{10}^{0}\Gamma_{11}^{1} + \Gamma_{11}^{1}\Gamma_{11}^{1} + \Gamma_{12}^{2}\Gamma_{11}^{1} + \Gamma_{13}^{3}\Gamma_{11}^{1}$$

$$= \frac{1}{2}\nu'.\frac{1}{2}\lambda' + \frac{1}{2}\lambda'.\frac{1}{2}\lambda' + \frac{1}{r}.\frac{1}{2}\lambda' + \frac{1}{r}.\frac{1}{2}\lambda'$$

$$= \frac{1}{4}\nu'\lambda' + \frac{1}{4}\lambda'^{2} + \frac{1}{r}\lambda'$$

$$\frac{\partial \Gamma_{11}^{i}}{\partial x^{i}} = \frac{\partial}{\partial r}\left(\Gamma_{11}^{1}\right) = \frac{\partial}{\partial r}\left(\frac{1}{2}\lambda'\right) = \frac{1}{2}\lambda''$$

$$So, \qquad R_{11} = \Gamma_{r1}^{i}\Gamma_{1i}^{r} + \frac{\partial \Gamma_{1i}^{i}}{\partial x^{1}} - \Gamma_{ri}^{i}\Gamma_{11}^{r} - \frac{\partial \Gamma_{11}^{i}}{\partial x^{i}}$$

$$= \frac{1}{4}\nu'^{2} + \frac{1}{4}\lambda'^{2} + \frac{2}{r^{2}} + \frac{1}{2}\nu'' + \frac{1}{2}\lambda'' - \frac{2}{r^{2}} - \frac{1}{4}\nu'\lambda' - \frac{1}{4}\lambda'^{2} - \frac{1}{r}\lambda' - \frac{1}{2}\lambda''$$

$$= \frac{1}{4}\nu'^{2} + \frac{1}{2}\nu'' - \frac{1}{4}\nu'\lambda' - \frac{1}{r}\lambda'$$

$$\therefore \quad R_{22} = \Gamma_{r2}^{i}\Gamma_{2i}^{r} + \frac{\partial \Gamma_{2i}^{i}}{\partial x^{2}} - \Gamma_{ri}^{i}\Gamma_{22}^{r} - \frac{\partial \Gamma_{22}^{i}}{\partial x^{i}}$$

Hence,

$$\Gamma_{r2}^{i}\Gamma_{2i}^{r} = \Gamma_{r2}^{0}\Gamma_{20}^{r} + \Gamma_{r2}^{1}\Gamma_{21}^{r} + \Gamma_{r2}^{2}\Gamma_{22}^{r} + \Gamma_{r2}^{3}\Gamma_{23}^{r}$$

$$= 0 + \Gamma_{22}^{1}\Gamma_{21}^{2} + \Gamma_{12}^{2}\Gamma_{22}^{1} + \Gamma_{32}^{3}\Gamma_{23}^{3}$$

$$= -re^{-\lambda}.\frac{1}{r} + \frac{1}{r}.\left(-re^{-\lambda}\right) + cot\theta.cot\theta \quad = -2e^{-\lambda} + cot^{2}\theta$$

$$\frac{\partial \Gamma_{2i}^{i}}{\partial x^{2}} = \frac{\partial}{\partial \theta}\left(\Gamma_{23}^{3}\right) = \frac{\partial}{\partial \theta}(cot\theta) = -cosec^{2}\theta$$

$$\Gamma^i_{ri}\Gamma^r_{22} = \Gamma^0_{r0}\Gamma^r_{22} + \Gamma^1_{r1}\Gamma^r_{22} + \Gamma^2_{r2}\Gamma^r_{22} + \Gamma^3_{r3}\Gamma^r_{22}$$

$$= \Gamma^0_{10}\Gamma^1_{22} + \Gamma^1_{11}\Gamma^1_{22} + \Gamma^2_{12}\Gamma^1_{22} + \Gamma^3_{13}\Gamma^1_{22}$$

$$= \frac{1}{2}\nu'.\left(-re^{-\lambda}\right) + \frac{1}{2}\lambda'.\left(-re^{-\lambda}\right) + \frac{1}{r}.\left(-re^{-\lambda}\right) + \frac{1}{r}.\left(-re^{-\lambda}\right)$$

$$= -\frac{re^{-\lambda}}{2}(\nu' + \lambda') - 2e^{-\lambda}$$

$$\frac{\partial\Gamma^i_{22}}{\partial x^i} = \frac{\partial\Gamma^1_{22}}{\partial x^1} = \frac{\partial}{\partial r}\left(\Gamma^1_{22}\right) = \frac{\partial}{\partial r}\left(-re^{-\lambda}\right) = -e^{-\lambda} + r\lambda' e^{-\lambda}$$

So,

$$R_{22} = \Gamma^i_{r2}\Gamma^r_{2i} + \frac{\partial\Gamma^i_{2i}}{\partial x^2} - \Gamma^i_{ri}\Gamma^r_{22} - \frac{\partial\Gamma^i_{22}}{\partial x^i}$$

$$= -2e^{-\lambda} + cot^2\theta - cosec^2\theta + \frac{re^{-\lambda}}{2}(\nu' + \lambda') + 2e^{-\lambda} + e^{-\lambda} - r\lambda' e^{-\lambda}$$

$$= -1 + \frac{re^{-\lambda}}{2}(\nu' - \lambda') + e^{-\lambda}$$

$$\therefore R_{33} = \Gamma^i_{r3}\Gamma^r_{3i} + \frac{\partial\Gamma^i_{3i}}{\partial x^3} - \Gamma^i_{ri}\Gamma^r_{33} - \frac{\partial\Gamma^i_{33}}{\partial x^i}$$

Now,

$$\Gamma^i_{r3}\Gamma^r_{3i} = \Gamma^0_{r3}\Gamma^r_{30} + \Gamma^1_{r3}\Gamma^r_{31} + \Gamma^2_{r3}\Gamma^r_{32} + \Gamma^3_{r3}\Gamma^r_{33}$$

$$= 0 + \Gamma^1_{33}\Gamma^3_{31} + 0 + \Gamma^2_{33}\Gamma^3_{32} + \Gamma^3_{13}\Gamma^1_{33} + \Gamma^3_{23}\Gamma^2_{33}$$

$$= -re^{-\lambda}sin^2\theta.\frac{1}{r} - sin\theta cos\theta.cot\theta + \frac{1}{r}.\left(-re^{-\lambda}sin^2\theta\right) + cot\theta.(-sin\theta cos\theta)$$

$$= -2e^{-\lambda}sin^2\theta - 2cos^2\theta$$

$$\frac{\partial\Gamma^i_{3i}}{\partial x^3} = \frac{\partial\Gamma^i_{3i}}{\partial\varphi} = 0$$

$$\Gamma^i_{ri}\Gamma^r_{33} = \Gamma^0_{r0}\Gamma^r_{33} + \Gamma^1_{r1}\Gamma^r_{33} + \Gamma^2_{r2}\Gamma^r_{33} + \Gamma^3_{r3}\Gamma^r_{33}$$

$$= \Gamma^0_{10}\Gamma^1_{33} + \Gamma^1_{11}\Gamma^1_{33} + \Gamma^2_{12}\Gamma^1_{33} + \Gamma^3_{13}\Gamma^1_{33} + \Gamma^3_{23}\Gamma^2_{33}$$

$$= \frac{1}{2}\nu'.\left(-re^{-\lambda}sin^2\theta\right) + \frac{1}{2}\lambda'.\left(-re^{-\lambda}sin^2\theta\right) + \frac{1}{r}.\left(-re^{-\lambda}sin^2\theta\right) + \frac{1}{r}.\left(-re^{-\lambda}sin^2\theta\right) + cot\theta.(-sin\theta cos\theta)$$

$$= -\frac{1}{2}\left(r\nu' e^{-\lambda}sin^2\theta\right) - \frac{1}{2}\left(r\lambda' e^{-\lambda}sin^2\theta\right) - \left(2e^{-\lambda}sin^2\theta\right) - cos^2\theta$$

$$\frac{\partial\Gamma^i_{33}}{\partial x^i} = \frac{\partial\Gamma^1_{33}}{\partial x^1} = \frac{\partial}{\partial r}\left(-re^{-\lambda}sin^2\theta\right) = -e^{-\lambda}sin^2\theta + r\lambda' e^{-\lambda}sin^2$$

So,

$$R_{33} = \Gamma^i_{r3}\Gamma^r_{3i} + \frac{\partial\Gamma^i_{3i}}{\partial x^3} - \Gamma^i_{ri}\Gamma^r_{33} - \frac{\partial\Gamma^i_{33}}{\partial x^i}$$

$$= -2e^{-\lambda}sin^2\theta - 2cos^2\theta + 0 + \frac{1}{2}\left(r\nu' e^{-\lambda}sin^2\theta\right) + \frac{1}{2}\left(r\lambda' e^{-\lambda}sin^2\theta\right) + \left(2e^{-\lambda}sin^2\theta\right) + cos^2\theta + e^{-\lambda}sin^2\theta - r\lambda' e^{-\lambda}sin^2\theta$$

$$= sin^2\theta\left\{-1 + \frac{re^{-\lambda}}{2}(\nu' - \lambda') + e^{-\lambda}\right\}$$

$$= R_{22}sin^2\theta$$

The components of the Ricci tensor read

$$\left.\begin{aligned} R_{00} &= e^{\nu-\lambda}\left(-\frac{1}{2}\nu'' - \frac{1}{4}\nu'^2 + \frac{1}{4}\lambda'\nu' - \frac{1}{r}\nu'\right) \\ R_{11} &= \frac{1}{4}\nu'^2 + \frac{1}{2}\nu'' - \frac{1}{4}\nu'\lambda' - \frac{1}{r}\lambda' \\ R_{22} &= -1 + \frac{re^{-\lambda}}{2}(\nu' - \lambda') + e^{-\lambda} \\ R_{33} &= R_{22}sin^2\theta \end{aligned}\right\} \qquad (6)$$

(where the prime denotes derivative with respect to r)

The curvature scalar is given by

$$R = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \qquad (7)$$

$$= e^{-\nu}.e^{\nu-\lambda}\left(-\frac{1}{2}\nu'' - \frac{1}{4}\nu'^2 + \frac{1}{4}\lambda'\nu' - \frac{1}{r}\nu'\right) - e^{-\lambda}.\left(\frac{1}{4}\nu'^2 + \frac{1}{2}\nu'' - \frac{1}{4}\nu'\lambda' - \frac{1}{r}\lambda'\right)$$
$$- \frac{1}{r^2}\left\{-1 + \frac{re^{-\lambda}}{2}(\nu' - \lambda') + e^{-\lambda}\right\}$$
$$- \frac{1}{r^2 sin^2\theta}.sin^2\theta\left\{-1 + \frac{re^{-\lambda}}{2}(\nu' - \lambda') + e^{-\lambda}\right\}$$

$$= e^{-\lambda}\left\{-\frac{1}{2}\nu'' - \frac{1}{4}\nu'^2 + \frac{1}{4}\lambda'\nu' - \frac{1}{r}\nu' - \frac{1}{4}\nu'^2 - \frac{1}{2}\nu'' + \frac{1}{4}\nu'\lambda' + \frac{1}{r}\lambda' - \frac{1}{2r}(\nu' - \lambda') - \frac{1}{r^2} - \frac{1}{2r}(\nu' - \lambda') - \frac{1}{r^2}\right\} + \frac{1}{r^2} + \frac{1}{r^2}$$

$$= e^{-\lambda}\left\{-\nu'' - \frac{1}{2}\nu'^2 + \frac{1}{2}\lambda'\nu' - \frac{2}{r}(\nu' - \lambda') - \frac{2}{r^2}\right\} + \frac{2}{r^2}$$

$$\therefore R = e^{-\lambda}\left\{-\nu'' - \frac{1}{2}\nu'^2 + \frac{1}{2}\lambda'\nu' - \frac{2}{r}(\nu' - \lambda') - \frac{2}{r^2}\right\} + \frac{2}{r^2} \qquad (7a)$$

In order to obtain a solution of equations (2) with non-zero curvature (unlike the standard Schwarzschild case, where $R = 0$), we have to impose, as is easily seen,

$$\lambda_0 = \lambda_2 = \lambda_3 = 0 \qquad (8)$$

Condition (8) amounts to assuming the staticity and the isotropy of the substratum. The only non-trivial equation (2) can be written as,

$$g^{11}\partial_1 R_{11} - R_{11}\left[2g^{11}\Gamma^1_{11} - g^{22}\Gamma^1_{22} - g^{11}\Gamma^1_{33} - g^{11}\Gamma^1_{00}\right] - g^{22}\Gamma^2_{12}R_{22} - g^{00}\Gamma^1_{10}R_{00} = \lambda_1 R$$

Where use has been made of the formula

$$D_a R_{ih} = \partial_a R_{ih} - \Gamma^s_{ia}R_{sh} - \Gamma^s_{ha}R_{is} \qquad (9)$$

Using the value of equation (6) and (7a) we get finally, after simple but lengthy calculations,

$$4 - 14r\lambda' + 2r\nu' + 8\lambda_1 r - 8\lambda_1 e^{\lambda} r - 4\lambda'\nu' r^2 - 8\lambda_1\lambda' r^2 + 8\lambda_1\nu' r^2 + 4\nu''^{r^2} + 4\lambda''^{r^2} - 4(\lambda')^2 r^2 + 4(\nu')^2 r^2 + 2\lambda_1 r^3\lambda'\nu' + 3\lambda'\nu''^{r^3} + \nu'\lambda''^{r^3} - 2\nu'''^{r^3} - r^3(\lambda')^2\nu' - 2\lambda_1 r^3(\nu')^2 + r^3(\nu')^3 - 4e^{\lambda} = 0 \qquad (10)$$

As it stands, equation (10) (a non-linear equation in the two unknown functions $\lambda(r) and\ \nu(r)$) is quite impossible to handle.

However, in order to simplify our task, we can seek solutions of (2) which describe small deviations from the Schwarzschild metric. To this end, let us assume the ansatz

$$\left.\begin{aligned} e^{\nu} &= 1-\frac{2GM}{c^2r}+\frac{1}{3}\Lambda r^2 \\ e^{\lambda} &= \left[1-\frac{2GM}{c^2r}+\frac{1}{3}\Lambda r^2\right]^{-1} \end{aligned}\right\} \quad (11)$$

Now using equation (11) in equation (4) we get,

$$ds^2=\left(1-\frac{2GM}{c^2r}+\frac{1}{3}\Lambda r^2\right)c^2dt^2-\left[1-\frac{2GM}{c^2r}+\frac{1}{3}\Lambda r^2\right]^{-1}dr^2 - r^2(d\theta^2+sin^2\theta d\varphi^2) \quad (12)$$

So that

$$g_{00}=\left(1-\frac{2GM}{c^2r}+\frac{1}{3}\Lambda r^2\right), \quad g_{11}=-\left[1-\frac{2GM}{c^2r}+\frac{1}{3}\Lambda r^2\right]^{-1},$$

$$g_{22}=-r^2, \qquad g_{33}=-r^2sin^2\theta$$

And

$$g^{00}=\left[1-\frac{2GM}{c^2r}+\frac{1}{3}\Lambda r^2\right]^{-1}, \quad g^{11}=-\left(1-\frac{2GM}{c^2r}+\frac{1}{3}\Lambda r^2\right),$$

$$g^{22}=-\frac{1}{r^2}, \qquad g^{33}=-\frac{1}{r^2sin^2\theta}$$

Using (5a), we have calculated the non-zero affine connections are

$$\Gamma^0_{01}=\Gamma^0_{10}=\frac{1}{2}g^{00}\left(\frac{\partial g_{10}}{\partial x^0}+\frac{\partial g_{00}}{\partial x^1}-\frac{\partial g_{10}}{\partial x^0}\right)$$

$$=\frac{1}{2}g^{00}\left(\frac{\partial g_{00}}{\partial x^1}\right)$$

$$=\frac{1}{2}\left[1-\frac{2GM}{c^2r}+\frac{1}{3}\Lambda r^2\right]^{-1}.\frac{\partial}{\partial r}\left(1-\frac{2GM}{c^2r}+\frac{1}{3}\Lambda r^2\right)$$

$$=\frac{1}{2}\left[1-\frac{2GM}{c^2r}+\frac{1}{3}\Lambda r^2\right]^{-1}\left(\frac{2GM}{c^2r^2}+\frac{2\Lambda r}{3}\right)$$

$$=\left[1-\frac{2GM}{c^2r}+\frac{1}{3}\Lambda r^2\right]^{-1}\left(\frac{GM}{c^2r^2}+\frac{\Lambda r}{3}\right)$$

$$\Gamma^1_{00} = \frac{1}{2} g^{11} \left(\frac{\partial g_{10}}{\partial x^0} + \frac{\partial g_{01}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^1} \right)$$

$$= \frac{1}{2} g^{11} \left(- \frac{\partial g_{00}}{\partial x^1} \right)$$

$$= \frac{1}{2} \left\{ - \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right) \right\} . \left\{ - \frac{\partial}{\partial r} \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right) \right\}$$

$$= \frac{1}{2} \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right) \left(\frac{2GM}{c^2 r^2} + \frac{2\Lambda r}{3} \right)$$

$$= \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right) \left(\frac{GM}{c^2 r^2} + \frac{\Lambda r}{3} \right)$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1} \right)$$

$$= \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial x^1} \right)$$

$$= \frac{1}{2} \left\{ - \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right) \right\} . \left[\frac{\partial}{\partial r} \left\{ - \left[1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right]^{-1} \right\} \right]$$

$$= - \frac{1}{2} \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right) \left[1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right]^{-2} \left(\frac{2GM}{c^2 r^2} + \frac{2\Lambda r}{3} \right)$$

$$= - \left[1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right]^{-1} \left(\frac{GM}{c^2 r^2} + \frac{\Lambda r}{3} \right)$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11} \left(\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1} \right)$$

$$= \frac{1}{2} g^{11} \left(- \frac{\partial g_{22}}{\partial x^1} \right)$$

$$= \frac{1}{2} \left\{ - \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right) \right\} . \left[- \frac{\partial}{\partial r} (-r^2) \right]$$

$$= - \frac{1}{2} \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right) . 2r$$

$$= -r \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right)$$

$$\Gamma^1_{33} = \frac{1}{2} g^{11} \left(\frac{\partial g_{31}}{\partial x^3} + \frac{\partial g_{13}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^1} \right)$$

$$= \frac{1}{2} g^{11} \left(- \frac{\partial g_{33}}{\partial x^1} \right)$$

$$= \frac{1}{2} \left\{ - \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right) \right\} \left[- \frac{\partial}{\partial r} (-r^2 sin^2\theta \ ) \right]$$

$$= -\frac{1}{2} \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right) (2r sin^2\theta)$$

$$= -r sin^2\theta \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right)$$

$$\Gamma^2_{33} = \frac{1}{2} g^{22} \left(\frac{\partial g_{32}}{\partial x^3} + \frac{\partial g_{23}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^2} \right)$$

$$= \frac{1}{2} g^{22} \left(- \frac{\partial g_{33}}{\partial x^2} \right)$$

$$= \frac{1}{2} \left(- \frac{1}{r^2} \right) \left\{ - \frac{\partial}{\partial \theta} (-r^2 sin^2\theta) \right\}$$

$$= -\frac{1}{2} . \frac{1}{r^2} . (2r^2 sin\theta cos\theta)$$

$$= -sin\theta cos\theta$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} g^{22} \left(\frac{\partial g_{12}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^2} \right)$$

$$= \frac{1}{2} g^{22} \left(\frac{\partial g_{22}}{\partial x^1} \right)$$

$$= \frac{1}{2} \left(- \frac{1}{r^2} \right) \left\{ \frac{\partial}{\partial r} (-r^2) \right\}$$

$$= \frac{1}{r}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2} g^{33} \left(\frac{\partial g_{13}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^1} - \frac{\partial g_{13}}{\partial x^3} \right)$$

$$= \frac{1}{2} g^{33} \left(\frac{\partial g_{33}}{\partial x^1} \right)$$

$$= \frac{1}{2} \left(-\frac{1}{r^2 sin^2\theta} \right) \left\{ \frac{\partial}{\partial r} (-r^2 sin^2\theta) \right\}$$

$$= \frac{1}{r}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2} g^{33} \left(\frac{\partial g_{23}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^2} - \frac{\partial g_{23}}{\partial x^3} \right)$$

$$= \frac{1}{2} g^{33} \left(\frac{\partial g_{33}}{\partial x^2} \right)$$

$$= \frac{1}{2} \left(-\frac{1}{r^2 sin^2\theta} \right) \left\{ \frac{\partial}{\partial \theta} (-r^2 sin^2\theta) \right\}$$

$$= cot\theta$$

This calculation leads the following expressions:

$$\Gamma^0_{01} = \Gamma^0_{10} = \left[1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right]^{-1}$$

$$\Gamma^1_{00} = \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right) \left(\frac{GM}{c^2 r^2} + \frac{\Lambda r}{3} \right),$$

$$\Gamma^1_{11} = -\left[1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right]^{-1} \left(\frac{GM}{c^2 r^2} + \frac{\Lambda r}{3} \right),$$

$$\Gamma^1_{22} = -r \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right),$$

$$\Gamma^1_{33} = -r sin^2\theta \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3} \Lambda r^2 \right)$$

$$\Gamma^2_{33} = -sin\theta cos\theta, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r},$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \qquad \Gamma^3_{23} = \Gamma^3_{32} = cot\theta$$

The Ricci tensors can be calculated from the equation (5b)

$$\therefore R_{00} = \Gamma^i_{r0} \Gamma^r_{0i} + \frac{\partial \Gamma^i_{0i}}{\partial x^0} - \Gamma^i_{ri} \Gamma^r_{00} - \frac{\partial \Gamma^i_{00}}{\partial x^i}$$

Now,

$$\Gamma^i_{r0}\Gamma^r_{0i} = \Gamma^0_{r0}\Gamma^r_{00} + \Gamma^1_{r0}\Gamma^r_{01} + \Gamma^2_{r0}\Gamma^r_{02} + \Gamma^3_{r0}\Gamma^r_{03}$$

$$= \Gamma^0_{10}\Gamma^1_{00} + \Gamma^1_{00}\Gamma^0_{01} + 0 + 0$$

$$= 2\left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right).\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)$$

$$= 2\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)^2$$

$$\frac{\partial\Gamma^i_{0i}}{\partial x^0} = \frac{\partial}{\partial t}\left(\Gamma^i_{0i}\right) = 0$$

$$\Gamma^i_{ri}\Gamma^r_{00} = \Gamma^0_{r0}\Gamma^r_{00} + \Gamma^1_{r1}\Gamma^r_{00} + \Gamma^2_{r2}\Gamma^r_{00} + \Gamma^3_{r3}\Gamma^r_{00}$$

$$= \Gamma^0_{10}\Gamma^1_{00} + \Gamma^1_{11}\Gamma^1_{00} + \Gamma^2_{12}\Gamma^1_{00} + \Gamma^3_{13}\Gamma^1_{00}$$

$$= \left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right).\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right) + \left\{-\left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)\right\}.\left\{\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)\right\} + \frac{1}{r}\left\{\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)\right\} + \frac{1}{r}\left\{\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)\right\}$$

$$= \left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)^2 - \left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)^2 + \frac{2}{r}\left\{\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)\right\}$$

$$= \frac{2}{r}\left\{\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)\right\}$$

$$\frac{\partial\Gamma^i_{00}}{\partial x^i} = \frac{\partial\Gamma^1_{00}}{\partial x^1} = \frac{\partial}{\partial r}\left\{\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)\right\}$$

$$= 2\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)^2 + \left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\left(-\frac{2GM}{c^2r^3} + \frac{\Lambda}{3}\right)$$

$$so, \quad R_{00} = \Gamma^{i}_{r0}\Gamma^{r}_{0i} + \frac{\partial \Gamma^{i}_{0i}}{\partial x^{0}} - \Gamma^{i}_{ri}\Gamma^{r}_{00} - \frac{\partial \Gamma^{i}_{00}}{\partial x^{i}}$$

$$= 2\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)^2 + 0 - \frac{2}{r}\left\{\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)\right\}$$
$$- 2\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)^2 - \left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\left(-\frac{2GM}{c^2r^3} + \frac{\Lambda}{3}\right)$$

$$= -\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\left(\frac{2GM}{c^2r^3} + \frac{2\Lambda}{3} - \frac{2GM}{c^2r^3} + \frac{\Lambda}{3}\right)$$

$$= -\Lambda\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)$$

$$\therefore R_{11} = \Gamma^{i}_{r1}\Gamma^{r}_{1i} + \frac{\partial \Gamma^{i}_{1i}}{\partial x^{1}} - \Gamma^{i}_{ri}\Gamma^{r}_{11} - \frac{\partial \Gamma^{i}_{11}}{\partial x^{i}}$$

Now

$$\Gamma^{i}_{r1}\Gamma^{r}_{1i} = \Gamma^{0}_{r1}\Gamma^{r}_{10} + \Gamma^{1}_{r1}\Gamma^{r}_{11} + \Gamma^{2}_{r1}\Gamma^{r}_{12} + \Gamma^{3}_{r1}\Gamma^{r}_{13}$$
$$= \Gamma^{0}_{01}\Gamma^{0}_{10} + \Gamma^{1}_{11}\Gamma^{1}_{11} + \Gamma^{2}_{21}\Gamma^{2}_{12} + \Gamma^{3}_{31}\Gamma^{3}_{13}$$
$$= \left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-2}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)^2$$
$$+ \left\{\left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-2}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)^2\right\} + \frac{1}{r}.\frac{1}{r} + \frac{1}{r}.\frac{1}{r}$$

$$= 2\left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-2}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)^2 + \frac{2}{r^2}$$

$$\frac{\partial \Gamma^{i}_{1i}}{\partial x^{1}} = \frac{\partial}{\partial r}\left(\Gamma^{0}_{10} + \Gamma^{1}_{11} + \Gamma^{2}_{12} + \Gamma^{3}_{13}\right)$$

$$= \frac{\partial}{\partial r}\Bigg[\left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)$$
$$+ \left\{-\left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)\right\} + \frac{1}{r} + \frac{1}{r}\Bigg]$$

$$= \frac{\partial}{\partial r}\left[\frac{2}{r}\right] = -\frac{2}{r^2}$$

$$\Gamma^i_{ri}\Gamma^r_{11} = \Gamma^0_{r0}\Gamma^r_{11} + \Gamma^1_{r1}\Gamma^r_{11} + \Gamma^2_{r2}\Gamma^r_{11} + \Gamma^3_{r3}\Gamma^r_{11}$$
$$= \Gamma^0_{10}\Gamma^1_{11} + \Gamma^1_{11}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^1_{11} + \Gamma^3_{13}\Gamma^1_{11}$$
$$= \left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right).\left[-\left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)\right] + \left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-2}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)^2 + \frac{1}{r}.\left[-\left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)\right] + \frac{1}{r}.\left[-\left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)\right]$$

$$= -\frac{2}{r}.\left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}$$

$$\frac{\partial \Gamma^i_{11}}{\partial x^i} = \frac{\partial}{\partial r}\left(\Gamma^1_{11}\right) = \frac{\partial}{\partial r}\left[-\left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)\right]$$
$$= 2\left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-2}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)^2 - \left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}\left(-\frac{2GM}{c^2r^3} + \frac{\Lambda}{3}\right)$$

So, $$R_{11} = \Gamma^i_{r1}\Gamma^r_{1i} + \frac{\partial \Gamma^i_{1i}}{\partial x^1} - \Gamma^i_{ri}\Gamma^r_{11} - \frac{\partial \Gamma^i_{11}}{\partial x^i}$$
$$= 2\left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-2}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)^2 + \frac{2}{r^2} - \frac{2}{r^2} + \frac{2}{r}.\left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right) - 2\left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-2}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)^2 + \left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}\left(-\frac{2GM}{c^2r^3} + \frac{\Lambda}{3}\right)$$

$$= \left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1} \left(\frac{2GM}{c^2r^3} + \frac{2\Lambda}{3} - \frac{2GM}{c^2r^3} + \frac{\Lambda}{3}\right)$$

$$= \Lambda\left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}$$

$$\therefore R_{22} = \Gamma^i_{r2}\Gamma^r_{2i} + \frac{\partial\Gamma^i_{2i}}{\partial x^2} - \Gamma^i_{ri}\Gamma^r_{22} - \frac{\partial\Gamma^i_{22}}{\partial x^i}$$

Now,

$$\Gamma^i_{r2}\Gamma^r_{2i} = \Gamma^0_{r2}\Gamma^r_{20} + \Gamma^1_{r2}\Gamma^r_{21} + \Gamma^2_{r2}\Gamma^r_{22} + \Gamma^3_{r2}\Gamma^r_{23}$$

$$= 0 + \Gamma^1_{22}\Gamma^2_{21} + \Gamma^2_{12}\Gamma^1_{22} + \Gamma^3_{32}\Gamma^3_{23}$$

$$= -r\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right).\frac{1}{r} + \frac{1}{r}.\left\{-r\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\right\} + cot^2\theta$$

$$= -2\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right) + cot^2\theta$$

$$\frac{\partial\Gamma^i_{2i}}{\partial x^2} = \frac{\partial}{\partial\theta}\left(\Gamma^3_{23}\right) = \frac{\partial}{\partial\theta}(cot\theta) = -cosec^2\theta$$

$$\Gamma^i_{ri}\Gamma^r_{22} = \Gamma^0_{r0}\Gamma^r_{22} + \Gamma^1_{r1}\Gamma^r_{22} + \Gamma^2_{r2}\Gamma^r_{22} + \Gamma^3_{r3}\Gamma^r_{22}$$

$$= \Gamma^0_{10}\Gamma^1_{22} + \Gamma^1_{11}\Gamma^1_{22} + \Gamma^2_{12}\Gamma^1_{22} + \Gamma^3_{13}\Gamma^1_{22}$$

$$= \left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right).\left\{-r\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\right\}$$
$$+ \left[-\left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)\right].\left\{-r\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\right\}$$
$$+ \frac{1}{r}.\left\{-r\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\right\} + \frac{1}{r}.\left\{-r\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\right\}$$

$$= -r\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right) + r\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right) - 2\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)$$

$$= -2\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)$$

$$\frac{\partial \Gamma_{22}^{i}}{\partial x^{i}} = \frac{\partial \Gamma_{22}^{1}}{\partial x^{1}} = \frac{\partial}{\partial r}\left(\Gamma_{22}^{1}\right) = \frac{\partial}{\partial r}\left\{-r\left(1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right)\right\}$$

$$= -\left(1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right) - 2r\left(\frac{GM}{c^2 r^2} + \frac{\Lambda r}{3}\right)$$

So,

$$R_{22} = \Gamma_{r2}^{i}\Gamma_{2i}^{r} + \frac{\partial \Gamma_{2i}^{i}}{\partial x^{2}} - \Gamma_{ri}^{i}\Gamma_{22}^{r} - \frac{\partial \Gamma_{22}^{i}}{\partial x^{i}}$$

$$= -2\left(1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right) + cot^2\theta - cosec^2\theta + 2\left(1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right) + \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right) + 2r\left(\frac{GM}{c^2 r^2} + \frac{\Lambda r}{3}\right)$$

$$= -1 + \frac{2GM}{c^2 r} + \frac{2}{3}\Lambda r^2 + 1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2$$

$$= \Lambda r^2$$

$$\therefore R_{33} = \Gamma_{r3}^{i}\Gamma_{3i}^{r} + \frac{\partial \Gamma_{3i}^{i}}{\partial x^{3}} - \Gamma_{ri}^{i}\Gamma_{33}^{r} - \frac{\partial \Gamma_{33}^{i}}{\partial x^{i}}$$

Now,

$$\Gamma_{r3}^{i}\Gamma_{3i}^{r} = \Gamma_{r3}^{0}\Gamma_{30}^{r} + \Gamma_{r3}^{1}\Gamma_{31}^{r} + \Gamma_{r3}^{2}\Gamma_{32}^{r} + \Gamma_{r3}^{3}\Gamma_{33}^{r}$$

$$= 0 + \Gamma_{33}^{1}\Gamma_{31}^{3} + \Gamma_{33}^{2}\Gamma_{32}^{3} + \Gamma_{13}^{3}\Gamma_{33}^{1} + \Gamma_{23}^{3}\Gamma_{33}^{2}$$

$$= -rsin^2\theta\left(1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right).\frac{1}{r} + (-sin\theta cos\theta).cot\theta + \frac{1}{r}\left\{-rsin^2\theta\left(1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right)\right\} + cot\theta.(-sin\theta cos\theta)$$

$$= -2\,sin^2\theta\left(1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right) - 2cos^2\theta$$

$$\frac{\partial \Gamma_{3i}^{i}}{\partial x^{3}} = \frac{\partial \Gamma_{3i}^{i}}{\partial \varphi} = 0$$

$$\Gamma^i_{ri}\Gamma^r_{33} = \Gamma^0_{r0}\Gamma^r_{33} + \Gamma^1_{r1}\Gamma^r_{33} + \Gamma^2_{r2}\Gamma^r_{33} + \Gamma^3_{r3}\Gamma^r_{33}$$

$$= \Gamma^0_{10}\Gamma^1_{33} + \Gamma^1_{11}\Gamma^1_{33} + \Gamma^2_{12}\Gamma^1_{33} + \Gamma^3_{13}\Gamma^1_{33} + \Gamma^3_{23}\Gamma^2_{33}$$

$$= \left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right).\left\{-rsin^2\theta\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\right\} + \left[-\left[1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right]^{-1}\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right)\right].\left\{-rsin^2\theta\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\right\} + \frac{1}{r}.\left\{-rsin^2\theta\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\right\} + \frac{1}{r}.\left\{-rsin^2\theta\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\right\} + cot\theta.(-sin\theta cos\theta)$$

$$= -rsin^2\theta\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right) + rsin^2\theta\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right) - 2sin^2\theta\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right) - cos^2\theta$$

$$= -2sin^2\theta\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right) - cos^2\theta$$

$$\frac{\partial\Gamma^i_{33}}{\partial x^i} = \frac{\partial\Gamma^1_{33}}{\partial x^1} + \frac{\partial\Gamma^2_{33}}{\partial x^2}$$

$$= \frac{\partial}{\partial r}\left\{-rsin^2\theta\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right)\right\} + \frac{\partial}{\partial\theta}(-sin\theta cos\theta)$$

$$= -sin^2\theta\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right) - 2rsin^2\theta\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right) + sin^2\theta - cos^2\theta$$

So,

$$R_{33} = \Gamma^i_{r3}\Gamma^r_{3i} + \frac{\partial\Gamma^i_{3i}}{\partial x^3} - \Gamma^i_{ri}\Gamma^r_{33} - \frac{\partial\Gamma^i_{33}}{\partial x^i}$$

$$= -2\,sin^2\theta\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right) - 2cos^2\theta + 0 + 2sin^2\theta\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right) + cos^2\theta + sin^2\theta\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2\right) + 2rsin^2\theta\left(\frac{GM}{c^2r^2} + \frac{\Lambda r}{3}\right) - sin^2\theta + cos^2\theta$$

$$= -sin^2\theta + sin^2\theta\left(1 - \frac{2GM}{c^2r} + \frac{1}{3}\Lambda r^2 + \frac{2GM}{c^2r} + \frac{2\Lambda r^2}{3}\right)$$

$$= -sin^2\theta + sin^2\theta(1 + \Lambda r^2)$$

$$= \Lambda r^2 sin^2\theta$$

$$\therefore R_{33} = R_{22} sin^2\theta$$

The components of the Ricci tensor read

$$\left.\begin{aligned} R_{00} &= \left\{-\Lambda\left(1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right)\right\} \\ R_{11} &= \Lambda\left[1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right]^{-1} \\ R_{22} &= \Lambda r^2 \\ R_{33} &= R_{22} sin^2\theta \end{aligned}\right\} \quad (13)$$

The curvature scalar is given by

$$\begin{aligned} R &= g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \\ &= \left[1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right]^{-1} . \left\{-\Lambda\left(1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right)\right\} \\ &\quad + \left\{-\left(1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right)\right\} . \Lambda\left[1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right]^{-1} + \left(-\frac{1}{r^2}\right) . \Lambda r^2 \\ &\quad + \left(-\frac{1}{r^2 sin^2\theta}\right) . \Lambda r^2 sin^2\theta \\ &= -\Lambda - \Lambda + \Lambda + \Lambda \\ &= 0 \end{aligned}$$

$$\therefore \qquad R = 0 \qquad (14)$$

Now the Einstein's field equation becomes for empty-space (i.e, $T_{ij} = 0$)

$$R_{ij} - \frac{1}{2} g_{ij} R = 0 \qquad (15)$$

Using (14) in equation (15) we get,

$$R_{ij} = 0 \qquad (16)$$

Thus, for empty-space,

$$R_{00} = 0$$

$$or, -\Lambda\left(1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right) = 0$$

$$\therefore \qquad -\Lambda = 0, \qquad \text{or,} \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right) \neq 0 \qquad (17)$$

Then the equation takes the value

$$r_s > \frac{2GM}{c^2}$$

$$or, \qquad r_s < \frac{2GM}{c^2}$$

Where r_s is the Schwarzschild radius.

9.3 CONCLUSION:

In the metric component, $g_{00} = \left(1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2\right)$ for $\Lambda = 0$ then the metric $g_{00} = \left(1 - \frac{2GM}{c^2 r}\right)$. For $r > \frac{2GM}{c^2}$, $g_{00} > 0$ and correspondingly, t is a legitimate time coordinate. But for $r < \frac{2GM}{c^2}$, $g_{00} < 0$ and therefore, t can no longer measure time. In this region a new time coordinate a mixture of t and r will have to be defined. The metric will then no longer be independent of this new time and hence the space will case to be static. Because of this, the surface $r = 2M$is called the **static limit**.

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