GEOMETRICAL ANALYSIS
IN
RIEMANNIAN AND WEYL SPACE

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DEDICATED TO MY
MOTHERLAND BANGLADESH

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PREFACE

This thesis is essentially a review work. As the title "**GEOMETRICAL ANALYSIS IN RIEMANNIAN AND WEYL SPACE"** is concerned, geometrical representations of Riemannian space and Weyl space are presented here in respect of general relativity. This paper is consisting of six chapters and an organizing out look is given below.

In the first chapter we discussed some algebraic concept of vector spaces and their duals. From these spaces, a new space is constructed by the process of tensor product. The process is quite general though confined to real finite-dimensional vector spaces. We also discussed the effect of components of vectors due to the change of basis. In the last of this chapter, we discussed the tensor algebra in short and test the orthogonality & diagonalization of the matrix g.

The second chapter is a review work mainly on the topic manifolds. In this chapter, the definition of manifold is constructed from the concept of topology in the following sequence: SET $\xrightarrow{topology\ (open\ set)}$ TOPOLOGICAL SPACE $\xrightarrow{locally\ \ like\ R^n}$ MANIFOLD $\xrightarrow{connection}$ MANIFOLD WITH CONNECTION $\xrightarrow{metric}$ RIEMANNIAN MANIFOLD**.** We also discussed differentiable manifold, diffeomorphism, tangent spaces in manifold, orientation, sub-manifold and maps of manifolds. We also discussed linear connection, Spin connection. At last we discussed the concept of covariant differentiation with some properties and parallelism with some consequences.

In the chapter three, after an establishment of geodesic equation and geodesic deviation equation, various properties for the congruence's of time like geodesic are discussed. Here we presented the Raychowdhury equation, Focusing theorem, Forbenius theorem and physical interpretation of the expansion scalar in respect of time like geodesics.

The chapter four is mainly expository and contains original calculations. In this chapter many latest concepts regarding hypersurface are presented. Firstly, induced metric on hyper - surface, differentiation of tangent tensor field, intrinsic covariant derivative and extrinsic curvature are discussed. Secondly Gauss-Codazzi equation (general form & contracted form), Einstein tensor on hypersurface and initial value problem are discussed. Finally we presented the possible discontinuities of metric and derivatives of metric on the hypersurface.

The chapter five is mainly conceptual and contains the original calculations. In this chapter we reviewed the Weyl geometry in the context of recent higher dimensional theorem of space time. We presented some results regarding the extensions of Riemannian theorems after proper introduction of Weyl theory in respect of modern geometrical language. We also presented the mechanism how a Riemannian space time may be locally & isometrically embedded in Weyl bulk. The problem regarding classical confinement & the stability of motion of particle or photon in the neighborhood of brane when Weyl bulk possess the geometry of warped product space. We constructed a classical analog of quantum confinement inspired in theoretical field models by considering a Weyl field which depends only on the extra co ordinate.

In the chapter six, we looked for exact solution of Einstein's field equations in rotating frame for empty space. As is well known, Einstein's field equations are highly non linear and it is extremely difficult to find any solution of these equations, let, alone physically meaningful solution .Beside the Schwarzschild solution (1916, after the advent of general relativity) which is spherically symmetric-the only physically reasonable rotating solution was found by Kerr (1963). Here we presented the original calculations of different sections of J.N. Islam's book [7] (Rotating field in General Relativity) and combinations that lead to the required Kerr solution in Boyer-Lindquist form.

CONTENTS

Chapter one

VECTOR AND TENSOR

1.INTRODUCTION:

This chapter is mainly divided into two parts i.e. vector and tensor. In the first part we discuss vector space. A vector is perfectly well defined geometric objects as it in vector field , defined a set of vectors with exactly one at each point in space time. We define vector space as a collection vectors(objects) which can be added together and multiplied by a real number-in a linear way. We also decompose vectors into components with respect to some set of basis vector while a basis is any set of vectors that both spans the vector space and linearly independent.

After the settlement of vector space, we discuss the dual vector space as an associated vector space to the original vector space. We define the dual space as the space of all linear maps from the original vector space to the real number.

In the second part we discuss the tensor as the generalization of the notion of vectors and dual vectors. We define the tensor as a multilinear map from a collection of dual vector and vector to real number. At last we also discuss some algebraic operations of tensors such as direct product,inner product, contraction etc.

To discuss this chapter the following books are used as references: [1][3][8][15][16].

VECTOR AND TENSOR

1.1 VECTOR SPACE:

To discuss vector space (i.e. a set of vectors) we are to need to involve ourselves with the field of scalars K (real field R) and with the given vector space V.

Let K be a given field and $\underline{V}$ is the set of vectors $\{v_1, v_2, ..., v_n\}$ on which two different operations namely addition of vectors and multiplication of vectors by scalars are defined i.e. for any $v_1, v_2 \in \underline{V}$ and $k_1 \in K$

* $v_1 + v_2 \in \underline{V}$; Addition of vectors
* $v_1 k_1 \in \underline{V}$; Multiplication of vectors by scalars

Then $\underline{V}$ is called the vector space over the field K if the following axioms are hold:

A_1. For any vectors $v_1, v_2, v_3 \in \underline{V}$

$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$$

A_2. There exists a vector, denoted by $0 \in V$ and called zero vector for which

$$0 + v_1 = v_1 \in \underline{V} \quad \text{for any } v_1 \in \underline{V}$$

A_3. For each $v_1 \in \underline{V}$ there exist a vector $-v_1 \in \underline{V}$ such that

$$v_1 + (-v_1) = 0$$

A_4. For any vectors $v_1, v_2 \in \underline{V}$

$$v_1 + v_2 = v_2 + v_1$$

B_1. For any scalar $k_1 \in K$ and any vectors $v_1, v_2 \in \underline{V}$

$$k_1(v_1 + v_2) = v_1 k_1 + v_2 k_1$$

B_2. For any scalar $k_1, k_2 \in K$ and any vectors $v_1 \in \underline{V}$

$$(k_1 + k_2)v_1 = k_1 v_1 + k_1 v_2$$

B_3. For any scalar $k_1, k_2 \in K$ and any vectors $v_1 \in \underline{V}$

$$(k_1 k_2) v_1 = k_1 (k_2 v_1)$$

B_4. For unit scalar $1 \in K$

$$1 v_1 = v_1 \quad \text{for any } v_1 \in \underline{V}$$

A few examples of vector spaces are given below.

1. The set of all complex numbers C is a vector space.

2. The set of all square matrix i.e. $n \times n$ matrix where the operation addition '+' corresponds to sum of corresponding elements in both matrix and operation multiplication '*' means multiplying each entry by real number.

3. Set of all polynomials:

$$a^0 + a^1 t + a^2 t^2 + ----+ a^s t^s$$

$a^i \in K$; is a vector space over K with respect to usual operations of addition of polynomials and multiplication of a polynomials by a constant.

To demonstrate the notion of linear dependence and independence of vectors and vector space we will proceed as follows:

A set of vectors $\{v_1, v_2, ---, v_n\}$ of vector space $\underline{V}$ are said linearly independent if there exist a set of scalar $\{a_1, a_2, ..., a_n\} \in K$ such that

$$a^1 v_1 + a^2 v_2 + ----+ a^n v_n = 0 \qquad 1.1$$

implies that all $a^1 = a^2 = ---- = a^n = 0$

Similarly a set of vectors which is not linearly independent is called linearly dependent i.e. a set of vectors $\{v_1, v_2, ---, v_n\}$ of vector space $\underline{V}$ are said to be linearly dependent if there exist a set of scalars $\{a_1, a_2, ..., a_n\} \in K$ such that

$$a^1 v_1 + a^2 v_2 + ----+ a^n v_n = 0 \qquad 1.2$$

implies that not all of the a 's are zero or one of the a 's is not zero.

If the null vector is an element of a set of vectors of a vector space $\underline{V}$ the set of vectors is linearly dependent i.e. if 0 is one of the vectors of set $\{v_1, v_2, ---, v_n\}$, say $v_1 = 0$ then

$$1.v_1 + 0.v_2 + ----+ 0.v_n = 0$$

and the coefficient of v_1 is not zero.

Again a set of vectors are linearly dependent if one of the vectors can be expressed as a linear combination of the others. Suppose $\{v_1 -- v_m -- v_n\}$ is set of vectors of vector space V . Then vectors will be linearly dependent if

$$V_m = \sum a^i v_i$$

By using Einstein's summation convention we can write the above vector as

$$V_m = a^i v_i$$

$$\Rightarrow \underline{V} = a^i v_i$$

i.e. $\underline{V}$ is called the linear combination of vectors $v_1, v_2, ---, v_n$. The set of all such linear combinations of finite elements of the set belonging the vectors is called linear span of that set.

A set of vectors of vector space $\underline{V}$ which are both linearly independent and spans the vector space is called the basis of the vector space $\underline{V}$. The number of vectors in any basis set of finite dimensional vector apace is called the dimension of the vector space.

A vector space may have two or more basis sets. Let $\{\underline{e}_1, \underline{e}_2, ..., \underline{e}_n\}$ or $\{\underline{e}_a\}$ and $\{\underline{e'}_1, \underline{e'}_2, ..., \underline{e'}_n\}$ or $\{\underline{e'}_a\}$ are two basis sets of a vector space $\underline{V}$. Then for any $\underline{v} \in \underline{V}$ it is possible to write

$$\underline{V} = v^a \underline{e}_a \text{ and } \underline{V} = v'^a \underline{e}_{a'} \text{ for some scalar.}$$

The expression for $\underline{V}$ in terms of $\underline{e}_a$ i.e. $\underline{V} = v^a \underline{e}_a$ is unique. The scalars v^a are called the components of $\underline{V}$ relative to the basis $\{\underline{e}_a\}$. Each basis vector of a basis set $\{\underline{e}_a\}$ can be written as a linear combination of the basis vectors of another different basis set $\{\underline{e'}_a\}$. Transformation law can for this be written as

$$\underline{e}_a = X_a^{c'} \underline{e}_{c'} \qquad 1.3$$

and conversely the primed basis can be written as

$$\underline{e}_{c'} = X_{c'}^{b} \underline{e}_b \qquad 1.4$$

where $X_a^{c'}$ and $X_{c'}^{b}$ are the matrices of $n \times n$ order i.e. each contain n^2 elements but both matrices are different. Now putting the value of $\underline{e}_{c'}$ in (1.3) we have

$$\underline{e}_a = X_a^{c'} X_{c'}^{b} \underline{e}_b \qquad 1.5$$

By uniqueness of components we write

$$X_a^{c'} X_{c'}^{b} = \delta_a^b \qquad 1.6$$

If the superscript and subscript in δ appears both same then we obtain the dimension of vector space.

$$\delta_b^b = n$$

In similar fashion we can write

$$X_b^{a'} X_{c'}^{b} = \delta_c^a \qquad 1.7$$

Now write the vectors $\underline{V} = v^a \underline{e}_a$ in terms of matrix $X_a^{b'}$ as

$$\underline{V} = v^a X_a^{b'} \underline{e}_{b'} \qquad 1.8$$

and by uniqueness of components

$$V^{b'} = X_{a'}^{b} V^{a'}$$

Then $$X_{b'}^{c} V^{b'} = X_{b'}^{c} X_{a}^{b'} V^{a} = \delta_{a}^{c} V^{a} = V^{c} \qquad 1.9$$

Thus we can summaries the relation between prime and unprimed basis as

$$\underline{e}_{a'} = X_{a'}^{b} \underline{e}_{b} \quad \text{and} \quad \underline{e}_{b} = X_{b}^{a'} \underline{e}_{a'} \qquad 1.10$$

and the components are related by

$$V^{a'} = X_{b}^{a'} V^{b} \quad \text{and} \quad V^{b} = X_{a'}^{a} V^{a'} \qquad 1.11$$

and $$X_{b}^{a'} X_{c'}^{b} = \delta_{c}^{a} \quad \text{and} \quad X^{b'} X_{b'}^{a} = \delta_{c}^{a} \qquad 1.12$$

1.2 DUAL SPACE:

Let V be a vector space over a field K. then a real valued function is defined as a rule that assigns each vector of V to an unique element in K. mathematically, if f is a real valued function on V then

$$f : V \rightarrow K\,,$$

The set of all such function satisfy the axioms of vector space and hence form a vector space. Now we are interested to define linear functional to demonstrate our key point dual space.

Linear functional is also real valued function on vector space such that

$$f\left(a\underline{u} + b\underline{\upsilon}\right) = af\left(\underline{u}\right) + bf\left(\underline{\upsilon}\right)$$

for all $a, b \in F$ and $\underline{u}, \underline{\upsilon} \in T$. In short we can say that the linear functional on vector space V is a linear mapping from V into K. For linearity of f , we can define addition and multiplication of linear functional by the following statements:

$$\left(f + g\right)\left(\underline{u}\right) = f\left(\underline{u}\right) + g\left(\underline{u}\right) \qquad 1.13$$

$$\left(af\right)\left(\underline{u}\right) = a\left[f\left(\underline{u}\right)\right] \qquad 1.14$$

Also the sum of linear functionals and multiplication of a linear functional by a scalar acts as a linear functional.

The set of all linear functionals on a vector space V also forms a vector space which is the dual of original vector space V and is generally denoted by V^*.

Now we will verify that linear functionals forms a vector space on V .

A$_1$. For each $f, g, h \in V^*$ and $\underline{u} \in V$

$$\left(\left(f + g\right) + h\right)\left(\underline{u}\right) = \left(f + g\right)\left(\underline{u}\right) + h\left(\underline{u}\right)$$

$$= [f(\underline{u}) + g(\underline{u})] + h(\underline{u})$$
$$= f(\underline{u}) + [g(\underline{u}) + h(\underline{u})]$$

(due to associatively of the elements of K for addition)

$$= f(\underline{u}) + (g+h)(\underline{u})$$
$$= (f + (g+h))(\underline{u})$$

Thus $(f+g)+h = f+(g+h)$ as above equality holds for each $\underline{u} \in V$.

A_2. For $\hat{0}, f \in V$ and $\underline{u} \in V$ ($\hat{0}$ means zero functional)

$$(\hat{0}+f)(\underline{u}) = \hat{0}(\underline{u}) + f(\underline{u})$$
$$= 0 + f(\underline{u})$$
$$= f(\underline{u})$$

Thus $\hat{0} + f = f$ because of equality holds for each $\underline{u} \in V$.

Similarly $(f+\hat{0}) = f$

A_3. For each $f \in V$ there exists $(-f) \in V$ such that

$$[f + (-f)](\underline{u}) = f(\underline{u}) + (-f)(\underline{u})$$
$$= f(\underline{u}) - f(\underline{u})$$
$$= 0$$
$$= \hat{0}(\underline{u})$$

Thus $f + (-f) = \hat{0}$ as above equality holds for each $\underline{u} \in V$.

Similarly $-f + f = \hat{0}$

A_4. $$(f+g)(\underline{u}) = f(\underline{u}) + g(\underline{u})$$
$$= g(\underline{u}) + f(\underline{u})$$
$$= (g+f)(\underline{u})$$

Thus $f + g = g + f$ as above equality holds for each $\underline{u} \in V$.

B_1. $$(a(f+g))(\underline{u}) = a[(f+g)(\underline{u})]$$
$$= a[f(\underline{u}) + g(\underline{u})]$$
$$= (af)(\underline{u}) + (ag)(\underline{u})$$
$$= [(af) + (ag)](\underline{u})$$

Thus $a(f+g) = (af) + (ag)$ because of equality holds for each $\underline{u} \in$ V.

B_2. $$((ab)f)(\underline{u}) = (ab)[f(\underline{u})]$$

$$= a\left(b\left[f\left(\underline{u}\right)\right]\right)$$
$$= a\left(\left(bf\right)\left(\underline{u}\right)\right)$$
$$= \left(a\left(bf\right)\right)\left(\underline{u}\right)$$

Thus $(ab)f = a(bf)$ as above equality holds for each $\underline{u} \in$ V.

B_3. $$\left(1f\right)\left(\underline{u}\right) = 1\left[f\left(\underline{u}\right)\right]$$
$$\Rightarrow \left(1f\right)\left(\underline{u}\right) = f\left(\underline{u}\right)$$

Therefore the elements of V^* satisfy all the axioms of vector space. To define the elements of V^* i.e. dual space we use a sign (~) called tilde over the elements i.e. $\tilde{\lambda}$ while to define element of V we use a sign (-) called bar over or below the element i.e. ($\bar{u}$) or ($\underline{u}$). Also the vectors in dual space V^* are called the covariant vector while the vectors in original vector space V are called the contravariant vector. Again basis vectors of V^* carry superscripts components relative to basis vector carry subscripts. Thus if $\tilde{\lambda}$ is a vector of dual space i.e. covariant vector then we can write it in terms of basis vector

$$\tilde{\lambda} = \lambda_a \underline{e}^a$$

where $\{\underline{e}_a\}$ is the basis set of V^*.

Let $\tilde{e}^a$ be the real valued function that assigns any vector $\underline{\lambda} \in$ V into a real number which is it's a'th component.

$$\tilde{e}^a\left(\underline{\lambda}\right) = \lambda^a .$$

In particular the basis vector $\underline{e}_b$ has only b' th component and all other vanishes. So we have

$$\tilde{e}^a\left(\underline{e}_b\right) = \delta^a_b \qquad 1.15$$

The dimension of V^* will be the same as V. and in order to define any $\tilde{e}^a$ all the vectors $\{\underline{e}_b\}$ must be known. A change in any $\underline{e}_k$ generally changes all the dual basis $\tilde{e}^a$.

Now consider the action of a co vector $\tilde{u} \in V^*$ on a arbitrary contravariant vector $\underline{\lambda} \in$ V

$$\tilde{\mu}\left(\underline{\lambda}\right) = \tilde{\mu}\left(\lambda^a \underline{e}_a\right)$$
$$= \lambda^a \tilde{\mu}\left(\underline{e}_a\right)$$
$$= \tilde{e}^a\left(\underline{\lambda}\right)\tilde{\mu}\left(\underline{e}_a\right)$$
$$= \mu_a \tilde{e}^a\left(\underline{\lambda}\right)$$

The quantity $\mu_a = \tilde{\mu}\left(\underline{e}_a\right)$ are called the components of $\tilde{\mu}$ on the basis dual to $\{\underline{e}_a\}$.

Now we show that $\{\tilde{e}^a\}$ form a basis set in V^*. Also we have the relation

$$x_a \tilde{e}^a = \tilde{0} \qquad \text{where } x_a \in K \text{ and } \tilde{0} \text{ is the zero functional.}$$

implies that

$0 = x_a \tilde{e}^a(\underline{e}_b) = x_a \delta^a_b = x_b$ for all b.

which shows that $\{\tilde{e}^a\}$ is linearly independent.

A change of basis {(1.3) and (1.4)} in vector space V induces a change of the dual basis. Let denote the dual of the prime basis $\{\underline{e}_{a'}\}$ by $\{\tilde{e}^{a'}\}$. So by definition

$$\tilde{e}^{a'}(\underline{e}_{b'}) = \delta^a_b \qquad 1.16$$

But by using (1.3) we can write

$$\tilde{e}^{a'}(\underline{e}_b) = \tilde{e}^{a'}\left(X^{c'}_b \underline{e}_{c'}\right) = X^{c'}_b \tilde{e}^{a'}(\underline{e}_{c'}) = X^{c'}_b \delta^a_c = X^{a'}_b \qquad 1.17$$

Now the matrix $X^{a'}_b$ has an inverse defined as $X^b_{a'}$

Then

$$X^{a'}_b X^b_{c'} = \delta^{a'}_{c'} \ , \ X^{a'}_b X^{a'}_d = \delta^d_b \qquad 1.18$$

Multiplying (1.17) by $X^c_{a'}$

$$X^c_{a'} \tilde{e}^{a'}(\underline{e}_b) = X^c_{a'} X^{a'}_b = \delta^c_b \qquad 1.19$$

Now comparing with (1.15) we get $\tilde{e}^b = X^b_{a'} \tilde{e}^a$

Thus we can easily obtain the transformation law for components

$$\lambda'^a = \tilde{e}^{a'}(\underline{\lambda}) = \left(X^{a'}_b \tilde{e}^b\right)(\underline{\lambda})$$

$$= X^{a'}_b \tilde{e}^b(\underline{\lambda}) = X^{a'}_b \lambda^b \qquad 1.20$$

Similarly

$$\mu'_k = \tilde{\mu}(\underline{e}'_k) = \tilde{\mu}\left(X^a_{k'} \underline{e}_a\right)$$

$$= X^a_{k'} \tilde{\mu}(\underline{e}_a) = X^a_{k'} \mu_a \qquad 1.21$$

Thus the dual basis of V^* transform according to

$$\tilde{e}^{a'} = X^{a'}_b \tilde{e}^b \quad \text{and} \quad \tilde{e}^a = X^a_{b'} \tilde{e}^{b'}$$

And component of $\tilde{\mu} \in V^*$ transform according to

$$\mu_{k'} = X^a_{k'} \mu_a \quad \text{and} \quad \mu_k = X^b_k \mu_b$$

By the procedure mentioned above to compute the dual V^{**} of V^* with dual basis $\{\underline{f}_a\}$ of V^{**} such that:

$$\underline{f}_a\left(\tilde{e}^b\right) = \delta_a^b$$

Let express any vector $\lambda \in V^{**}$ in terms of components as

$$\underline{\lambda} = \lambda^a \underline{f}_a$$

Under a change of basis of V, components of vectors in V transform according to $\lambda^{a'} = X_b^{a'} \lambda^b$. This induces a change of dual basis of V^*, under which components of vectors in V^* transform according to $\mu_{a'} = X_{a'}^b \mu_b$. In turn this induces a change of basis of V^{**} under which the components of vector in V^{**} transform according to $\lambda^{a'} = X_b^{a'} \lambda^b$ (Because the inverse of the inverse of a matrix is the matrix itself). That is the components of vectors in V^{**} transform in exactly the same way as the components of vectors in V.

This means that if we set up a one to one correspondence between vectors in V and V^{**} by making $\lambda^a \underline{e}_a$ in V correspond to $\lambda^a \underline{f}_a$ in V^{**}, where $\{\underline{f}_a\}$ is the dual of the dual of $\{\underline{e}_a\}$, then this correspondence is basis independent.

A basis independent one to one correspondence between vector spaces is called natural isomorphism and naturally isomorphic vector space identified by identifying corresponding vectors. Consequently we shall identify T^{**} with T.

1.3 TENSOR PRODUCT:

Let T and U be two vector space over R . Then T^* and U^* indicates the duals of T and U respectively. From these two vector spaces we can construct a new vector space under an operation called “tensor product” i.e. the Cartesian product $T \times U$ is the set of all ordered pairs of the form (v, w) ; $v \in T$, $w \in U$. Thus the space of all sets of ordered pairs forms a vector space. [5]

A bilinear functional f on $T \times U$ is a real valued function $f : T \times U \to R$ which is bilinear i.e. satisfy the following condition:

$$f(m\underline{u}_1 + n\,\underline{u}_2, \underline{v}) = mf(\underline{u}_1, \underline{v}) + nf(\underline{u}_2, \underline{v})$$

$$f(\underline{u}, k\underline{v}_1 + l\underline{v}_2) = k\, f(\underline{u}, \underline{v}_1) + l\, f(\underline{u}, \underline{v}_2)$$

Where $m, n, l \in R$ and $\underline{u}, \underline{u}_1, \underline{u}_2 \in T$ *and* $\underline{v}, \underline{v}_1, \underline{v}_2 \in U$.

As we have seen that linear functional on a vector space forms a vector space under the operation addition and multiplication by scalar whose set is known as the dual of original vector space so it is easy to show that the set of all bilinear functional on vector space $T \times U$ forms another vector space under the operation additions and multiplication by a scalar which is the dual $T^* \times U^*$ of the original vector space $T \times U$. Hence we can conclude that the tensor product $T \times U$ of T and U as the vector space of all bilinear functional on $T^* \times U^*$.

Alternatively,

A vector which is a member of the tensor product space is called a tensor. Since a tensor product means product spaces it is possible to define a tensor which

is the tensor product $\lambda \otimes \mu$ of individual vectors $\underline{\lambda} \in T$ and $\underline{\mu} \in U$ by setting

$$\underline{\lambda} \otimes \underline{\mu} = \lambda^a \mu^b \underline{e}_{ab} \qquad 1.22$$

where λ^a *and* μ^b are the components of $\underline{\lambda}$ and $\underline{\mu}$ respectively relative to the basis of T and U which induces the basis of $T \otimes U$. Though this definition is given via bases, it is in fact basis independent.

In particular, $\underline{e}_a \otimes \underline{f}_b = \underline{e}_{ab}$ 1.23

The tensors in $T \otimes U$ having no form like $\underline{\lambda} \otimes \underline{\mu}$ are called decomposable. [3]

The dimension of $T \otimes U$ is the product of the dimensions of T and U also in a natural way bases of T and $\{\underline{e}_b\}$ of U induces a basis $\{\underline{e}_{ab}\}$ of $T \otimes U$. The components of any $P \in T \otimes U$ relative to the basis given in terms of the dual bases of T^* and U^* by

$$P^{ab} = P(\tilde{e}^a, \tilde{e}^b)$$

Let find out the transformation rule for the component P^{ab} and induced basis vector $\underline{e}_{ab}$ when new bases are introduced into T and U. Let the bases of T & U are transformed according to

$$\underline{e}_{a'} = X^c_{a'} \underline{e}_c \quad \textit{and} \quad \underline{e}_{b'} = X^d_{b'} \underline{e}_d \qquad 1.24$$

This induces a new basis $\{\underline{e}_{a'b'}\}$ in $T \otimes U$ and for any $(\tilde{\lambda}, \tilde{\mu}) \in T^* \otimes U^*$ we get

$$\underline{e}_{a'b'}(\tilde{\lambda}, \tilde{\mu}) = \lambda_{a'} \mu_{b'} = X^c_{a'} X^d_{b'} \lambda_c \mu_d \qquad 1.25$$

$$or, \ \underline{e}_{a'b'}(\tilde{\lambda}, \tilde{\mu}) = X^c_{a'} X^d_{b'} \underline{e}_{cd}(\tilde{\lambda}, \tilde{\mu})$$

Thus we obtain

$$\underline{e}_{a'b'} = X^c_{a'} X^d_{b'} \underline{e}_{cd} \qquad 1.26$$

Now for any basis vector $\underline{e}_{a'b'}$ of $T \otimes U$ and for any $\underline{P} \in T \otimes U$ we get

$$\underline{P} = P^{a'b'} \underline{e}_{a'b'} \qquad 1.27$$

Substituting the value of (1.26) in (1.27)

$$\underline{P} = P^{a'b'} X^{c}_{a'} X^{d}_{b'} \underline{e}_{c'd'}$$

By uniqueness of components

$$P^{cd} = X^{c}_{a'} X^{d}_{b'} P^{a'b'}$$

In similar fashion we can show that

$$P^{a'b'} = X^{a'}_{c} X^{b'}_{d} P^{cd}$$

Also a tensor showing N contravariant vectors and M co variant vectors(dual) is said to have valence $\binom{N}{M}$.Again vectors are tensors of type $\binom{1}{0}$and they are linear function of one-form(dual).Similarly one-form(dual) are tensors of type $\binom{0}{1}$.

1.4 METRIC TENSOR:

The components g_{ab} of a symmetric covariant tensor having valence $\binom{0}{2}$ is called metric tensor while it must keep the following properties.

a) Symmetric i.e. $g_{ab} = g_{ba}$

b) Non singular i.e. $|g_{ab}| \neq 0$

Equivalently has an inverse i.e. $|g_{ab}|$ has an inverse.

Let T is a vector space. Then by virtue of the theory of vector space, a metric tensor provides T with an inner product $\langle \underline{\lambda}, \underline{\mu} \rangle$ of vectors $\underline{\lambda}, \underline{\mu} \in T$ defined by

$$\langle \underline{\lambda}, \underline{\mu} \rangle \equiv g(\underline{\lambda}, \underline{\mu}) = g(\underline{\mu}, \underline{\lambda}) = g(\mu^a \underline{e}_a, \lambda^b \underline{e}_b) = \mu^a \lambda^b g(\underline{e}_a, \underline{e}_b) = \mu^a \lambda^b g_{ab} \qquad 1.28$$

In particular, $g(\underline{e}_a, \underline{e}_b) = g_{ab}$

Since the matrix $[g_{ab}]$ is non singular, its inverse must exist. Let $[g^{ab}]$ be the a'th row b'th column of this inverse. Then we obtain

$$g^{ab} g_{bc} = \delta^a_c$$

Also due to the property $g_{ab} = g_{ba}$ we have $g^{ab} = g^{ba}$

In tensor algebra metric tensor also serve as a mapping. It maps a vector into one form (linear real valued function of vectors) in a 1-1 correspondence

Let $\underset{\sim}{\lambda} \in T$ then $g(\underset{\sim}{\lambda}, \ )$ for some fixed $\underset{\sim}{\lambda}$ is a one forms. Thus

$$\tilde{\lambda} = g(\underset{\sim}{\lambda}, \ ) \qquad 1.29$$

Let us take the component version of the equation

$$\lambda_a = \tilde{\lambda}(\underset{\sim}{e}_a) = g(\underset{\sim}{\lambda}, \underset{\sim}{e}_a)$$
$$= g\ (\lambda^b \underset{\sim}{e}_b, \underset{\sim}{e}_a) = \lambda^b g(\underset{\sim}{e}_b, \underset{\sim}{e}_a) = \lambda^b g_{ba} = g_{ab}\lambda^b$$

In the above equation last equality follows from the symmetry in g_{ab} .Similarly

$$g^{cd}\lambda_d = g^{cd}\, g_{df}\, \lambda^f$$
$$= \delta^c_f\, \lambda^f$$
$$= \lambda^c$$

Which shows that the map is invertible .The metric provides a unique pairing between one forms and vectors.

Let us define the length of some vector in terms of metric tensor. Let $\underset{\sim}{\lambda} \in T$ be any contra variant vector. Then the length denoted by $|\lambda|^2$ is the inner product $\langle \lambda, \lambda \rangle$ defined as

$$|\underset{\sim}{\lambda}|^2 = \langle \underset{\sim}{\lambda}, \underset{\sim}{\lambda} \rangle \equiv g(\underset{\sim}{\lambda}, \underset{\sim}{\lambda}) = g_{ab}\, \lambda^a \lambda^b$$

Thus we obtain, $\quad |\underset{\sim}{\lambda}| = \left| g_{ab}\, \lambda^a \lambda^b \right|^{1/2} = \left| \lambda_a \lambda^a \right|^{1/2}$

The modulus signs are used due to g may be indefinite for any covariant vector $\tilde{\mu}$ its length is defined similarly

$$|\tilde{\mu}|^2 = \langle \tilde{\mu}, \tilde{\mu} \rangle \equiv g(\tilde{\mu}, \tilde{\mu}) = g^{ab}\, \mu_a \mu_b$$

Hence $\quad |\tilde{\mu}| = \left| g_{ab}\, \mu^a \mu^b \right|^{1/2}$

By the definition of inner product we can also find the angle between two non null contra variant vectors $\underset{\sim}{\lambda}, \underset{\sim}{\mu}$ as

$$\cos\theta = \frac{\langle \lambda, \mu \rangle}{|\underline{\lambda}||\underline{\mu}|}$$

$$\cos\theta = \frac{g_{ab}\,\lambda^a\,\mu^b}{|g_{cd}\lambda^c\lambda^d||g_{mn}\mu^m\mu^n|}$$

In case of indefinite metric tensor we get $|\cos\theta| \rangle 1$ giving as it were a complex angle between the vectors.

Again we are always free to choose a new basis $\{\underline{e}_{j'}\}$ in which the new metric components

$$g_{i'\,j'} = g(\bar{e}_{i'}, \bar{e}_{j'}) = g(X^k_{i'}\,\bar{e}_k, X^l_{j'}\,\bar{e}_l) = X^k_{i'}\,X^l_{j'}\,g(\bar{e}_k, \bar{e}_l) = X^k_{i'}\,X^l_{j'}\,g_{kl}$$

$$or, g_{\,i'\,j'} = X^k_{i'}\,g_{kl}\,X^l_{j'}$$

Consider the above equation as a matrix equation. Then it is convenient to rewrite this equation as

$$g_{i'\,j'} = X^k_{i'}\,g_{k\,l}\,X^l_{j'}$$

Again by imposing the matrix algebra, it is easy to see this matrix equation

$$g' = X^T g\, X$$

where X^T is the transpose of the matrix X , where entries are $X^k_{i'}$ we will now see that a claver choice of X will reduce the matrix g' to a very simple form. Since X is arbitrary, we ill take it to be the product of two matrices

$$X = OD$$

where O is the orthogonal matrix $(O^{-1} = O^T)$ and D is the diagonal matrix (in particular $D^T = D$).Then we get

$$X^T = (OD)^T = D^T O^T = D\,O^{-1} \qquad \text{(by using matrix algebra)}$$

And

$$g' = D\,O^{-1} g\;OD$$

It is well known that any symmetric matrix such as g can be reduced to diagonal form, g_d by a similarity transformation using an orthogonal matrix. So let us choose O to do this:

$$g_d = O^{-1} g\,O$$

$$g' = D\, g_d\, D$$

If g_d is the matrix diag $(g_1, g_2, g_{.3}..........g_n)$ and as yet our undermined matrix D is diag $(d_1, d_2, d_{.3}..........d_n)$ Then g' is

$$g' = (\, g_1 d_1^2, g_2 d_2^2,\; g_3 d_3^2 ....g_n d_n^2)$$

We now choose $d_j = (\left|g_j\right|)^{-1/2}$ so that each element the diagonal of g' is either 1or -1. We cannot use d_j to change the sign of g_j, only its magnitude. Now the diagonal elements of g_d are the eigen values of g, and are unique apart from the order in which they appear. Moreover, since g has an inverse, none of the eigen values is zero. If we choose O to make all the negative ones appear first, then we have proved the theorem that any vector space with a metric tensor has a basis on which the metric tensor has the canonical form diag (-1……………-1, 1………1) Such a basis is said to be orthonormal. If s is the number of +1"s and t is the number of -1"s in the canonical form then s-t is the signature of the metric (the difference in the number of minus and plus signs) and s+ t is the rank of the metric (the number of non zero eigen values) .If the a metric is continuous, the rank and signature of the of the metric tensor field are same at every point [15].

The metric tensor may be differentiable as one requires but it must at least be continuous. This implies that its canonical form must be constant everywhere since it is composed of only integers and integers cannot change continuously. So we speak the signature of the field g. As long as one can choose the basis transformation matrix X freely at each point, one can transform from any given basis field to a globally orthonormal basis in which the components of g are its canonical one. But this transformation field X is not usually coordinate transformation and in fact it is generally impossible to find a coordinate basis which is also orthonormal in any open region U of a manifold.

1.5 TENSOR ALGEBRA: In this context we shortly discuss the algebraic operations of tensors.

(A) LINEAR COMBINATION:

Two tensors of type (p,q) can be added and the some produces another tensor of same type i.e. (p,q). Then we can write

$$A^{\mu_1 \mu_2 ....\mu_p}_{\lambda_1 \lambda_2 ....\lambda_q} + B^{\mu_1 \mu_2 ....\mu_p}_{\lambda_1 \lambda_2 ....\lambda_q} = C^{\mu_1 \mu_2 ....\mu_p}_{\lambda_1 \lambda_2 ....\lambda_q}$$

i.e $C^{\mu_1\mu_2.............\mu_p}_{\lambda_1\lambda_2...............\lambda_q}$ is the linear combination of $A^{\mu_1\mu_2.............\mu_p}_{\lambda_1\lambda_2...............\lambda_q}$ and $B^{\mu_1\mu_2.............\mu_p}_{\lambda_1\lambda_2...............\lambda_q}$

(B) DIRECT PRODUCT:

Given a tensor of type (p,q) i.e. $A^{\mu_1\mu_2.............\mu_p}_{\lambda_1\lambda_2...............\lambda_q}$ and a tensor of type (p',q') i.e. $B^{a_1a_2.............a_{p'}}_{\nu_1\nu_2...............\nu_{q''}}$ then their direct product is given by

$$A^{\mu_1\mu_2.............\mu_p}_{\lambda_1\lambda_2...............\lambda_q}\ B^{a_1a_2.............a_{p'}}_{\nu_1\nu_2...............\nu_{q''}} = A^{\mu_1\mu_2.............\mu_p\,a_1.........a_{p'}}_{\lambda_1\lambda_2...............\lambda_q\,\nu_1............\nu_{q'}}$$

is a tensor of type $(p+p',q+q')$ .This process is also known as outer product .

(C) CONTRACTION OF TENSOR:

The algebraic operation by which the rank of a mixed tensor (covariant & contravariant) is lowered by 2 is known as contraction. In the contraction process one contravariant index and one covariant index of a mixed tensor are set equal and the repeated index summed over. The resulting tensor is of rank lowered by two than the original tensor i.e.

$$A^{\mu_1\mu_2..........\mu_p}_{\lambda_1\lambda_2..........\lambda_q} = A^{\mu_1\mu_2..........\mu_{p-1}\lambda}_{\lambda_1\lambda_2..........\lambda_{q-1}\lambda} \rightarrow B^{\mu_1\mu_2..........\mu_{p-1}}_{\lambda_1\lambda_2..........\lambda_{q-1}}$$

(D) INNER PRODUCT:

The direct product of two tensor followed by a contraction is known as inner product i.e.

$$A^{\mu_1,\mu_2......\mu_p}_{\lambda_{11},\lambda_2.........\lambda_q}\ B^{\nu_1,\nu_2......\nu_{p'-1}\lambda_q}_{\sigma_1\sigma_2........\sigma_{q'}} = C^{\mu_1,\mu_2......\mu_p\nu_1......\nu_{p'-1}}_{\lambda_{11},\lambda_2.........\lambda_{q-1}\sigma_1.........\sigma_{q'}}.$$

But this operation also be performed by two arbitrary tensor followed by same process. i.e.

$$A^{\mu_1,\mu_2......\mu_p}_{\lambda_{11},\lambda_2.........\lambda_q}\ \ B^{\lambda_1,\nu_2......\nu_{p'}}_{\mu_1\sigma_2........\sigma_{q'}} = C^{\mu_2....\mu_p.\nu_2.....\nu_{p'}}_{\lambda_2....\lambda_q\sigma_2.......\sigma_{q'}}$$

F) LOWERING & RAISING OF INDICES:

his process can be of course be combined in various ways. A particular important operation is given by a metric tensor, the raising and lowering of indices with the metric. Let us consider a tensor $A^{\mu_1\mu_2.............\mu_p}_{\lambda_1\lambda_2...............\lambda_q}$ and the direct product plus contraction with the metric tensor $g_{\mu_1\nu}$ gives

$$g_{\mu_1\nu}\ A^{\mu_1\mu_2.............\mu_p}_{\lambda_1\lambda_2...............\lambda_q} = A^{\mu_2.............\mu_p}_{\nu\lambda_1\lambda_2...............\lambda_q}$$

which is a (p-1,q+1) tensor,

1.6 TENSOR DENSITIES:

While tensors are the objects which in a sense transform in the nicest and the simplest possible way under coordinate transformations, they are not only the relevant objects. An important class of non- tensors is so called tensor densities. The prime example of tensor density is the determinant $g = -\det g_{\mu\nu}$ of the metric tensor (-ve sign included only to make g + ve in signature (- +++). [10][6][18]

Tensor densities are needed in volume and surface integral as well as in formulating an action principle from which field equation can be derived in a convenient way.

Consider a transformation from coordinates x^{μ} and x'^{μ}. An element of four dimensional volume element transform as

$$dx'^0 dx'^1 dx'^2 dx'^3 = J\, dx^0 dx^1 dx^2 dx^3 \qquad 1.30$$

where J is the Jacobean of transformation given by

$$J = \frac{\partial(x'^0 x'^1 x'^2 x'^3)}{\partial(x^0 x^1 x^2 x^3)} = \begin{vmatrix} \frac{\partial x'^0}{\partial x^0} & \frac{\partial x'^0}{\partial x^1} & \frac{\partial x'^0}{\partial x^2} & \frac{\partial x'^0}{\partial x^3} \\ \frac{\partial x'^1}{\partial x^0} & \frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^1}{\partial x^2} & \frac{\partial x'^1}{\partial x^3} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x'^3}{\partial x^0} & \frac{\partial x'^3}{\partial x^1} & \frac{\partial x'^3}{\partial x^2} & \frac{\partial x'^3}{\partial x^3} \end{vmatrix} \qquad 1.31$$

In short we can write J as

$$J = \left|\frac{\partial x'}{\partial x}\right| \;;\; J^{-1} = \left|\frac{\partial x}{\partial x'}\right| \qquad 1.32$$

where the 2[nd] equation follows by taking matrix of both sides of the identity

$$\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right)\left(\frac{\partial x^{\lambda}}{\partial x'^{\mu}}\right) = \delta^{\lambda}_{\nu} \qquad \text{(matrix equation)} \qquad 1.33$$

Now we can write the equation (1.30) as

$$d^4x' = Jd^4x \qquad 1.34$$

We get the transformation of covariant metric tensor as follows:

$$g_{ab} = X'^{\mu}_{a} \;\; g'_{\mu\nu} \;\; X'^{\nu}_{b} \qquad 1.35$$

Let us consider (1.35) as a matrix equation and take determinant on both sides of (1.35)

$$g = J\,g'\,j \qquad \text{where } X'^{\mu}_{a} = \frac{\partial x'^{\mu}}{\partial x^{a}}$$
$$or,\; g = J^{2} g' \qquad 1.36$$

where $g' = \det(g'_{\mu\nu})$.In general g is negative quantity, so take a square root of the negative of (1.36)

$$\sqrt{-g} = J\sqrt{-g'}$$
$$or,\; \zeta = J\zeta' \qquad 1.37$$

where $\zeta = \sqrt{-g}$ *and* $\zeta' = \sqrt{-g'}$ and ζ is called curly of g.

Consider a scalar quantity s that remains invariant under co-ordinate transformation i.e.

$$s = A^{\mu}B_{\mu} = A(dx^{\mu})B(\frac{\partial}{\partial x^{\mu}}) = A(\frac{\partial x^{\mu}}{\partial x^{\nu'}}dx^{\nu'})\,B(\frac{\partial x^{r'}}{\partial x^{\mu}}\frac{\partial}{\partial x^{r'}}) = \frac{\partial x^{\mu}}{\partial x^{\nu''}}A'^{\nu}\,\frac{\partial x^{r'}}{\partial x^{\mu}}B'_{r} = \delta'^{r}_{\nu}A'^{\nu}B'_{r}$$

$$or, s = A'^{\nu}B'_{\nu} = s'$$

Also consider the following volume integral over some four dimensional region Ω

$$\int_{\Omega} s\sqrt{-g}\; d^{4}x = \int_{\Omega} s\zeta\, d^{4}x = \int_{\Omega} s\zeta'\,J\,d^{4}x = \int_{\Omega'} s'\zeta'\,d^{4}x' \qquad 1.38$$

Where Ω' is the region in the co-ordinate x'^{μ} that correspond to x^{μ}.Equation (1.37) implies that

$$\int_{\Omega} s\zeta\, d^{4}x = \text{Invariant} \qquad 1.39$$

For this reason $s\zeta$ is called scalar quantity that is its volume integral is an invariant. From (1.36) and (1.37) we see that ζ is a scalar density of weight -1; so that ζ^{w} is a scalar density of weight w.In general a tensor density of weight W is an object that transform as

$$T^{\mu'_1\mu'_2\ldots\ldots\mu'_p}_{\nu'_1\nu'_2\ldots\ldots\nu'_q} = \det(\frac{\partial x'}{\partial x})^{W}\;\frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}}\;\frac{\partial x^{\mu'_2}}{\partial x^{\mu_2}} - - - - - - \frac{\partial x^{\mu'_q}}{\partial x^{\mu_q}}\,T^{\mu_1\mu_2\ldots\ldots\mu_p}_{\nu_1\nu_2\ldots\ldots\nu_q}$$

There is one more tensor density which like the kornecker tensor has the same component in all coordinate systems. This is the totally antisymmetric Levi-civita tensor $\in^{\mu\nu\rho\sigma}$ defined by

$$\in^{\mu\nu\rho\sigma} = \begin{cases} +1 \;; \textit{if } \mu\nu\rho\sigma \textit{ is the even permutation of reference order.} \\ -1 \;; \textit{if } \mu\nu\rho\sigma \textit{ is the odd permutation of reference orde.} \\ 0 \quad ; \textit{if any two or more indices are equal..} \end{cases}$$

<u>**Chapter two**</u>

MANIFOLDS

2.0 INTRODUCTION:

In general relativity the mathematics of curved space where the curvature is created by energy and momentum is closely related to the concept of manifold. So a manifold (in which a curve is considered as a set of points) is an essential tool. After discussing some preliminary topics we will begin with the notion of manifold which generalizes the concept of a surface or a curve in R^3.However the definition will be given without reference to an embedding in R^n .Rather it will generalizes the idea of a parametric representation of a surface i.e. homeomorphic map from an open piece the surface in the plane R^2.Such a parametric representation is called a chart or a co-ordinate system. The surface is then covered by the domains of the charts .Charts are used to define on manifolds objects and attributes originally defined on R^n .

The concept of differentiable manifold generalizes the idea of differentiable surface in Euclidian space i.e. R^3 and has enough structure so that the basic concepts of calculus can be carried out.From the notion of directional derivative in Euclidian space we will obtain the notion of tangent vector to a differentiable manifold .We will study the the mapping between manifolds and the effect that mappings have on the tangent vector. Also we will discuss covariant differentiation of vectors and parallel displacement in manifold.

To study this chapter I have to deal with the following books: [1], [2], [3], [8], [11], and [15].

MANIFOIDS

2.1 TOPOLOGY:

To discuss manifolds, we need to have basic knowledge of topology.

A Topological space is a set with structure allowing for the definition of neighboring points and continuous functions

Definition: A system U of subsets of a set X defines a topology on X if U contains

(a)The null set and the set X itself.

(b)The union of every one of its subsystems

(c)The interaction of every one of its finite subsystems

The sets in U are called the open sets of the topological space (X,U) often abbreviated to X.

Example: The open sets of R, defined by unions of open intervals $a \langle \ x \langle \ b$ and the null set is a topology on R. Let us test this:

The properties (a) and (b) are obviously satisfied and straight forward. To verify (c) let us consider

$$A = \bigcup_{i \in I} A_i \qquad B == \bigcup_{j \in J} B_j$$

A_i and B_j are open sets. Then

$$A \bigcap B = \bigcup_{\substack{i \in I \\ j \in J}} (A_i \bigcap B_j)$$

is open since the intersection of two open intervals is either an empty set or an open interval. This topology is called usual topology on R.

Let X be a non empty set and let the open set consist of φ and X; This topology is called trivial.

Let X is a non empty set and let the open set consist of all subsets of X, φ and X included. This Topology is called the discrete topology.

A topological space is a Hausdorff (separated) if any two distinct points possess disjoint neighborhoods. In a Hausdorff space the points are closed subsets. The usual topology on R is Hausdorff. The discrete topology is Hausdorff. The trivial topology is not Hausdorff.

2.2 COVERING:

A system $\{U_i\}$ of (open) subsets of X is a (open) covering if each element in X belongs to at least one . $\{U_i\}$ $i.e.(UU_i = X)$ If the system $\{U_i\}$ has a finite number of elements the covering is said to be finite. Unless otherwise specified a covering will always be as open covering.

A sub covering of the covering U is a subset of U which is itself a covering. A covering U is locally finite if for every point x, there exist a neighborhood N(x), which has a non empty intersection with only a finite number of members of U.

A subset $A \subset X$ is compact if it is Hausdorff and if every covering of A has a finite sub covering.

2.3 MANIFOLDS:

A manifold is one of the most fundamental concepts in mathematics and physics which captures the idea of a space that may be curved or may have complicated topology. Then a manifold is defined as a Hausdorff topological space such that every point has a neighborhood homeomorphic to R^n i.e. a set of points M is defined to be a manifold if each point of M has an open neighborhood which has continuous 1-1 map onto an open set in R^n .(By R^n we mean the set of all n'tuples of real numbers $(x^1, x^2, x^3 ... x^n)$).But in local region 'M is look like R^n (By local like we don't mean that the metric is same but only basic notion of analysis like open sets, functions and coordinates). The entire manifold is constructed by smoothly sewing together those local regions. The dimension of the manifold is essentially n. the definition of manifold involves only open sets and not the whole of M and R^n because we don't want to restrict to global topology of M.

Example of manifolds:

(1) E_n is an n-dimensional manifold with a single identity chart defined by

$$x^i(y_1......y_n) = y_i$$

(2) The set of all (pure boost) Lorenz transformations is like wise a three dimensional manifold; the parameters are the three components of the velocity of the boost.

(3) For R-particles, the numbers consisting of all their position (3R number) and velocities (3R numbers) define a point in 6R-dimensional manifold, called phase space.

(4) For particularly common manifold is a vector space. To show such a space is a manifold we draw a map from it to some R^n. Let the vector space be n-dimensional and choose any basis $\{\bar{e}_1,......,\bar{e}_n\}$. Any vector $\underline{u}$ is then represent able as a linear combination

$$\underline{u} = a_1\underline{e}_1 + a_2\underline{e}_2 + .----+ a_n\underline{e}_n$$

But $\underline{u}$ is a point $\underline{V}$, so this establish a map from V to R^n, $\underline{u} \mapsto (a_1......a_n)$. In fact every point in R^n correspond to a unique vector in V under this map. So not only is $\underline{V}$ covered entirely by the single coordinate system, we have just constructed, but $\underline{V}$ is identical as a manifold with R^n.

2.4 DIFFERENTIABLE MANIFOLD:

A differentiable manifold is essentially a topological space with certain structure which is locally homeomorphic to R^n.

Let M be a manifold and ψ is a one to one map from a neighborhood U of M onto an open set in R^n i.e. ψ assigns to every point $p \in U$ an n-tuple of real numbers $(x^1, x^2,..........., x^n)$.

A chart (U,ψ) of a manifold M is an open set U of M, called the domain of the chart, together with a homeomorphism $\psi : U \to V$ of U onto an open set V in R^n. The coordinates $(x^1, x^2,..........., x^n)$ of the image $\psi(x) \in R^n$ of the point $x \in U \subset M$ are called the coordinates of x (local coordinate) in the chart (U,ψ). A chart (U,ψ) is also called a local coordinate system.

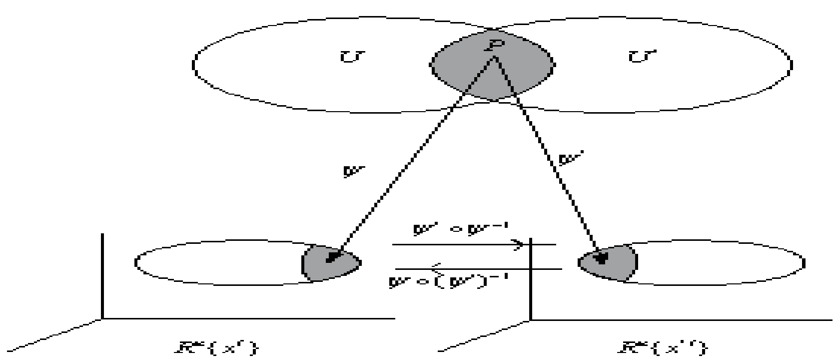

FIG: 1

Consider a number of open neighborhood which covers M and define them by U_α and for each neighborhood there is a distinct coordinate function which may be denoted by ψ_α. The open neighborhood must have overlaps if all points of M are to be included in at least one and these overlaps enables us to give a more characterization of manifold.

Suppose U' is an open neighborhood overlapping U and U' has a map ψ' onto an open region of R^n. The intersection of U and U' is open and is given two different coordinate systems by the two maps. To relate this coordinate system, pick a point $(x^1, x^2, ..., x^n)$ from the image of overlap under the map ψ. As defined before, ψ has an inverse ψ^{-1}, so there is a unique point p in the overlap which has these coordinates under ψ. Also let, ψ' assigns the point p of overlaps into another point $(y^1, y^2, ..., y^n)$ in R^n.

In this way we obtain function relationship (coordinate transformation)

$$y^1 = y^1(x^1, x^2, ..., x^n)$$

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……………………

$$y^n = y^n(x^1, x^2, ..., x^n)$$

If the partial derivatives of order k or less of all these functions $\{y^i\}$ with respect to all the $\{x^i\}$ exist and continuous then the maps ψ and ψ' strictly the charts $(U, \psi), (U', \psi')$ are said to be C^k related.

Consider two charts (U, ψ) and (U', ψ') which are said to be compatible if the combined map $\psi' \circ \psi^{-1}$ on the image $\psi(U \cap U')$ of the overlap of U and U' is a homeomorphism (continuous one-one and having continuous inverse).

An atlas of class C^k on a manifold M is a collection of sets $\{(U_\alpha, \psi_\alpha)\}$ of charts of M such that the domains $\{U_\alpha\}$ cover M and homeomorphism satisfy the compatibility condition.

A topological manifold M together with an equivalence class or compatible class of C^k atlases is a C^k structure on M and we say that M is a C^k manifold.

Strictly speaking a differentiable manifold is a manifold such that the maps $\psi' \circ \psi^{-1}$ of open sets of R^n into R^n are differentiable but not necessarily continuously differentiable. Very often the expression, differentiable manifold, smooth manifold are used to mean a C^k manifold where k is large enough for the given context, eventually $k = \infty$. A manifold of class C^1 (which includes C^k for $k > 1$) is called a differentiable manifold. In most cases it is impossible to cover the manifold with a single co-ordinate neighborhood such as the upper part of a sphere in a stereographic projection. The differentiability of a manifold endows it with an enormous amount of structure: the possibility of defining tensors, differential forms and lie derivatives.

Two co-ordinate system x^i and y^j on an open set of R^n are said to define same orientation if the Jacobean determinant $J = D(x^i)/D(y^j)$ is positive at all points of the set. A chart (U,φ) on a manifold M defines a orientation of U by means of the orientation provided by the co-ordinates $(\varphi^i(x) = x^i)$ on $\varphi(U) \in R^n$.A differentiable manifold is said to be orientable if there exists an atlas such that on the overlap $U \cap V$ of any two charts (U,φ) and (V,ψ); $D(\varphi^i)/D(\psi^j) \rangle 0$. A manifold defined in terms of such an atlas is said to be oriented.

An orientation on a manifold i.e. at a point $p \in M$ can also defined in terms of the orientation of the tangent vector space $T_p(M)$.If the manifold is orientable a frame transported along any path in the tangent bundle of the manifold comes back to its starting point with the same orientation.

2.5 DIFFEOMORPHISM:

Let M and N be two differentiable (C^k) manifold of dimension m and n respectively. Let $f : M \to N$. The function $\psi \circ f \circ \varphi^{-1}$ represents f in the local charts (U,φ), (W,ψ) of M and N. the differentiability of $f : M \to R$ is simply a particular case of the situation now considered.

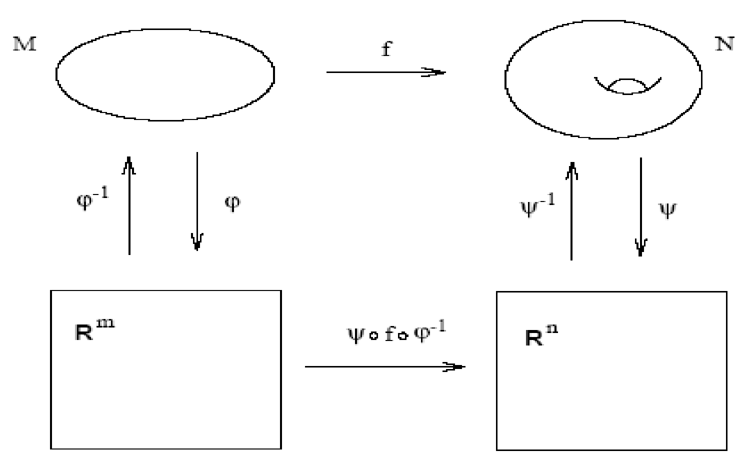

FIG:2

f is C^r,differentiable at x for $r \le k$ if $\psi \circ f \circ \varphi^{-1}$ is C^r differentiable at $\varphi(x)$. In other words f is differentiable $\{C^r\}$ at x if the coordinates $\left(y^\alpha = f^\alpha\left(x^i\right)\right)$ of y are differentiable $\{C^r\}$ functions of the coordinates $\left(x^i\right)$ of x. f is C^r mapping from M to N if f is C^r at every point $x \in M$.

In particular f is a $\{C^r\}$ diffeomorphism if f is bisection and f and f^{-1} are continuously $\{C^r\}$ differentiable. Diffeomorphism are to differentiable manifold what homeomorphism is to topological space and what isomorphism are to vector space.

The composition of deffeomorphism is again a deffeomorphism. Thus the relation of being deffeomorphic is an equivalence relation of the collection of differentiable manifolds. It is quite possible for a locally Euclidian space to possess distinct differentiable structures which are deffeomorphic. In a remarkable paper Milner showed the existence of locally Euclidian space (S^7 is an example)- which possess non diffeomorphic structure .There are also locally Euclidian space which possess no differentiable structure at all .

Now let find a relation between the coordinate system x^i and x'^i. Take a point p in $U \cap U'$ which gives the image $\left(x^1, x^2, ..., x^n\right)$ under the map ψ and $\left(x'^1, x'^2, ..., x'^n\right)$ under the map ψ'. The primed coordinates can be written in terms of unprimed coordinates by the equations

$$x'^i = f^i\left(x^1, x^2, ..., x^n\right)$$

where $(f^1, f^2, f^3 .... f^n) = f = \psi \circ \psi^{-1}$

Similarly

$$x^i = g^i\left(x'^1, x'^2, ..., x'^n\right)$$

where $\left(g^1, g^2, ..., g^n\right) = g = \psi \circ (\psi')^{-1}$.

The function f and its inverse g are both one to one and differentiable and it follows that the Jacobean $\left|\frac{\partial x'^i}{\partial x^j}\right|$ and $\left|\frac{\partial x^i}{\partial x'^j}\right|$ are non-zero.

2.6 SUBMANIFOLD:

A subset S of a manifold M of dimension n is a sub manifold of M if every point $x \in S$ is in the domain of a chart (U, φ) of M such that

$$\phi : U \cap S \to R^2 \times \{a\} \text{ by } \phi(x) = \left(x^1, ..., x^q, a^1, ...a^{n-q}\right).$$

where a is a fixed element of R^{n-q}. It is easy to check that the charts $\left(\overline{u}, \overline{\phi}\right)$ where $\overline{u} = U \cap S$ and $\overline{\phi} : \overline{u} \to R^q$ by $\phi(x) = \left(x^1, ..., x^q\right)$, form an atlas on S of the same class as the atlas $\{(u, \phi)\}$ of M.

If S already has a manifold structure, it is called a sub manifold of M if it can be given a sub manifold structure which is equivalent to the already existing structure. Sub-manifolds are defined by a system of equations. Thus we can say a sub manifold of a manifold M is a manifold which is a smooth subset of M.

An m-dimensional sub-manifold S of an n-dimensional manifold M is a set of points of M which have the following property: in some open neighborhood in M of any point p of S there exists a coordinate system for M in which the points of S in that neighborhood are the points characterized by $x^1 = x^2 = ... = x^{n-m} = 0$.

If M is ordinary three dimensional Euclidian space, then ordinary smooth surfaces and curves are sub-manifolds. In four dimensional Minkowski space-time, the 3-dimensional space of events simultaneous to a given event in the view of a particular observer (same time coordinate) is a sub manifold.

The solutions of differential equations are usually relations say $\left\{y_i = f_i\left(x^1 ....x^m\right)\right\}$, $i = 1.......p$, can be thought of as sub-manifolds with coordinates $\left\{x^1 ....x^m\right\}$ of larger manifold whose coordinates are $\left\{y_1 ....y_p x^1 ....x^m\right\}$.

Suppose p is a point of a sub manifold S (of dimension m) of M (of dimension n). a curve in S through p is also a curve in M through p, so naturally a tangent vector to each curve at p is a element of both T_p, the tangent space of M at p and V_p, the tangent space to S at p. in fact, V_p is a vector subspace of T_p not in V_p has no unique projection onto V_p.

The solution of one-forms at p is just the reverse. Let T_p^* be the dual of T_p, the set of one forms at p which are functions defined on all T_p. similarly let at V_p^* be the dual of V_p, the one-forms S itself has at p. any one-form in T_p^* defines one in V_p^*: this only involves restricting its domain from all of T_p down to its subspace V_p. but there is no unique element T_p^* corresponding to a given element of V_p^*, since simply knowing the values of a one-form on V_p does not tell us what its value will be on a vector not in V_p, thus a vector defined on a sub manifold S is also a vector on M and a one-form on M is also a one-form on S. but neither statement is reversible.

2.7 A LITTLE MORE GEOMETRY ON MANIFOLD [15]:

We have introduced maps between two different manifolds and how maps could be composed. We now turn to use of such maps in carrying along tensor fields from one manifold to another. Let us consider two manifolds M and N, possibly of different dimension, with the coordinate system x^{μ} and y^{α} respectively. We imagine that we have a map $\phi : M \to N$ and a function $f : N \to R$.[11]

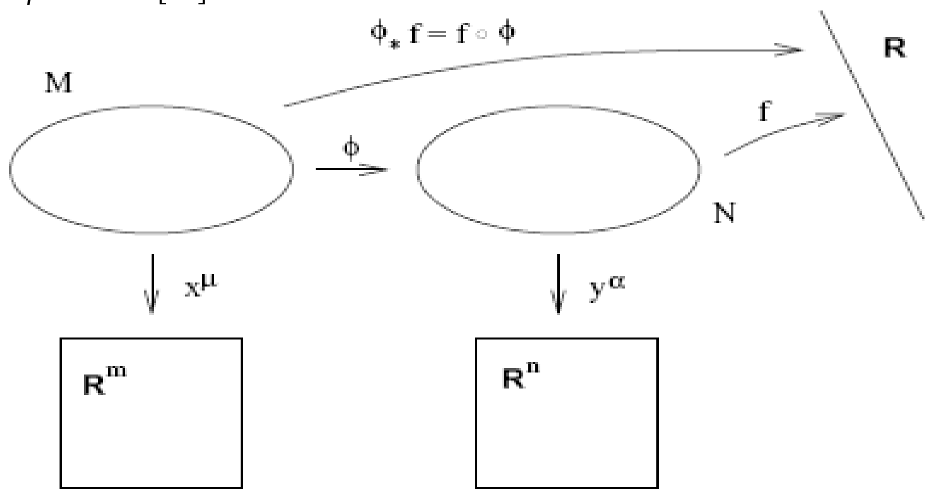

FIG: 3

It is obvious that we can compose ϕ with f to construct a map$(f \circ \phi) : M \to R$, which is simply a function on M. such a construction is sufficiently useful that it gets its own name; we define the pullback of f by ϕ denoted $\phi_* f$, by:

$$\phi_* f = (f \circ \phi)$$

The name makes sense, since we think of φ_* as “pulling back” the function f from N to M.

We can pull function back but we can’t push them forward. If we have a function $g : M \to R$, there is no way we can compose g with φ to create a function on N; the arrow sign don’t fit together correctly. But recall that a vector can be thought of as a derivative operator that maps smooth functions to real numbers. This allows us to define the push forward of a vector. If V(p) is a vector at the point p on M, we define the push forward vector $\phi^* V$ at the point $\phi(p)$ on N by giving its action on functions on N:

$$(\phi^* V)(f) = V(\phi_* f)$$

So to push forward a vector field we say “the action of $\phi^* V$ on any function is simply the action of V on the pullback of that function”.

This is a little abstract and it would be nice to have a more concrete description. We know that a basis for vectors on M is given by the set of partial derivatives $\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$ and a basis of N is given by the set of partial derivatives $\partial_\alpha = \dfrac{\partial}{\partial y^\alpha}$. Therefore we would like to relate the components of $V = V^\mu \partial_\mu$ to those of $(\phi^* V) = (\phi^* V)^\alpha \partial_\alpha$. We can find the sought- after relation by applying the pushed- forward vector to a test function and using the chain rule

$$\begin{aligned}(\phi^* V)^\alpha \partial_\alpha f &= V^\mu \partial_\mu (\phi_* f) \\ &= V^\mu \partial_\mu (f \circ \phi) \\ &= V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \partial_\alpha f\end{aligned}$$

The simple formulae makes it irresistible to think of the push forward operation ϕ^* as a matrix operator, $(\phi^* V)^\alpha = (\phi^*)^\alpha_\mu V^\mu$, with the matrix given by $(\phi^*)^\alpha_\mu = \dfrac{\partial y^\alpha}{\partial x^\mu}$.

The behavior of a vector under a push forward thus bear an unmistakable resemblance to the vector transformation law under change of coordinate. In fact it is a generalization, since when M and N are the same manifold the construction are (as we shall discuss) identical, but don't be fooled, since in general μ and α have different allowed values and there is no reason for the matrix $\dfrac{\partial y^\alpha}{\partial x^\mu}$ to be invertible

2.8 TANGENT VECTOR AND TANGENT SPACE ON MANIFOLD:

The tangent vector space $T_x(M)$ on a manifold M at a point $x \in M$ is used to define differential properties of objects in a neighborhood of x independently of local coordinates. The tangent vector space 'models' the manifold at x, most approximation in physics and mathematics consist in replacing locally a given manifold by its tangent vector space at a point x, such an approximation can be called local linearization. $T_x(M)$ is a isomorphic to R^n if M is a manifold of dimension n.

Let us imagine that we want to construct the tangent space at a point p in a manifold M, using only things that are intrinsic to M (no-embedding in a higher dimensional space etc). Consider the set of all parameterized curves through p-that is the space of all (non-degenerate) maps $\gamma : R \to M$ such that P is in the image of γ. Let $\gamma(\lambda)$ be a curve passing through the point p of M described by the equation $x^a = x^a(\lambda),\ a = 1.......n$. Also consider a differentiable function

$f\left(x^1.......x^n\right)$ i.e. $f\left(x^a\right)$ on M. At each point of the curve f has a value. Therefore along the curve there is a differentiable function $g(\lambda)$ which gives the value of f at the point whose parametric value is λ . Hence

$$g(\lambda)= f\left(x^1(\lambda)........x^n(\lambda)\right)= f\left(x^a\right)$$

This implies

$$\frac{dg}{d\lambda}=\frac{dx^a}{d\lambda}\frac{\partial f}{\partial x^a}$$

which is true for any function g so we can write

$$\frac{d}{d\lambda}=\frac{dx^a}{d\lambda}\frac{\partial}{\partial x^a} \qquad 2.1$$

One would say that the set of numbers $\left\{\frac{dx^a}{d\lambda}\right\}$ are components of a vector tangent to the curve $x^a(\lambda)$ since curve has a unique parameter, so to every curve there is a unique set $\left\{\frac{dx^a}{d\lambda}\right\}$ which are then said to be components of the tangent vector to the curve. Thus every curve has a unique tangent vector. If p be the point whose parametric value is 0 then the tangent vector $\underline{V}$ at p can be written as

$$\bar{V} = v^a \frac{\partial}{\partial x^a}.$$

The real coefficients v^a are the components of vectors $\underline{V}$ at p with respect to the local coordinate system $\left(x^1.......x^n\right)$ in the neighborhood of p. Now we will make the following claim:

"The tangent space T_p can be identified with the space of directional derivative properties along the curves through the point p".

To establish this idea we must demonstrate two things: first that the space of directional derivatives is a vector space and the second that it is the vector space we want (It has the same dimensionality as M, yields a natural idea of a vector pointing along a certain direction and so on) [15].

The first claim, that directional derivative forms a vector space, seems straightforward enough. Imagine two operators $\frac{d}{d\lambda}$ and $\frac{d}{d\eta}$ representing derivatives along two curves through p. there is no problem adding these and scaling by real numbers, to obtain a new operator $a\frac{d}{d\lambda}+b\frac{d}{d\eta}$.

It is not immediately, obvious, however, that the space is closed i.e. that the resulting operator is itself a derivative operator. A good derivative operator is one that acts linearly on functions and obeys the conventional Leibniz (product) rule on product of functions. One new operator is manifestly linear, so we need to verify that it obeys the Leibniz rule. We have

$$\left(a\frac{d}{d\lambda}+b\frac{d}{d\eta}\right)(fg)=af\frac{dg}{d\lambda}+ag\frac{df}{d\lambda}+bf\frac{dg}{d\eta}+bg\frac{df}{d\eta}$$

$$=\left(a\frac{df}{d\lambda}b\frac{df}{d\eta}\right)g+\left(a\frac{dg}{d\lambda}+b\frac{dg}{d\eta}\right)f.$$

Thus the product rule is satisfied and the set of directional derivatives is therefore a vector space.

Is it the vector space that we would like to identify with the tangent space? The easiest way to become convinced is to find a basis for the space. Consider again a coordinate chart with the coordinate x^{μ}. Then there is a obvious set of n directional derivatives at p, namely the partial derivatives ∂_{μ} at p.

We are now going to claim that the partial derivative operators $\{\partial_{\mu}\}$ at p form a basis for the tangent space T_p.(it follows immediately that T_p is n-dimensional since that is the number of

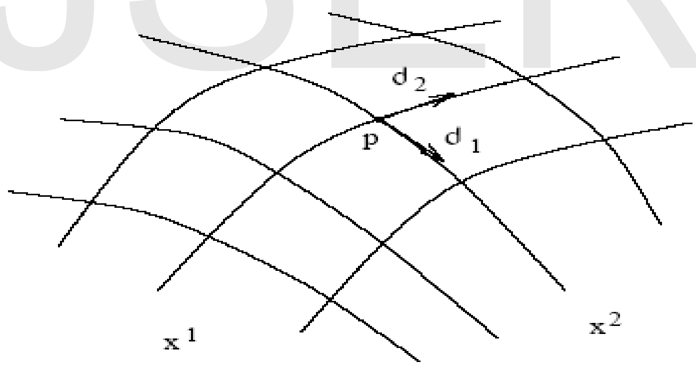

FIG: 4

basis vectors). To see this we will show that any directional derivative can be decomposed into a sum of real number time's partial derivatives.

Consider an n-manifold M, a co ordinate chart $\phi : M \to R^{n}$, a curve $\gamma : R \to M$ and a function $f : M \to R$. If λ is the parameter along γ, we want to expand the vector operator $\frac{d}{d\lambda}$ in terms of partial derivatives ∂_{μ} [15].

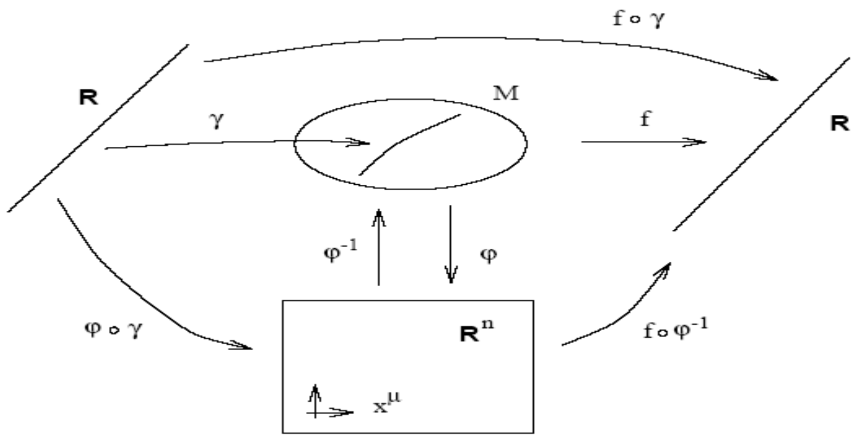

FIG: 5

Using the chain rule

$$\frac{d}{d\lambda}f = \frac{d}{d\lambda}(f \circ \gamma)$$

$$= \frac{d}{d\lambda}\left[\left(f \circ \phi^{-1}\right) \circ (\phi \circ \gamma)\right]$$

$$= \frac{d(\phi \circ \gamma)^{\mu}}{d\lambda}\frac{\partial\left(f \circ \phi^{-1}\right)}{\partial x^{\mu}}$$

$$= \frac{dx^{\mu}}{d\lambda}\partial_{\mu} f$$

The first line simply takes the informal expression on the left hand side and rewrite it as an honest derivative of the function $(f \circ \gamma): R \to R$. The second line just comes from the definition of the inverse map ϕ^{-1}. The third line is the formal chain rule and the last line is a return to the informal notation of the first. Since the function f is arbitrary

$$\frac{d}{d\lambda} = \frac{dx^{\mu}}{d\lambda}\partial_{\mu} \qquad 2.2$$

Thus the partials $\{\partial_{\mu}\}$ do indeed represent a good basis for the vector space of directional derivatives which we can therefore safely identify with the tangent space. We already know that the vector represented by $\frac{d}{d\lambda}$ is a tangent vector to the curve with parameter λ.

Thus equation (2.2) can be thought of as a restatement of (2.1) where we claimed that the components of tangent vectors were simply $\frac{dx^{\mu}}{d\lambda}$.

The only difference is that we are working on an arbitrary manifold and we have specified our basis vector to be $\hat{e}_{(\mu)} = \partial_{\mu}$.

One of the advantages of the rather abstract point of view we have taken towards vectors is that the transformation law is immediate. Since the basis vectors are $\hat{e}_{(\mu)} = \partial_{\mu}$, the basis vector in some new coordinate system x'^{μ} are given by the chain rule

$$\partial_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \partial_{\mu}$$

We get the transformation law of the vector components by the same technique used in flat space, demanding the vector $\underline{V} = v^{\mu}\partial_{\mu}$ be unchanged by a change of basis. We have

$$V^{\mu}\partial_{\mu} = V^{\mu'}\partial_{\mu'} = V^{\mu'} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \partial_{\mu}$$

And hence (as the matrix $\frac{\partial x^{\mu'}}{\partial x^{\mu}}$ is the inverse of $\frac{\partial x^{\mu}}{\partial x'^{\mu}}$)

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu} \tag{2.3}$$

Since the basis vector is usually not written explicitly, the rule (2.3) for transforming components is what we call the “vector transformation law”. We notice that it is compatible with the transformation of vector components in special relativity under Lorentz transformations, $V^{\mu'} = \Delta^{\mu'}_{\mu} V^{\mu}$ since a Lorentz transformation is a special kind of coordinate transformation with $x^{\mu'} = \Delta^{\mu'}_{\mu} x^{\mu}$. But equation (2.3) is much more general, as it encompasses the behavior of vectors under arbitrary changes of coordinates (and therefore bases), not just linear transformation. As usual we are trying to emphasize a some what subtle on-tological direction- “tensor component do not change when we change coordinate, they changes if we change the basis in the tangent space” but we have decided to use the coordinates to define our basis. Therefore a change of coordinates induces a change of basis.

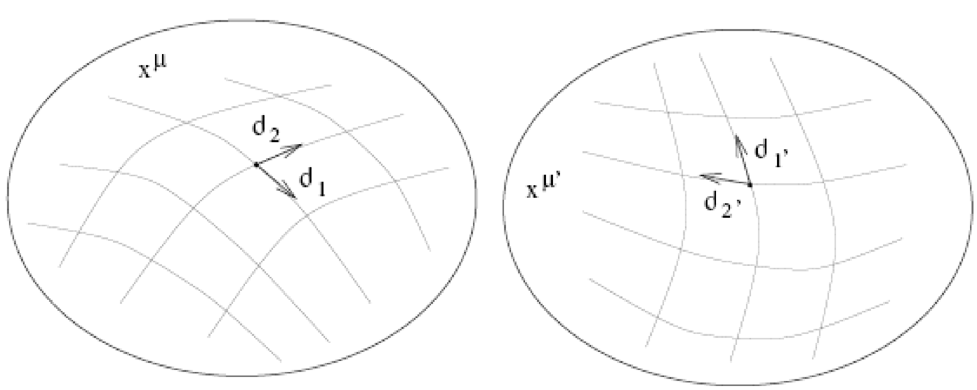

FIG: 6

If the tangent vector $\underline{V}$ has ambient coordinate ($v_1 \ldots v_s$) and local coordinate ($v^1 \ldots v^n$), then they are related by

$$v_i = \sum_{k=1}^{n} \frac{\partial y_i}{\partial x^k} v^k$$

$$And \quad v^i = \sum_{k=1}^{s} \frac{\partial x^i}{\partial y_k} v_k$$

Definition of $\frac{\partial}{\partial x^i}$ **[**2]: Take a point $p \in M$. Then $\frac{\partial}{\partial x^i}$ is the vector at p whose local coordinate under x is given by

j'th coordinate $\quad = \left(\frac{\partial}{\partial x^i}\right)^j = \delta_i^j = \begin{cases} 1 & if \;\; j = i \\ 0 & if \;\; j \neq i \end{cases}$

$$= \frac{\partial x^j}{\partial x^i}$$

Its ambient coordinates are given by

$$\text{j th coordinates } = \frac{\partial y_j}{\partial x^i}$$

2.9 RIEMANNIAN MANIFOLD:

A smooth inner product on a manifold M is a function $\langle -,- \rangle$ that associates to each pair of smooth contravariant vector fields X and Y a smooth scalar (field) $\langle X, Y \rangle$ satisfying the following properties:

Symmetry: $\quad \langle X, Y \rangle = \langle Y, X \rangle \quad \forall X$ and Y

Bilinearity: $\quad \langle \alpha X, \beta Y \rangle = \alpha\beta \langle X, Y \rangle \quad \forall \; X, Y$ and scalars α, β

$$\langle X, Y + Z \rangle = \langle X, Y \rangle + \langle X, Z \rangle$$

$$\langle X + Y, Z \rangle = \langle X, Z \rangle + \langle Y, Z \rangle$$

Non-degeneracy: If $\langle X, Y \rangle = 0$ for every Y, then $X = 0$

A Riemannian manifold is a manifold M together with a continuous 2-covariant tensor field g, called metric tensor, such that

(i) g is symmetric

(ii) for each $x \in M$, the bilinear form g_x is non degenerate;

Since M is finite dimensional in this chapter this means $g_x(v,w)=0$ for all $v \in T_x$ if and only if $w=0$. Such a manifold is said to posses a Riemannian structure.

Before we look at some examples, let us see how these things can be specified. First notice that, if x is any chart and p is any point in the domain of x then

$$\langle X,Y\rangle = X^i Y^j \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j} \right\rangle$$ This gives us smooth function

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j} \right\rangle \text{ such that } \langle X,Y\rangle = g_{ij} X^i Y^j$$

Which constitutes the coefficients of type $(0,2)$ symmetric tensor. This tensor is called the fundamental tensor or metric tensor of the Riemannian manifold.

A Riemannian manifold is called proper if

$$g_x(v,w) > 0 \text{ for all } v \in T_x \quad v \neq 0 \quad x \in M$$

Otherwise the manifold is called pseudo-Riemannian or is said to be possess an infinite metric. The index of a proper Riemannian manifold M^n is n. On such a manifold a basis (frame) (e_i) is called orthonormal if

$$\langle e_i, e_j \rangle = \delta_{ij}$$

Example:

(i) $M = E_n$ with the inner product $g_{ij} = \delta_{ij}$

(ii) $M = E_s$, with given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -c^2 \end{bmatrix}$$

where c is the light velocity. We call this Riemannian manifold flat Minkowski space M^4.

2.10 COVARIANT DIFFERENTIATION:

By a parallel field we mean a vector field with the property that the vectors at different points are parallel. But on a manifold, what does the nation of parallel field mean? For instance, in E_n there is an obvious notion: Just take a fixed vector V and translate it around. On the torus there are good candidates for parallel fields (as in fig 7) but not on the two sphere. [2]

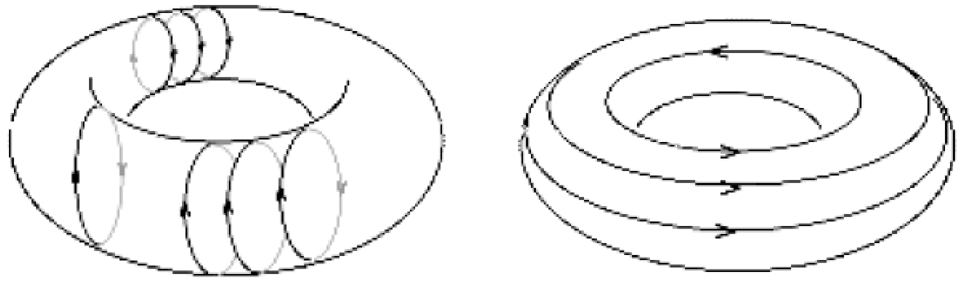

FIG: 7

Let us restrict attention to parallel fields of constant length; we can recognize such a field by taking the derivatives of its co-ordinate or by following a path and taking the derivative of the vector field with respect to t: we should come up with zero. But we wouldn't always come up zero if the co-ordinates are no rectilinear since the vector field may change direction as we move along the curved co-ordinate axes.

Let, x^j is such field and check its parallelism by taking the derivatives $\frac{dx^j}{dt}$ along some path $x^i = x^i(t)$ .However there are two catches to this approach: one is geometric and the other is algebraic.

Geometric look, for example, at the field on either torus in the above figure. Since it is circulating and hence non- constant so $\frac{dX}{dt} \neq 0$ which is not what we want. However the projection of $\frac{dX}{dt}$ parallel to the manifold does vanish, we will make this precise below:

Algebraic since, $\overline{X}^j = \frac{\partial \overline{x}^j}{\partial x^h} X^h$ then by product rule

$$\frac{d\overline{X}^j}{dt} = \frac{\partial^2 \overline{x}^j}{\partial x^k \partial x^h} X^h \frac{\partial x^k}{dt} + \frac{\partial \overline{x}^j}{\partial x^h} \frac{dX^h}{dt} \qquad 2.4$$

Showing that unless the second derivatives vanish $\frac{dX}{dt}$ does not transform as a vector field. What this means in practical terms is that we can check for parallelism at present – even in E_3 if the co-ordinates are not linear.

First let us restrict to M is embedded in E_s with the metric inherited from the embedding. The projection of $\frac{dX}{dt}$ along M will be called the co-variant derivative of X (with respect to t) and written $\frac{DX}{dt}$ or $\nabla_t X$.

Again we would like to define covariant derivative operator ∇ to reform the functions of partial derivatives but in way of independent of co-ordinates. We therefore require that ∇ be a map from (k, l) tensor fields to (k, l+1) tensor fields which has the following two properties.[15]

(1) Linearity: $\nabla(T+S) = \nabla T + \nabla S$

(2) Leibniz (product) rule: $\nabla\,(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$

If ∇ is going to obey the Leibniz rule, it can always be written as the partial derivative plus some linear transformation. Thus to make a covariant derivative we first take the partial derivative and then apply a correction to make the result covariant.

Consider the co-variant derivative of a vector V^{μ}. It means that for each direction μ, the co-variant derivative ∇_{μ} will be given by the partial derivative ∂_{μ} Plus a correction specified by a matrix $(\Gamma_{\mu})^{\rho}_{\sigma}$ (an n x n matrix, where n is the dimension of manifold for each μ). In fact the parenthesis are dropped usually and write there matrices, known as the connection co-efficient as $\Gamma^{\rho}_{\mu\sigma}$. Which is the rule of parallel displacement of vector. We therefore have,

$$\nabla_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \Gamma^{\nu}_{\mu\lambda} V^{\lambda}$$

$\nabla_{\mu} V^{\nu}$ defined in this way is indeed a (1, 1) tensor and the usual transformation rule is

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^{\mu}}{\partial y^{\mu'}} \frac{\partial y^{\nu'}}{\partial x^{\nu}} \nabla_{\mu} V^{\nu}$$

Frequently, the covariant derivative $\nabla_{\mu} V^{\nu}$ is also denoted by a semicolon $\nabla_{\mu} V^{\nu} = V^{\nu}{}_{;\mu}$. Just as for functions, one can also define the covariant directional derivative of a vector field V along another vector field X by

$$\nabla_X V^{\mu} = X^{\nu} \nabla_{\nu} V^{\mu}$$

Similarly the co-variant derivative of a co-vector is given by

$$\nabla_{\mu} V_{\nu} = \partial_{\mu} V_{\nu} - \Gamma^{\lambda}_{\mu\nu} V_{\lambda}$$

An important distinction between co-variant derivative and lie derivative is that: Dragging back a vector for the lie derivative required the entire congruence so that two vectors $\overline{U}$ and $\overline{W}$ had to be defined in a neighborhood of curve χ; parallel transport by contrast requires only the curve χ, the fields $\overline{U}$ and $\overline{W}$ on the curve and of curve the connection on the curve. Thus we can write now

$$\nabla_{\overline{U}} (f\,\overline{W}) = f\,\nabla_{\overline{U}} \overline{W} + \overline{W}\,\nabla_{\overline{U}} f$$

The notion of parallel transport along a curve requires that it must be independent of the parameter on the curve. Therefore we conclude that for any function g

$$\nabla_{g\bar{U}} \overline{W} = g \nabla_{\bar{U}} \overline{W}$$

Again at a point the covariant derivatives in different directions should have the additive property

$$(\nabla_{\bar{U}} \overline{W})_P + (\nabla_{\bar{V}} \overline{W})_P = (\nabla_{\bar{U}+\bar{V}} \overline{W})_P$$

The connection can't be regarded as a tensor field.

2.11 INVARIANT INTERPRETATION OF THE COVARIANT DERIVATIVES: [6]

The appearance of the Christoffel symbol in the definition of covariant derivative may at first sight appear a bit unusual (even though it also appears when one just transforms Cartesians partial derivatives to polar co-ordinate etc). There is a more invariant way of explaining the appearance of this term, related to the more co-ordinate independent way of looking at tensors explained above. Namely since $V^{\mu}(x)$ are really just the co-efficient of the vector field $\overline{V}(x) = V^{\mu}(x)\partial_{\mu}$ when expanded in the basis ∂_{μ}, a meaningful definition of the derivative of a vector field must take into account not only the change in the co-efficient but also the fact that the basis changes from point to point and this is precisely what the Christoffel symbol do . Writing

$$\nabla_{\mu} V = \nabla_{\mu}(V^{\nu}\partial_{\nu}) = (\partial_{\mu} V^{\nu})\partial_{\nu} + V^{\nu}(\nabla_{\mu}\partial_{\nu})$$

We see that we reproduce the definition of the covariant derivative if we set

$$\nabla_{\nu}\partial_{\mu} = \Gamma^{\lambda}_{\mu\nu}\partial_{\lambda}$$

Then we have

$$\nabla_{\mu} V = (\partial_{\mu} V^{\lambda} + \Gamma^{\lambda}_{\mu\nu} V^{\nu})\partial_{\lambda}$$

which agree with the above definition .

In some example the Christoffel symbol indeed describe the change of the tangent vectors ∂_{μ} For instance on the plane, in polar co-ordinates one has

$$\nabla_{r}\partial_{r} = \Gamma^{\mu}_{rr}\partial_{\mu} = 0$$

which is correct because ∂_r indeed does not change when moves in the radial direction. ∂_r changes however, when one moves in the angular direction given by ∂_{φ}. In fact it changes its direction proportional to ∂_{φ} but this change as stronger for small values of r than for larger one. This is precisely captured by non zero Christoffel Symbol.

$$\nabla_{\varphi}\partial_{r} = \Gamma^{\varphi}_{\varphi r}\,\partial_{\varphi} = \frac{1}{r}\partial_{\phi}$$

2.12 COVARIANT DERIVATIVE OF TENSOR AND SOME PROPERTIES:

(A) COVARIANT DERIVATIVE OF TENSOR OF TYPE (2,2):

If the (p, q) tensor is the direct product of p vectors and q co-vectors, then we already know is covariant derivative. We simply adopt the formulae for an arbitrary (p,q) tensor is the sum the partial derivative. A Christoffel symbol with positive sign for each upper indices p and a Christoffel symbol with a negative sign for each of the lower indices q. Then in equation

$$\nabla_{\bar{e}_\mu}\{T(\tilde{e}^{\nu_1},\tilde{e}^{\nu_2};\bar{e}_{\lambda_1},\bar{e}_{\lambda_2})\} = (\nabla_{\bar{e}_\mu}T)(\tilde{e}^{\nu_1},\tilde{e}^{\nu_2};\bar{e}_{\lambda_1},\bar{e}_{\lambda_2}) + T(\nabla_{e_\mu}\tilde{e}^{\nu_1},\tilde{e}^{\nu_2};\bar{e}_{\lambda_1},\bar{e}_{\lambda_2}) + T(\tilde{e}^{\nu_1},\nabla_{e_\mu}\tilde{e}^{\nu_2};\bar{e}_{\lambda_1},\bar{e}_{\lambda_2})$$

$$+T(\tilde{e}^{\nu_1},\tilde{e}^{\nu_2};\nabla_{e_\mu}\bar{e}_{\lambda_1},\bar{e}_{\lambda_2}) + T(\tilde{e}^{\nu_1},\tilde{e}^{\nu_2};\bar{e}_{\lambda_1},\nabla_{e_\mu}\bar{e}_{\lambda_2})$$

$$or,\ \partial_\mu T^{\nu_1,\nu_2}_{\lambda_1,\lambda_2} = \nabla_\mu T^{\nu_1,\nu_2}_{\lambda_1,\lambda_2} - \Gamma^{\nu_1}_{\mu\sigma}T^{\sigma,\nu_2}_{\lambda_1,\lambda_2} - \Gamma^{\nu_2}_{\mu\sigma}T^{\nu_1,\sigma}_{\lambda_1,\lambda_2} + \Gamma^{\sigma}_{\mu\lambda_1}T^{\nu_1,\nu_2}_{\sigma,\lambda_2} + \Gamma^{\sigma}_{\mu\lambda_2}T^{\nu_1,\nu_2}_{\lambda_1,\sigma}$$

$$or,\nabla_\mu T^{\nu_1,\nu_2}_{\lambda_1,\lambda_2} = \partial_\mu T^{\nu_1,\nu_2}_{\lambda_1,\lambda_2} + \Gamma^{\nu_1}_{\mu\sigma}T^{\sigma,\nu_2}_{\lambda_1,\lambda_2} + \Gamma^{\nu_2}_{\mu\sigma}T^{\nu_1,\sigma}_{\lambda_1,\lambda_2} - \Gamma^{\sigma}_{\mu\lambda_1}T^{\nu_1,\nu_2}_{\sigma,\lambda_2} - \Gamma^{\sigma}_{\mu\lambda_2}T^{\nu_1,\nu_2}_{\lambda_1,\sigma}$$

(B) COVARIANT DERIVATIVE FOR TENSOR DENSITY:

As we know that, if T is a $(p,q;w)$ tensor density, then $g^{w/2}T$ is a (p,q) tensor. Thus $\nabla_\mu(g^{w/2}T)$ is a $(p,q+1)$ tensor. To map this back to a tensor density of weight w we multiply this by $g^{-w/2}$, arriving at the definition [6]

$$\nabla_\mu T = g^{-w/2}\,\nabla_\mu(g^{w/2}T)$$

$$= \frac{w}{2g}(\partial_\mu g)T + \nabla^{tensor}_\mu T$$

where ∇^{tensor}_μ just means the usual covariant derivative for (p,q) tensor defined above.

For example. For a scalar density φ one has

$$\nabla_\mu\varphi = \partial_\mu\varphi + \frac{w}{2g}(\partial_\mu g)\varphi$$

In particular, since the determinant **g** is a scalar density of weight -2, it follows that

$$\nabla_\mu g = 0$$

which obviously simplifies the integration by parts in integrals defined with the measure $\sqrt{g}\ d^4x$.

(c) THE COVARIANT CURL OF A COVECTOR: If U_ν is a covariant vector then its covariant curl is

$$\nabla_\mu U_\nu - \nabla_\nu U_\mu = \partial_\mu U_\nu - \Gamma^\lambda_{\mu\nu} U_\lambda - \partial_\nu U_\mu + \Gamma^\lambda_{\nu\mu} U_\lambda$$
$$= \partial_\mu U_\nu - \partial_\nu U_\mu$$

(Symmetric Christoffel symbol drop out in ant- symmetric Linear Combination). Thus the Maxwell field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is a tensor under general co-ordinate transformations; no metric of covariant derivative is needed to make it a tensor in general space time.

(D) THE COVARIANT CURL OF AN ANTI SYMMETRIC TENSOR:

Let $A_{\nu\lambda..........}$. be completely ant symmetric. Then as for the curl of covector the metric and Christoffel symbol drop out of the expression for the curl, we get

$$\nabla_{[\mu} A_{\nu\lambda........]} = \partial_{[\mu} A_{\nu\lambda......]}$$

The square bracket denotes the complete anti symmetrization.

(e) THE COVARIANT DIVERGENCE OF A VECTOR:

By covariant divergence of a vector field one means the scalar

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda \qquad 2.5$$

Again a useful identity for the contracted Christoffel symbol is

$$\Gamma^\mu_{\mu\lambda} = g^{-1/2}\,\partial_\lambda (g^{+1/2}) \qquad 2.6$$

Here is an elementary proof for this identity. The standard expansion formula for the determinant

$$g = \sum_\nu (-1)^{\mu\nu} g_{\mu\nu} \left| m_{\mu\nu} \right| \qquad 2.7$$

where $\left| m_{\mu\nu} \right|$ is the determinant of the minor of $g_{\mu\nu}$ i.e. of the matrix obtained by removing the μ the row and ν the column from $g_{\mu\nu}$. If follows that

$$\frac{\partial g}{\partial g_{\mu\nu}} = (-1)^{\mu+\nu} \left| m_{\mu\nu} \right| \qquad 2.8$$

Again another consequence of (2.7) is

$$\sum_{\nu}(-1)^{\mu+\nu} g_{\lambda\nu}\left|g_{\mu\nu}\right| = 0 \qquad ;\lambda \neq \mu \tag{2.9}$$

Since this is in particular, the determinant of a matrix with $g_{\mu\nu} = g_{\nu\mu}$ i.e of matrix with two equal rows.

Together these two results can be written as

$$\sum_{\nu}(-1)^{\mu+\nu} g_{\lambda\nu}\left|g_{\mu\nu}\right| = \delta_{\mu\nu} g \tag{2.10}$$

Multiplying (2.8) by $g_{\mu\nu}$ and using (2.10)

$$g_{\lambda\nu}\,\frac{\partial g}{\partial g_{\mu\nu}} = \delta_{\lambda\nu} g \tag{2.11}$$

or the simply identity

$$\frac{\partial g}{\partial g_{\mu\nu}} = g^{\mu\nu} g \tag{2.12}$$

Thus,

$$\partial_\lambda g = \frac{\partial g}{\partial g_{\mu\nu}} \partial_\lambda g_{\mu\nu} = g\, g^{\mu\nu} \partial_\lambda g_{\mu\nu} \tag{2.13}$$

$$or,\; g^{-1}\partial_\lambda g = g^{\mu\nu}\partial_\lambda g_{\mu\nu}$$

On the other hand the contracted Christoffel symbol is

$$\Gamma^{\mu}_{\mu\lambda} = \frac{1}{2} g^{\mu\nu}\partial_\lambda g_{\mu\nu}$$

which establishes the equation (2.6)

Thus the covariant divergence can be written as

$$\nabla_\mu V^\mu = g^{-1/2}\partial_\mu (g^{1/2} V^\mu) \tag{2.14}$$

and one only needs to calculate g and its derivative, not the Christoffel symbol themselves, to calculate the covariant divergence of a vector field

(F) COVARIANT DERIVATIVE COMMUTES ON SCALAR:

This is of course a familiar property of the ordinary partial derivative but it is also true for the second covariant derivatives of a scalar and is a consequences of the symmetry of the Christoffel symbols in the second and third indices and is also known as the no torsion property of the covariant derivative. Namely, we have

$$\nabla_\mu \nabla_\nu \Phi - \nabla_\nu \nabla_\mu \Phi = \nabla_\mu \partial_\nu \Phi - \nabla_\nu \partial_\mu \Phi$$
$$= \partial_\mu \partial_\nu \Phi - \Gamma^\lambda_{\mu\nu} \Phi - \partial_\nu \partial_\mu \Phi + \Gamma^\lambda_{\nu\mu} \Phi$$
$$= 0$$

But the second covariant derivatives as higher rank tensors don't commute.

(g) ∇_μ COMMUTES WITH CONTRACTION:

This means that if A is a (p,q) tensor and B is the $(p-1,q-1)$ tensor obtained by contraction over two particular indices, then the covariant derivative of B is the same as the covariant derivative of A followed by contraction over these two indices. This comes about because of cancellation between the corresponding two Christoffel symbols with opposite sign. Consider a (1,1) tensor A^ν_ρ and its contraction A^ν_ν .The latter is just the partial derivative. This can also be obtained by taking first covariant derivative of A.

$$\nabla_\mu A^\nu_\rho = \partial_\mu A^\nu_\rho + \Gamma^\nu_{\mu\lambda} A^\lambda_\rho - \Gamma^\lambda_{\mu\rho} A^\nu_\lambda$$

and then contracting:

$$\nabla_\mu A^\nu_\nu = \partial_\mu A^\nu_\nu + \Gamma^\nu_{\mu\lambda} A^\lambda_\nu - \Gamma^\lambda_{\mu\nu} A^\nu_\lambda$$

The most transparent way of stating this property is that the kornecker delta is covariantly constant i.e.

$$\nabla_\mu \delta^\nu_\lambda = 0$$

(G) THE METRIC IS CO-VARIANTLY CONSTANT:

Since $\nabla_\mu g_{\nu\lambda}$ is tensor we can choose any co-ordinate system we like to establish if this tensor is zero or not at a given point x. Choose an inertial co-ordinate system at x .Then the partial derivatives of the metric and the Christoffel symbol zero there. Therefore the covariant derivative of the metric is zero. Since $\nabla_\mu g_{\nu\lambda}$ is a tensor, this is then true is every co-ordinate system.

2.13 PARALLELISM:

In a diffentiable manifold there is no intrinsic notion of parallelism between two vectors defined at two different points. However given a metric and a curve connecting these two points, one can compare the two by dragging one along the curve to the other using the covariant derivative.

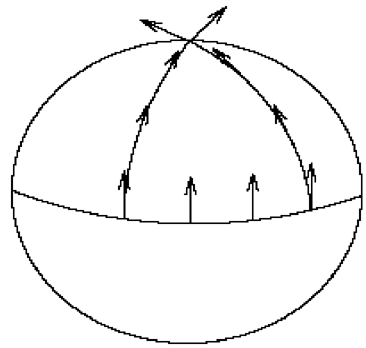

FIG: 8

Thus the concept of moving a vector along a path, keeping constant all the while is known as parallel transport. As we shall see, parallel transport is defined whenever we have connection (rule of parallel transport); the intuitive manipulation of vectors in flat space makes implicit use of this Christoffel connection on this space. The crucial difference between flat and curved space is that, in a curved space, the result of parallel transporting a vector from one point to another will depend on the path taken between the points. Let us consider a two sphere to see the case of parallel transport. Start with a vector on the equator, pointing along a line of constant longitude. Parallel transport it up to the North Pole along a line of longitude in the devious way. Then take the original vector. Parallel transport it along the equator by an angle θ and then move it up to the North Pole as before. It is clear that the vector, parallel transported along two paths, arrived at the same destination with two different values (rotated by θ).[15]

It therefore appears as if there is no natural way to uniquely move a vector from tangent space to another. We can always parallel transport it but the result depends on the path and there is no natural choice of which path to take.

Parallel transport is supposed to be the curved space generalization of the concept of "Keeping the vector constant" as we move it along path; similarly for a tensor of arbitrary rank. Given a curve $x^{\mu}(\lambda)$, then the requirement of constancy of a tensor T along this curve, in flat space is simply

$$\frac{dT}{d\lambda} = \frac{dx^{\mu}}{d\lambda}\frac{dT}{dx^{\mu}} = 0$$

We therefore define the covariant derivative along the path to be given by the operator

$$\frac{D}{d\lambda} = \frac{dx^{\mu}}{d\lambda}\nabla_{\mu}$$

We therefore define the parallel transport of the tensor T along the path $x^{\mu}(\lambda)$ to be the requirement that, along the path

$$\left(\frac{DT}{d\lambda}\right)^{\mu_1\mu_2......\mu_k}_{\nu_1\nu_2.........\nu_m} = \frac{dx^\sigma}{d\lambda}\nabla_\sigma T^{\mu_1\mu_2.....\mu_k}_{\nu_1\nu_2.........\nu_m} = 0 \qquad 2.15$$

This is well defined tensor equation. Since the both tangent vector $\frac{dx^\mu}{d\lambda}$ and the covariant derivative ∇T are tensors. This is known as the equation of parallel transport. For a vector it takes the form

$$\frac{d}{d\lambda}V^\mu + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0$$ (Absolute derivative along a curve) 2.16

We can consider the parallel transport equation as a first order differential equation defining an initial value problem; given a tensor at some point along the path, there will be a unique continuation of the tensor to other points along the path such that the continuation solves equation (2.16). We say that such a tensor is parallel transported.

2.14 SOME CONSEQUENCES OF PARALLEL TRANSPORT:

(a) Taking T to be the tangent Vector $X^\mu = \dot{x}^\mu(\tau)$ to the curve itself. The condition for parallel transport becomes

$$\frac{DX^\mu}{d\tau} = 0 \Leftrightarrow \ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0$$

i.e. precisely the geodesic equation. Thus geodesics, as we have already seen these are curves with zero acceleration can equivalently be characterized by the property that their tangent vectors are parallel transported (do not Change) along the curve. For this reason geodesic are known as auto parallels.

(b) The notion of parallel transport is obviously dependent on the connection and different connections lead to the different answers. If the connection is metric compatible (metric is co variantly constant) the metric is always parallel transported with respect to it. Thus

$$\frac{D}{d\lambda}g_{\mu\nu} = \frac{dx^\sigma}{d\lambda}\nabla_\sigma(g_{\mu\nu}) = 0$$

It follows that the inner product of two parallel-transported vectors is preserved i.e. if V^μ and W^ν are parallel transported along a curve $x^\alpha(\lambda)$ we have

$$\frac{D}{d\lambda}(g_{\mu\nu}V^\mu W^\nu) = (\frac{D}{d\lambda}g_{\mu\nu})V^\mu W^\nu + g_{\mu\nu}(\frac{D}{d\lambda}V^\mu)W^\nu + g_{\mu\nu}V^\mu(\frac{D}{d\lambda}W^\nu) = 0$$

This means that parallel transport with respect to a metric compatible connection preserves the norms of vectors, the sense of orthogonanlity and so on.

(c) We can write down an explicit and general solution to the parallel transport equation. First notice that for some path $\gamma : \lambda \rightarrow x^{\sigma}(\lambda)$ solving the parallel transport equation for a vector V^{μ} amounts to find a matrix $P^{\mu}_{\rho}(\lambda, \lambda_0)$ which relates the vector at its initial value $V^{\mu}(\lambda_0)$ to its value somewhere later down the path

$$V^{\mu}(\lambda) = p^{\mu}_{\rho}(\lambda, \lambda_0) V^{\rho}(\lambda_0) \qquad 2.17$$

The matrix $P^{\mu}_{\rho}(\lambda, \lambda_0)$ is known as parallel propagator that depends on the path γ. If we define

$$A^{\mu}_{\rho}(\lambda) = -\Gamma^{\mu}_{\sigma\rho} \frac{dx^{\sigma}}{d\lambda} \qquad 2.18$$

Where the quantities on the right hand side are evaluated at $x^{\nu}(\lambda)$, then the parallel transport equation become

$$\frac{d}{d\lambda} V^{\mu} = A^{\mu}_{\rho} V^{\rho}$$

Since the parallel propagator must works for any vector, substituting (2.17) in (2.18) shows that $P^{\mu}_{\rho}(\lambda, \lambda_0)$ also obeys the equation

$$\frac{d}{d\lambda} P^{\mu}_{\rho}(\lambda, \lambda_0) = A^{\mu}_{\sigma}(\lambda) \;\; P^{\sigma}_{\rho}(\lambda, \lambda_0) \qquad 2.19$$

To solves this equation first integrate both sides

$$P^{\mu}_{\rho}(\lambda, \lambda_0) = \delta^{\mu}_{\rho} + \int_{\lambda_0}^{\lambda} A^{\mu}_{\sigma}(\eta)\, p^{\sigma}_{\rho}(\eta, \lambda_0) d\eta \qquad 2.20$$

The kornecker delta, it is easy to see, provides the correct normalization for $\lambda = \lambda_0$ [6].

We can solve (3.17) by iteration, taking the right hand side and plugging it into itself repeated by giving

$$P^{\mu}_{\rho}(\lambda, \lambda_0) = \delta^{\mu}_{\rho} + \int_{\lambda_0}^{\lambda} A^{\mu}_{\rho}(\eta) d\eta + \int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\eta} A^{\mu}_{\sigma}(\eta)\; A^{\sigma}_{\rho}(\eta')\; d\eta' d\eta + ----------- \qquad 2.21$$

The n’th term in this series is an integral over n- dimensional right angle triangle or n-simplex.

2.15 LINEAR CONNECTION ON MANIFOLD:

A linear connection ∇ on a manifold M is a mapping that sends every pair of smooth vector fields (X, Y) to a vector field $\nabla_X Y$ such that

$$\nabla_X (aY + Z) = a\nabla_X Y + \nabla_X Z$$

For any constant scalar a .But

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y$$

When f is a function and it is linear on X .

$$\nabla_{X+fY} Z = \nabla_x z + f\nabla_Y Z$$

Action on a function f , ∇_X is defined by

$$\nabla_X f = Xf$$

Let $\{e_a\}$ be the basis for the vector fields and denote ∇_{e_a} by ∇_a. Because of $\nabla_a e_b$ being a vector there exist scalars Γ^c_{ba} such that

$$\nabla_a e_b = \Gamma^c_{ba} e_c$$

To get component version, let $X = X^a e_a$ and then from definition of connection

$$\begin{aligned}
\nabla_X Y &= \nabla_{X^a e_a}(Y^b e_b) \\
&= X^a \nabla_a (Y^b e_b) \\
&= X^a \nabla_a (Y^b)\, e_b + X^a Y^b (\nabla_a e_b) \\
&= X(Y^b) e_b + X^a Y^b \Gamma^c_{ba} e_c \\
or,\ \nabla_X Y &= [\, X(Y^b) + X^a Y^c \Gamma^b_{ca} \,] e_b \\
or,\ \nabla_X Y &= [e_a (Y^b) + Y^c \Gamma^b_{ca}] X^a e_b \\
or,\ (\nabla_X Y)^b &= [e_a (Y^b) + Y^c \Gamma^b_{ca}] X^a \\
or,\ (\nabla_X Y)^b &= (\text{covariant derivative of } Y^b) X^a \\
or,\ (\nabla_X Y)^b &= Y^b_{;a} X^a
\end{aligned}$$

In above Γ^b_{ca} are called the components of the connection which are also called a rule for parallel displacement of a vector along a curve. Again $\nabla_X Y$ is completely specified by giving the components of the connection .In above equation $Y^b_{;a}$ are components of the (1,1) tensor ∇Y .Neither of the two terms in $Y^b_{;a}$ transform like the tensor components but the sum i.e

$$\begin{aligned}
Y^b_{;a} &= \frac{\partial}{\partial x^a}(Y^b) + \Gamma^b_{ca} Y^c \\
\text{or, } Y^b_{;a} &= Y^b_{,a} + \Gamma^b_{ca} Y^c
\end{aligned}$$

transform like the tensor components.

THEOREM [14]: If a manifold possess a metric g then there exist a unique symmetric connection, Levi-Civita connection or metric connection ∇ such that

$$\nabla g = 0$$

Proof: Suppose g is a metric .Let X, Y, Z be the vector fields. Since $g(X,Y)$ is a function then we get

$$X(g(Y,Z)) = \nabla_X (g(Y,Z))$$
$$= (\nabla_X g)(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$
$$or,\ X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \qquad 2.22$$

Similarly

$$Y(g(Z,X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \qquad 2.23$$

and $$Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \qquad 2.24$$

Adding the first two equations and then subtracting the third equation we get

$$X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(\nabla_Y Z, X) + g(Z, \nabla_Y X) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)$$

$$or,\ g(Z, \nabla_Y X) = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) - g(\nabla_Y Z, X) + g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

$$or,\ g(Z, \nabla_Y X) + g(Z, \nabla_Y X) = -Z(g(X,Y)) + Y(g(Z,X) + X(g(Y,Z)) + g(Z, \nabla_Y X) - g(\nabla_X Y, Z) + g(\nabla_Z X, Y) - g(Y, \nabla_X Z) + g(X, \nabla_Z Y) - g(\nabla_Y Z, X)$$

$$or,\ 2g(Z, \nabla_Y X) = -Z(g(X,Y)) + Y(g(Z,X) + X(g(Y,Z)) + g(Z, \nabla_X Y - \nabla_Y X) + g(Y, \nabla_Z X - \nabla_X Z) - g(X, \nabla_Y Z - \nabla_Z Y)$$

$$or, g(Z, \nabla_Y X) = \frac{1}{2}\{-Z(g(X,Y)) + Y(g(Z,X) + X(g(Y,Z)) + g(Z,[X,Y]) + g(Y,[Z,X]) - g(X,[Y,Z])\}$$

Here the symmetry of the connection has been used to set

$$\nabla_X Y - \nabla_Y X = [X,Y]$$

If (e_a) be the vector basis we may set $Z = e_a$ $X = e_c$ *and* $Y = e_b$ .Then we obtain

$$or,\ g(e_a, \nabla_c e_b) = \frac{1}{2}\{ e_b(g_{ac}) + e_c(g_{ba}) - e_a(g_{cb}) + g(e_a[e_c, e_b]) + g(e_b[e_a, e_c]) - g(e_c[e_b, e_a])\}$$

$$or,\ g_{ad}\ \Gamma^d_{bc} = \frac{1}{2}\{ g_{ac,b} + g_{ba,c} - g_{cb,a} + \gamma^d_{cb}\, g_{ad} + \gamma^d_{ac}\, g_{bd} - \gamma^d_{ba}\, g_{cd}\}$$

Where $[e_b, e_c] = \gamma^a_{bc}\, e_a$ and the quantity γ^a_{bc} are called commutator co-efficient or structure constant.

If the basis is co-ordinate induced then the last three terms will be vanishes .and hence we obtain the formulae defining the connection co-efficient or Christoffel symbol as

$$g_{ad}\ \Gamma^{d}_{bc} = \frac{1}{2}(g_{ac,b} + g_{ba,c} - g_{cb,a})$$

$$or,\ \Gamma^{d}_{bc} = g^{ad}\frac{1}{2}(g_{ac,b} + g_{ba,c} - g_{cb,a})$$

Whenever a manifold M possess a metric we will usually use the metric connection without explicitly saying so. However in a metric manifold not all the connections are metric.

2.16 SPIN CONNECTION:

The co-variant derivative of a tensor is given by its partial derivative plus a correction term for each index involving the tensor i.e. Connection co-efficient. The same procedure will be true for non-co-ordinate basis but we replace the ordinary connection co-efficient $\Gamma^{\lambda}_{\mu\nu}$ by the spin connection denoted by $w_{\mu}{}^{a}{}_{b}$ .Each Latin index gets a factor of the spin connection in the usual way:

$$\nabla_{\mu} X^{a}_{\ b} = \partial_{\mu} X^{a}_{\ b} + w_{\mu}{}^{a}{}_{c} X^{c}_{b} - w_{\mu}{}^{c}{}_{b} X^{a}_{c}$$

The name spin connection comes from the fact this can be used to take co-variant derivatives of spinors which is actually impossible using conventional connection co-efficient.

The usual demand that a tensor be independent of the way it is written allows us to derive a relationship between the spin connection (vielbeins) and the $\Gamma^{\lambda}_{\mu\nu}$'s .Consider the co-variant derivative of a vector X , first in a purely co-ordinate basis:

$$\begin{aligned}\nabla X &= (\nabla_{\mu} X^{\nu})\, dx^{\mu} \otimes \partial_{\nu} \\ &= (\partial_{\mu} X^{\nu} + \Gamma^{\nu}_{\mu\lambda} X^{\lambda}) dx^{\mu} \otimes \partial_{\nu}\end{aligned} \qquad 2.25$$

Now find the same object in mixed basis and convert into the co-ordinate basis.

$$\begin{aligned}\nabla X &= (\nabla_{\mu} X^{a})\, dx^{\mu} \otimes \hat{e}_{(a)} \\ &= (\partial_{\mu} X^{a} + w_{\mu}{}^{a}{}_{b} X^{b}) dx^{\mu} \otimes \hat{e}_{(a)} \\ &= \{\partial_{\mu}(e^{a}_{\nu} X^{\nu}) + w_{\mu}{}^{a}{}_{b}\, e^{b}_{\lambda} X^{\lambda}\} dx^{\mu} \otimes (e^{\sigma}_{a}\partial_{\sigma}) \\ &= e^{\sigma}_{a}(e^{a}_{\nu}\partial_{\mu} X^{\nu} + X^{\nu}\partial_{\mu} e^{a}_{\nu} + w_{\mu}{}^{a}{}_{b}\, e^{b}_{\lambda} X^{\lambda}) dx^{\mu} \otimes \partial_{\sigma}\end{aligned}$$

$$or,\ \nabla X = (\partial_{\mu} X^{\nu} + e^{\nu}_{a}\partial_{\mu} e^{a}_{\lambda} X^{\lambda} + e^{\nu}_{a} e^{b}_{\lambda}\ w_{\mu}{}^{a}{}_{b} X^{\lambda}) dx^{\mu} \otimes \partial_{\nu} \qquad 2.26$$

$as(\sigma \to \nu, \nu \to \lambda)$

Comparing with (2.25) we obtain

$$\Gamma^{\nu}_{\mu\lambda} = e^{\nu}_{a}\partial_{\mu} e^{a}_{\nu} + e^{\nu}_{a} e^{b}_{\lambda}\ w_{\mu}{}^{a}{}_{b}$$

Or equivalently we can write

$$w_{\mu}{}^{a}{}_{b} = e^{a}_{\nu} e^{\lambda}_{b} \Gamma^{\nu}_{\mu\lambda} - e^{\lambda}_{b} \partial_{\mu} e^{a}_{\nu}$$

A bit manipulation allows us to write this relation as the vanishing of the co-variant derivative of the vielbein (German word)

$$\nabla_{\mu} e_{\nu} = 0$$

which is sometimes known as the 'tetrad Postulate' and always true.

Chapter three

GEODESIC CONGRUENCE

3.0 INTRODUCTION:

In this chapter we have developed mathematical techniques required in the description of congruence's- the term designating an entire system of non-intersecting geodesics while consider only the cases of time like geodesics as it is virtually identical to the space like geodesics. To discuss the behavior of congruence's we introduce the expansion scalar as well as shear & rotation tensors. We have derived a useful evolution equation for the expansion which is well known Raychowdhury equation. On the basis of Raychowdhury equation we have shown that the gravity tends to forces geodesics, in the sense that an initially diverging congruence's (geodesic flying a part) will be found to diverge less rapidly in the future and an initially converging congruence's (geodesics coming together) will converge more rapidly in the future. Also we have presented Forbenius theorem which states that a congruence is hyper surface orthogonal – the geodesics are everywhere orthogonal to a family of hyper- surface if only if its rotation tensor vanishes.

The following books are used as references to study this chapter: [5], [6],[18], [19],[20].

GEODESICS CONGRUENCES

3.1 GEODESICS:

In a flat space time a geodesic is the shortest distance between two points i.e. a straight line .It has the property that its tangent vector is parallely transported along itself. But in a manifold geodesic is a curve analogous to straight line in a flat space which extrimizes the distance between two fixed points.

Consider a non-null curve γ on M described by the relation $x^{\alpha}(\lambda)$ where λ an arbitrary parameter is and let P and Q be two points on this curve. The distance between P and Q i.e. arc length along γ is given by

$$l = \int_P^Q \sqrt{\pm g_{\alpha\beta} \frac{dx^{\alpha}}{d\lambda} \frac{dx^{\beta}}{d\lambda}}\, d\lambda$$

$$or,\ l = \int_P^Q \sqrt{\pm g_{\alpha\beta}\, \dot{x}^{\alpha}\, \dot{x}^{\beta}}\, d\lambda$$

where $\dot{x}^{\alpha} = \frac{dx^{\alpha}}{d\lambda}$ and in the square root the positive (negative) sign is chosen if the curve is space like (time like).It is clear that l is invariant under a reparametarization of the curve $\lambda \to u(\lambda)$, using the arc length

$$s(t) = \int_a^Q \sqrt{\pm g_{\alpha\beta} \frac{dx^{\alpha}}{du} \frac{dx^{\beta}}{du}}\, du$$

(where s (t) =length of path from t=a to Q, setting P=a)

The reason for choosing to do this is that the tangent vector $T^{\alpha} = \frac{dx^{\alpha}}{ds}$ is then a unit vector in the sense that $\|T\|^2 = \pm 1$.

If we consider a curve in E_3, then the derivative of the unit tangent vector (again with respect to s is normal to the curve) and its magnitude is a measure of how fast the curve is turning and so we call the derivative of T^{α}, the curvature of γ.

If γ happens to be on a manifold, then the unit tangent vector is still

$$T^{\alpha} = \frac{dx^{\alpha}}{ds} = \frac{dx^{\alpha}}{dt} \Big/ \frac{ds}{dt} = \frac{dx^{\alpha}/dt}{\sqrt{\pm g_{pq} \frac{dx^{p}}{dt} \frac{dx^{q}}{dt}}}$$

Since the differential form of arc length is, $ds^2 = \pm g_{\alpha\beta} dx^\alpha dx^\beta$.

But to get the curvature, it is needed to take the co-variant derivative of the tangent vector T^α .So

$$\nabla(T^\alpha) = \nabla(\frac{dx^\alpha}{ds})$$

$$or,\ \nabla(T^\alpha) = \frac{d}{ds}(\frac{dx^\alpha}{ds}) + \Gamma^\alpha_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}$$

$$or,\ \nabla(T^\alpha) = \frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}$$

$$or,\ P^{\dot{\alpha}} = \frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds} \qquad where\ P^{\dot{\alpha}} = \nabla(T^\alpha)$$

But we get the first curvature vector P of the curve γ is given by

$$P^{\dot{\alpha}} = \frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}$$

A curve on M whose first curvature is zero is called the geodesic .Thus a geodesic is a curve that satisfy the system of second order differential equation

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds} = 0 \qquad 3.1$$

which is well known geodesic equation .Here we note that P is a tangent vector at right angles to the curve γ which measures its change relative to M. If the distance between any two points on a geodesic is zero then the geodesic is called the null geodesic .It is characterized by

$$g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$$

and also by equation (3.1).

3.2 GEODESIC DEVIATION EQUATON [5]:

In a certain sense the main effect of the curvature (gravity) is that initially parallel trajectories of freely falling non-interacting particles (dust, pebbles) do not remain parallel i.e. the gravity has the tendency to focus (defocus) the matter. This statement finds its mathematically precise formulation in the geodesic deviation equation.

The geometrical picture of the Riemannian tensor is best illustrated by examining the behavior of the neighboring geodesics. Let γ_0 and γ_1 are two such geodesics described by the relation $x^\alpha(t)$ in which t is an affine parameter .The geodesic may be time like or space like or null. We are to develop the notion of deviation vector between these two geodesics and derive an evolution

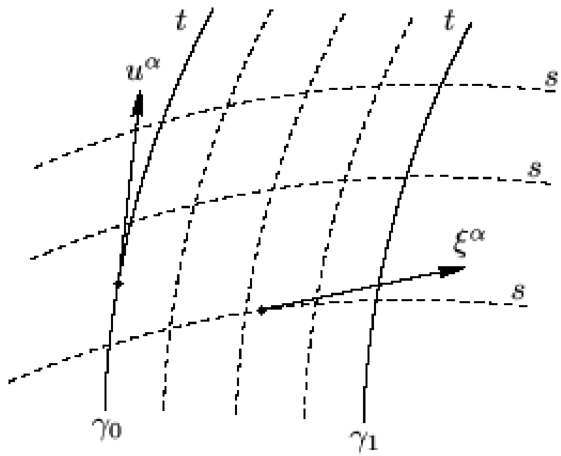

FIG: 1

(Deviation vector between two neighboring geodesics)

equation for this vector. Let us introduce an entire family of interpolating geodesics between γ_0 and γ_1 in the space(as in figure1). To each geodesic we assign a level $s \in [0,1]$ such that γ_0 comes with the level s= 0 and γ_1 with s=1. Let us describe the whole geodesic system with relation $x^\alpha(s,t)$, in which s serves to specify which geodesic and t is an affine parameter along the specified geodesic. The vector field $U^\alpha = \dfrac{\partial x^\alpha}{\partial t}$ is tangent to the geodesic and satisfy the equation $U^\alpha_{\ ;\beta} U^\beta = 0$. If we keep t fixed in the relation $x^\alpha(s,t)$ and vary s instead, we obtain another family of curves labeled by t and parameterized by s .In general these curves will not be geodesics .The family has $\xi^\alpha = \dfrac{\partial x^\alpha}{\partial s}$ as its tangent vector field and the restriction of this vector to γ_0, $\xi^\alpha\big|_s = 0$, gives a significant notion of deviation vector between γ_0 and γ_1 which characterized by the condition

$$
\begin{aligned}
& [U,\xi]^\alpha = U^\beta \nabla_\beta \xi^\alpha - \xi^\beta \nabla_\beta U^\alpha = 0 \\
\Rightarrow\ & L_U \xi^\alpha - L_\xi U^\alpha = 0 \\
\Rightarrow\ & \xi^\alpha_{\ ;\beta} U^\beta - U^\alpha_{\ ;\beta} \xi^\beta = 0 \\
\Rightarrow\ & \xi^\alpha_{\ ;\beta} U^\beta = U^\alpha_{\ ;\beta} \xi^\beta
\end{aligned} \qquad 3.2
$$

We wish to derive an expression for its acceleration.

$$
\frac{D^2 \xi^\alpha}{dt^2} \equiv (\xi^\alpha_{\ ;\beta} U^\beta)_{;\nu} U^\nu = U^\nu \nabla_\nu (U^\beta \nabla_\beta \xi^\alpha) \qquad 3.3
$$

In which it is understood that all the quantities are to be evaluated on γ_0 .In flat space time the geodesics γ_0 *and* γ_1 are straight although their separation may change with t; this change is necessarily linear i.e.

$$\frac{D^2\xi^\alpha}{dt^2} = 0 \quad \text{in flat space time.} \qquad 3.4$$

A non-zero result for $\frac{D^2\xi^\alpha}{dt^2}$ will therefore reveal the presence of curvature and indeed this vector will be found to be proportional to the Riemannian tensor.

Considering the condition $U^\alpha_{;\beta}U^\beta = 0$ and $\xi^\alpha_{;\beta}U^\beta = U^\alpha_{;\beta}\xi^\beta$ it is possible to show that $\xi^\alpha U_\alpha$ is constant along γ_0 i.e.

$$\begin{aligned}
\frac{d}{dt}\xi^\alpha U_\alpha &= (\xi^\alpha U_\alpha)_{;\beta}U^\beta \\
&= \xi^\alpha_{;\beta}U_\alpha U^\beta + \xi^\alpha U_{\alpha;\beta}U^\beta \\
&= U^\alpha_{;\beta}\xi^\beta U_\alpha \\
&= \frac{1}{2}(U^\alpha U_\alpha)_{;\beta}\xi^\beta \\
&= 0
\end{aligned}$$

Because $U_\alpha U^\alpha = \in =$ constant. The parameterization of the interpolating geodesics can therefore be turned so that on γ_0 , ξ^α is everywhere orthogonal to U^α i.e.

$$\xi^\alpha U_\alpha = 0$$

This means that the curves $t =$ constant cross γ_0 orthogonally. This adds weight to the interpretation of ξ^α as a deviation vector. Now calculate the relative acceleration of γ_1 with respect to γ_0 and let us start from the equation (3.3)

$$\begin{aligned}
\frac{D^2\xi^\alpha}{dt^2} &= \left(\xi^\alpha_{;\beta}U^\beta\right)_{;\nu}U^\nu \\
&= (U^\alpha_{;\beta}\xi^\beta)_{;\nu}U^\nu \qquad \textit{By the help of } (3.2)
\end{aligned}$$

$$or,\ \frac{D^2\xi^\alpha}{dt^2} = U^\alpha_{;\beta\nu}\xi^\beta U^\nu + U^\alpha_{;\beta}\xi^\beta_{;\nu}U^\nu \qquad 3.5$$

But the Riemannian tensor is given by

$$U^\alpha_{;\beta\nu} - U^\alpha_{;\nu\beta} = -R^\alpha_{\mu\beta\nu}U^\mu$$

Thus the equation (3.5) becomes

$$\frac{D^2\xi^\alpha}{dt^2} = (U^\alpha_{;\nu\beta} - R^\alpha_{\mu\beta\nu}U^\mu)\xi^\beta U^\nu + U^\alpha_{;\beta}\xi^\beta_{;\nu}U^\nu\xi^\nu$$
$$= (U^\alpha_{;\nu}U^\nu)_{;\beta}\xi^\beta - U^\alpha_{;\nu}U^\nu_{;\beta}\xi^\beta - R^\alpha_{\mu\beta\nu}U^\mu\xi^\beta U^\nu + U^\alpha_{;\beta}\xi^\beta_{;\nu}U^\nu\xi^\nu$$

The first term vanishes by virtue of geodesic equation and the second and fourth terms cancel each other.

Thus we obtain

$$\frac{D^2\xi^\alpha}{dt^2} = -R^\alpha_{\mu\beta\nu}U^\mu\xi^\beta U^\nu$$

$$or,\ \frac{D^2\xi^\alpha}{dt^2} = -R^\alpha_{\beta\gamma\delta}U^\beta\xi^\gamma U^\delta \qquad 3.6$$

Equation (3.6) is the required geodesic deviation equation. It shows that curvature produce a relative acceleration between two neighboring geodesics even if they start parallel, curvature prevents the geodesics from remaining parallel.

3.3 CONGRUENCE OF TIME LIKE GEODESIC:

Consider an open region $\Re$ on space time. Then congruence is defined as a nonintersecting family of curves such that through each point in $\Re$ there passes one and only one curve from this family. We would like to determine the behavior of the deviation vector ξ^α between two neighboring geodesics (as in figure 2) in the congruence as a function of proper time τ along the reference geodesic.

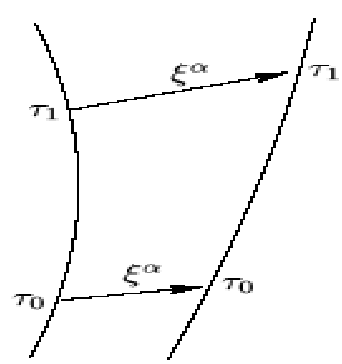

Fig: 2

Here we would consider the geometric set-up as same as considered in the previous section and the relations given below:

$$U^\alpha U_\alpha = -1 \qquad U^\alpha_{;\beta} U^\beta = 0$$

$$And \quad U^\alpha_{;\beta}\,\xi^\beta = \xi^\alpha_{\ ;\beta} U^\beta \qquad U^\alpha\ \xi_\alpha = 0$$

Where $U^\alpha = \dfrac{dx^\alpha}{d\tau}$ tangent to the geodesic–will be assumed to hold. In particular ξ^α is orthogonal to U^α i.e. the deviation vector points in the directions transverse to the flow of congruence.

3.4 TRANSVERSE METRIC:

Let U^α be the associated time like vector field of a given congruence and the space time metric $g_{\alpha\beta}$ can be decomposed into a longitudinal part $-U_\alpha U_\beta$ and a transverse part $h_{\alpha\beta}$ given by

$$g_{\alpha\beta} = -U_\alpha U_\beta + h_{\alpha\beta}$$
$$or, \quad h_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta$$

The transverse metric is purely spatial in the sense that it is orthogonal to U^α i.e.

$$U^\alpha h_{\alpha\beta} = 0 = h_{\alpha\beta} U^\beta$$

It is effectively three dimensional: in a co-moving Lorentz frame at some point P within the congruence,

$$U_\alpha =^* (-1,0,0,0)\ :\quad g_{\alpha\beta} = diag\,(-1,1,1,1)\ \ and\ \ h_{\alpha\beta} =^* diag\ (0,1,1,1)$$

where $=^*$ means equal in the specified co-ordinate system.

3.5 KINEMATICS:

Let introduce a tensor field, $B_{\alpha\beta} = U_{\alpha;\beta}$ which is purely transverse like $h_{\alpha\beta}$ since

$$U^\alpha B_{\alpha\beta} = U^\alpha U_{\alpha;\beta} = \frac{1}{2}(U_\alpha U^\alpha)_{;\beta} = 0$$

and

$$B_{\alpha\beta} U^\beta = U_{\alpha;\beta} U^\beta = 0$$

It determines the evaluation of the deviation vector. From $\xi^\alpha_{\ ;\beta} U^\beta = U^\alpha_{;\beta}\xi^\beta$ we obtain

$$\xi^\alpha_{;\beta}\, U^\beta = B^\alpha_\beta\, \xi^\beta \qquad 3.7$$

and we see that B^α_β measures the failure of the ξ^α to be parallely transported along the congruence. We now decompose the tensor field $B_{\alpha\beta}$ into trace, symmetric trace free and anti-symmetric parts. [5]

This gives

$$B_{\alpha\beta} = \frac{1}{3}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + w_{\alpha\beta}$$

where $\theta = B^{\alpha}_{\alpha} = U^{\alpha}_{;\alpha}$ is the expansion scalar.

$\sigma_{\alpha\beta} = B_{(\alpha\beta)} - \frac{1}{3}\theta h_{\alpha\beta}$ is the shear tensor .

$w_{\alpha\beta} = B_{[\alpha\beta]}$ is the rotation tensor.

In particular the congruence is diverging (geodesics flying a part) if the expansion scalar is greater than zero i.e. $\theta \rangle 0$ and it will be converging (geodesics coming together) if $\theta \langle 0$.

3.6 RAYCHOWDHURY EQUATION:

Let us derive an evolution equation for expansion scalar θ and so begin by developing an equation for $B_{\alpha\beta}$ itself. We get

$$\begin{aligned} B_{\alpha\beta;\mu}U^{\mu} &= U_{\alpha;\beta\mu}U^{\mu} \\ &= (U_{\alpha;\mu\beta} - R_{\alpha\nu\beta\mu})U^{\mu} \\ &= (U_{\alpha;\mu}U^{\mu})_{;\beta} - U_{\alpha;\mu}U^{\mu}_{;\beta} - R_{\alpha\nu\beta\mu}U^{\nu}U^{\mu} \\ or, B_{\alpha\beta;\mu}U^{\mu} &= -B_{\alpha\mu}B^{\mu}_{\beta} - R_{\alpha\mu\beta\nu}U^{\mu}U^{\nu} \end{aligned} \qquad 3.8$$

Again from the definition of curvature tensor we obtain

$$(\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha})U^{\mu} = R^{\mu}_{\nu\alpha\beta}U^{\nu}$$

Contracting on the indices α *and* μ we get

$$\begin{aligned} (\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha})U^{\alpha} &= R_{\nu\beta}U^{\nu} \\ or,\ (\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha})U^{\alpha} &= R_{\alpha\beta}U^{\alpha} \end{aligned}$$

Multiplying on both sides by U^{β} we obtain

$$U^{\beta}\nabla_{\alpha}\nabla_{\beta}U^{\alpha} - U^{\beta}\nabla_{\beta}\nabla_{\alpha}U^{\alpha} = R_{\alpha\beta}U^{\alpha}U^{\beta}$$

The first term in the above equation can be written as

$$U^{\beta}\nabla_{\alpha}\nabla_{\beta}U^{\alpha} = \nabla_{\alpha}(U^{\beta}\nabla_{\beta}U^{\alpha}) - (\nabla_{\alpha}U^{\beta})(\nabla_{\beta}U^{\alpha})$$

Thus the above equation can be written as

$$U^{\beta}\nabla_{\beta}(\nabla_{\alpha}U^{\alpha}) + (\nabla_{\alpha}U^{\beta})(\nabla_{\beta}U^{\alpha}) - \nabla_{\alpha}(U^{\beta}\nabla_{\beta}U^{\alpha}) + R_{\alpha\beta}U^{\alpha}U^{\beta} = 0$$

But the third term vanishes due to $U^{\beta}\nabla_{\beta}U^{\alpha} = 0$ .So we obtain

$$U^{\beta}\nabla_{\beta}(\nabla_{\alpha}U^{\alpha}) + (\nabla_{\alpha}U^{\beta})(\nabla_{\beta}U^{\alpha}) + R_{\alpha\beta}U^{\alpha}U^{\beta} = 0$$

$$or, \quad U^{\beta}\nabla_{\beta}(\nabla_{\alpha}U^{\alpha}) + (\nabla_{\alpha}U_{\beta})(\nabla^{\beta}U^{\alpha}) + R_{\alpha\beta}U^{\alpha}U^{\beta} = 0$$

Here the first term the rate of change of divergence $\theta = \nabla_{\alpha}U^{\alpha}$ along U^{α} i.e.

$$U^{\beta}\nabla_{\beta}(\nabla_{\alpha}U^{\alpha}) = \frac{d\theta}{d\tau}$$

Then the equation for θ is obtained by taking the trace [6]

$$\frac{d\theta}{d\tau} = -(\nabla_{\alpha}U_{\beta})(\nabla^{\beta}U^{\alpha}) - R_{\alpha\beta}U^{\alpha}U^{\beta}$$

$$= -U_{\beta;\alpha}U^{\alpha;\beta} - -R_{\alpha\beta}U^{\alpha}U^{\beta}$$

$$or, \ \frac{d\theta}{d\tau} = -B_{\beta\alpha}B^{\alpha\beta} - R_{\alpha\beta}U^{\alpha}U^{\beta}$$

But we get ,

$$B_{\alpha\beta} = \frac{1}{3}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + w_{\alpha\beta}$$

(From the definition of shear tensor)

Then $B^{\alpha\beta}B_{\alpha\beta} = \frac{1}{3}\theta^{2} + \sigma^{\alpha\beta}\sigma_{\alpha\beta} - w^{\alpha\beta}w_{\alpha\beta}$

Substituting these values we get

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^{2} - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + w^{\alpha\beta}w_{\alpha\beta} - R_{\alpha\beta}U^{\alpha}U^{\beta} \qquad 3.9$$

which is well known Raychowdhury equation for congruence of time like geodesics. Since the shear and rotation tensor are purely spatial (transverse), $\sigma^{\alpha\beta}\sigma_{\alpha\beta} \geq 0 \ and \ w^{\alpha\beta}w_{\alpha\beta} \geq 0$ with the equality sign holding if only if the tensor is zero.

3.7 FOCUSING THEOREM:

The importance of Raychowdhury equation for congruence of time like geodesic is revealed by the following theorem:

Let a congruence of time like geodesic be hyper-surface orthogonal so that the rotation tensor $w_{\alpha\beta} = 0$ and let the strong energy condition $\rho + P_i \geq 0$ hold for the statement

$$(T_{\alpha\beta} - \frac{1}{2}T\, g_{\alpha\beta})\, V^{\alpha}V^{\beta} \geq 0$$

Then the Raychowdhury equation become

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^{2} - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + w^{\alpha\beta}w_{\alpha\beta} - R_{\alpha\beta}U^{\alpha}U^{\beta}$$

The first two terms in R.H.S are non- positive and one then assume the geometry such that

$$R_{\alpha\beta}U^{\alpha}U^{\beta} \geq 0$$

Then the above equation implies

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + w^{\alpha\beta}w_{\alpha\beta} - R_{\alpha\beta}U^{\alpha}U^{\beta} \le 0 \qquad 3.10$$

This means that the expansion must therefore decrease during the congruence evolution. Thus an initially diverging $(\theta \rangle 0)$ congruence will diverge less rapidly in the future while an initially converging $(\theta \langle 0)$ congruence will converse more rapidly in the future. This statement is known as **FOCUSING** theorem.

The interpretation of this result is that gravity is an attractive force whose effect is to focus geodesics. According to the equation (3.10) $\frac{d\theta}{d\tau}$ is not only negative but actually bounded from above by:

$$\frac{d\theta}{d\tau} \le -\frac{1}{3}\theta^2$$

Rewriting this equation as:

$$\frac{d}{d\tau}\theta \ge \frac{1}{3}\theta^2 \Rightarrow \frac{d}{d\tau}\frac{1}{\theta} \ge \frac{1}{3}$$

Now integrating the above equation we obtain

$$\frac{1}{\theta(\tau)} \ge \frac{1}{\theta(0)} + \frac{\tau}{3}$$

$$\Rightarrow \theta^{-1}(\tau) \ge \theta^{-1}(0) + \frac{\tau}{3} \qquad where\ \theta_0 \equiv \theta(0)$$

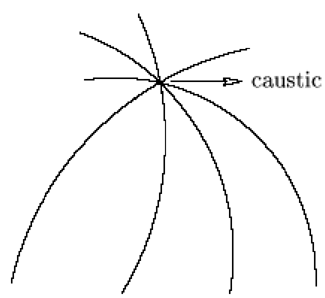

FIG: 3

(Geodesic convergence into a caustic of the congruence)

This shows that if the congruence is initially converging $(\theta_0 \langle 0)$, then $\theta(\tau) \to -\infty$ within a proper time $\tau \le \frac{3}{|\theta_0|}$.The interpretation of this result is that the congruence will develop a caustic, a point at which some of geodesics come together. Obviously a caustic is singularity of the congruence and Raychowdhury equation loses their meaning at such point.

3.8 FORBENIUS THEOREM:

Some congruence having a vanishing rotation tensor. i.e. $w_{\alpha\beta} = 0$ are said to be hypersurface orthogonal which means that the congruence is everywhere orthogonal to a family of space like hyper-surface $\Re$ (as in figure 4)

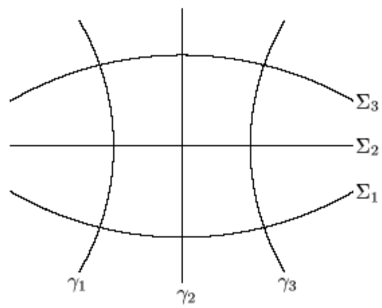

FIG: 4

(Family of hyper surfaces orthogonal to a congruence of time like geodesic)

The congruence will be hyper-surface orthogonal if U^{α} is everywhere proportional to the normal n^{α} to the hyper-surface. Let these are described by the equation of form $\Phi(x^{\alpha}) = c$, where c is constant specific to each hyper-surface. Then

$$n_{\alpha} \propto \Phi_{,\,\alpha} \quad and \;\; U_{\alpha} = -\mu \Phi_{,\alpha}$$

for some proportionality constant μ.(suppose that Φ is increases towards the future and the positive quantity μ can be determined from the normalization condition $U^{\alpha}U_{\alpha} = -1$).

Differentiating the above equation we obtain

$$U_{\alpha;\beta} = -\mu \, \Phi_{;\alpha\beta} - \Phi_{,\alpha} \, \mu_{,\,\beta}$$

Now consider the completely anti-symmetric tensor

$$U_{[\alpha;\beta}U_{\gamma]} = \frac{1}{3!}(U_{\alpha;\beta}U_{\gamma} + U_{\beta;\gamma}U_{\alpha} + U_{\gamma;\alpha}U_{\beta} - U_{\beta;\alpha}U_{\gamma} - U_{\gamma;\beta}U_{\alpha} - U_{\alpha;\gamma}U_{\beta})$$

$$or,\; U_{[\alpha;\beta}U_{\gamma]} = \frac{1}{3!}\{(-\mu\Phi_{;\alpha\beta} - \Phi_{,\alpha}\mu_{,\beta})U_{\gamma} + (-\mu\Phi_{;\beta\gamma} - \Phi_{,\beta}\mu_{,\beta})U_{\alpha} + (-\mu\Phi_{;\gamma\alpha} - \Phi_{,\gamma}\mu_{,\alpha})U_{\beta}$$
$$-(-\mu\Phi_{;\beta\alpha} - \Phi_{,\beta}\mu_{,\alpha})U_{\gamma} - (-\mu\Phi_{;\gamma\beta} - \Phi_{,\gamma}\mu_{,\beta})U_{\alpha} - (-\mu\Phi_{;\alpha\gamma} - \Phi_{,\alpha}\mu_{,\gamma})U_{\beta}\}$$
$$= 0$$

Using the fact that $(\Phi_{;\alpha\beta} = \Phi_{;\beta\alpha})$.

We therefore have hyper-surface orthogonal means

$$U_{[\alpha;\beta}U_{\gamma]} = 0. \qquad 3.11$$

The converse of this statement that $U_{[\alpha;\beta}U_{\gamma]}=0$ implies the existence of a scalar field Φ such that $U_\alpha \propto \Phi_{,\alpha}$ is also true. Equation (3.11) is very useful because whether or not U^α hyper-surface is orthogonal can be decided on the basis of the vector field alone without having to find Φ explicitly .Again we have never used geodesic equation in derivation of the equation (3.11) and also we did not use the fact that U^α was normalized .So equation (3.11) is quite general i.e.

"A congruence of curves is hyper-surface orthogonal if $U_{[\alpha;\beta}U_{\gamma]}=0$ where U^α is tangent to the curves."

This statement is known as the **FROBENIUS** theorem.

We see that μ must be constant on each hypersurface because it varies only in the direction orthogonal to the hyper-surfaces. Thus μ can be expressed as a function of Φ and defining a new function $\Psi=\int \mu(\Phi)\,d\Phi$, we find that U_α is not only proportional to a gradient, it is equal to one: $U_\alpha=-\Psi_{,\alpha}$ .It is remarkable that if U_α can be expressed in this form, then it automatically satisfy the geodesic equation:

$$U_{\alpha;\beta}U^\beta=\psi_{;\alpha\beta}\Psi^\beta=\Psi_{;\beta\alpha}\Psi^\beta=\frac{1}{2}(\Psi^{'\beta}\,\Psi_{,\beta})_{;\alpha}=\frac{1}{2}(U^{'\beta}\,U_{,\beta})_{;\alpha}=0$$

Thus we can summarize as, a vector field U^α (time like, space like or null not necessarily geodesic)is hyper-surface orthogonal if there exist a scalar field Φ such that $U_\alpha \propto \Phi_{,\alpha}$ which implies that $U_{[\alpha;\beta}U_{\gamma]}=0$ .If a vector field is time like and geodesic , then it is hyper-surface orthogonal if there exist a scalar field Ψ such that $U_\alpha=-\Psi_{,\alpha}$ which implies that $w_{\alpha\beta}=U_{[\alpha;\beta]}=0$.

3.9 INTERPRETATION OF EXPANSION SCALAR (θ) :

Here we will show that expansion scalar θ is equal to the rate of change of congruence's cross sectional volume δV i.e.

$$\theta=\frac{1}{\delta V}\frac{d}{d\tau}\delta V \qquad 3.12$$

Let us introduce the notion of cross sections and cross sectional volume. Select a particular geodesic γ from the congruence and on this geodesic pick a point P at which $\tau=\tau_P$ .Construct - in a small neighborhood around P , a small set $\delta\Sigma(\tau_P)$ of points P' such that :

(a) Through each of these points there passes another geodesic of the congruence.

(b) At each point P' , τ is also equal to τ_P .i.e. $\tau = \tau_P$.

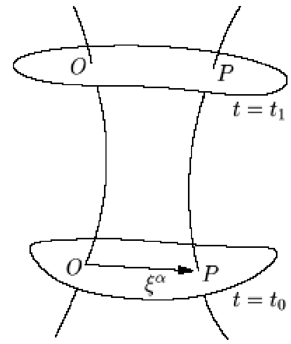

Fig: 5

(Congruence cross section about a reference geodesic)

This set forms a three dimensional region, a small segment of hyper-surface $\tau = \tau_P$ (as in figure 5). We assume that the parameterization has been adjusted so that γ intersect $\delta\sum(\tau_P)$ orthogonally. We will call $\delta\sum(\tau_P)$ the congruence's cross section around the geodesic γ at proper time $\tau = \tau_P$ .We want to calculate the volume of this hyper-surface segment and compare it with the volume of $\delta\sum(\tau_Q)$ where Q is the neighboring point on γ .

Let us introduce co-ordinates on $\delta\sum(\tau_P)$ by labeling y^α $(\alpha = 1,2,3)$ to each point P' in the set. We use y^α to label the geodesic since through each point of these there passes a geodesic from the congruence. By demanding that each geodesic keep its label as its moves away from $\delta\sum(\tau_P)$,we simultaneously obtain a co-ordinate system y^α in $\delta\sum(\tau_Q)$ or any other cross section .This construction therefore defines a co-ordinate system (τ, y^α) in a neighborhood of the geodesic γ and there exist a transformation between this system such that

$$x^a = x^a(\tau, y^\alpha)$$

Because y^α is constant along the geodesic, we have

$$U^a = \left(\frac{\partial x^a}{\partial \tau}\right)_{y^\alpha} \qquad 3.13$$

On the other hand we have the vectors

$$e^a_\alpha = \left(\frac{\partial x^a}{\partial y^\alpha}\right)_\tau \qquad 3.14$$

are tangent to the cross section .These relation implies that the lie derivative of e^a_α along U vanishes i.e. $L_U\, e^a_\alpha = 0$

Also we have $U_a\, e^a_\alpha = 0$ holding on γ .

Let introduce a three tensor h_{ab} defined by

$$h_{ab} = g_{\alpha\beta}\, e_a^\alpha\, e_b^\beta \qquad 3.15$$

A three tensor is a tensor with respect to co-ordinate transformations $y^\alpha \to y'^\alpha$ but a scalar with respect to transformations $x^a \to x'^a$. This acts as a metric tensor on $\delta\Sigma(\tau)$.For displacements contained within the set (so that $d\tau = 0$); $x^\alpha = x^\alpha(y^a)$ and

$$\begin{aligned} ds^2 &= g_{\alpha\beta}\, dx^\alpha\, dx^\beta \\ &= g_{\alpha\beta}\, \left(\frac{\partial x^\alpha}{\partial y^a} dy^a\right) \left(\frac{\partial x^\beta}{\partial y^b} dy^b\right) \\ &= g_{\alpha\beta}\, e_a^\alpha\, e_b^\beta\, dy^a\, dy^b \\ &= h_{ab}\, dy^a dy^b \end{aligned}$$

Thus h_{ab} is the three dimensional metric on the congruence's cross sections. Because γ is orthogonal to its cross section ($U_\alpha e_a^\alpha = 0$), we have that $h_{ab} = h_{\alpha\beta}\, e_a^\alpha\, e_b^\beta$ on γ where $h_{\alpha\beta} = g_{\alpha\beta} - U_\alpha U_\beta$ is the transverse metric. If we define h^{ab} to be the inverse of h_{ab}, then it is expressed as $h^{\alpha\beta} = h^{ab}\, e_a^\alpha\, e_b^\beta$ on γ.

The three dimensional volume element on the cross section or cross sectional volume is given by $\delta V = \sqrt{h}\, d^3 y$ where $h = \det[h_{ab}]$.Because the co-ordinates y^α are co-moving (as each geodesic moves with a constant value of its co-ordinates), $d^3 y$ does not change as the cross section $\delta\Sigma(\tau)$ evolves from $\tau = \tau_P$ to $\tau = \tau_Q$. A change in δV therefore comes entirely from a change on $\sqrt{h}$:

$$\frac{1}{\delta V}\frac{d}{d\tau}\delta V = \frac{1}{\sqrt{h}}\frac{d}{d\tau}\sqrt{h} = \frac{1}{2}\, h^{ab}\, \frac{d}{d\tau} h_{ab} \qquad 3.16$$

Let us calculate the rate of change of three metric:

$$\begin{aligned} \frac{d}{d\tau} h_{ab} &= (g_{\alpha\beta}\, e_a^\alpha e_b^\beta)_{;\mu} U^\mu \\ &= g_{\alpha\beta} (e^\alpha_{a;\mu} U^\mu) e_b^\beta + g_{\alpha\beta}\, e_a^\alpha\, (e^\beta_{b;\mu} U^\mu) \\ &= g_{\alpha\beta}\, (U^\alpha_{;\mu}\, e_a^\mu)\, e_b^\beta + g_{\alpha\beta}\, e_a^\alpha\, (U^\beta_{;\mu} e_b^\mu) \\ &= U_{\beta;\alpha}\, e_a^\alpha\, e_b^\beta + U_{\alpha;\beta}\, e_a^\alpha\, e_b^\beta \\ &= (U_{\beta;\alpha} + U_{\alpha;\beta})\, e_a^\alpha\, e_b^\beta \end{aligned}$$

$$or,\ \frac{d}{d\tau} h_{ab} = (B_{\alpha\beta} + B_{\beta\alpha}) e_a^\alpha\, e_b^\beta$$

Multiplying the above equation by h^{ab} and evaluating on γ , we obtain

$$h^{ab}\, \frac{d}{d\tau} h_{ab} = (B_{\alpha\beta} + B_{\beta\alpha})\, (h^{ab}\, e_a^\alpha\, e_b^\beta)$$

$$= 2B_{\alpha\beta}h^{\alpha\beta}$$
$$= 2B_{\alpha\beta}g^{\alpha\beta}$$
$$= 2\theta$$

$$\text{i.e. } \theta = \frac{1}{2}h^{ab}\frac{d}{d\tau}h_{ab} \qquad 3.17$$

From (3.17) we get, $\theta = \frac{1}{2}h^{ab}\frac{d}{d\tau}h_{ab}$ which is as same as equation (3.12) .

Chapter four

HYPERSURFACE

4.0 INTRODUCTION:

Mainly three topics related to hypersurface are discussed in this chapter. With sufficient pre-requisite ideas the first part is concerned the intrinsic geometry of a hypersurface in which we studied an induced three dimensional metric h_{ab} on a particular hypersurface after the embedding of the space time with metric $g_{\alpha\beta}$.

The second part is concerned with the extrinsic geometry of a hypersurface or how the is embedded in the enveloping space time manifold .We studied how the space time curvature tensor can be decomposed into a purely intrinsic part (the curvature tensor of the hypersurface) and an extrinsic part that measures the bending of the hyper surface in space time (this bending is described by a three dimensional tensor K_{ab} known as extrinsic curvature). We also found the Einstein tensor in terms of induced metric and extrinsic curvature.

The third part is concerned with possible discontinuities of the metric and its derivative on a hypersurface. In this topic we studied how the hypersurface partitions space time into two regions and distinct metric tensor in each region.

This chapter is mainly quoted from the book [5].Also the following books are used as references [2],[19][18].

HYPERSURFACE

To discuss the concept of hypersurface first we should define scalar field.

SCALAR FIELD: A smooth scalar field on a manifold M is just a smooth real valued map $\Phi : M \to E_1$ .In other words it is a smooth function of the co ordinates of M as a subset of E_r. Thus Φ associates to each point m of M a unique scalar Φ (m). If U is a subset of M, then a smooth scalar field on U is a real valued map $\Phi : U \to E_1$. If U$\neq$M, then such a scalar field is called local .[2]

If Φ is a scalar field on M and X is a chart then we express Φ as a smooth function φ of the associated parameters $X^1, X^2 ----X^{n.}$. If the chart is $\overline{X}$ we will write $\overline{\varphi}$ for the function of the parameters $\bar{X}^{1.}\overline{X}^2 --- \overline{X}^n$. But we must have $\varphi = \overline{\varphi}$ at each point of the manifold.

4.1 HYPERSURFACES:

In a four-dimensional space-time manifold, a hyper surface is three-dimensional sub manifolds that can be either time like or space like or null like. A particular hypersurface Σ is selected either by putting a restriction on the co ordinates

$$\Phi(x^\alpha) = 0$$

or by giving a parametric equation of the form

$$x^\alpha = x^\alpha(y^a)$$

where $y^a (a = 1,2,3)$ are the intrinsic co ordinates to the hypersurface.

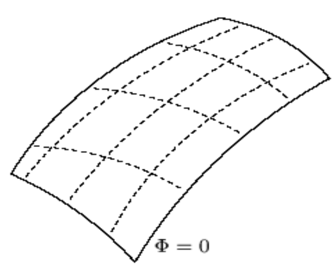

FIG: 1

For example, a two sphere in a three dimensional flat space is described either by

$$\Phi(x.y, z) = x^2 + y^2 + z^2 - R^2 = 0$$

where R is a radius of sphere or given by

$x = RSin\theta Cos\Phi$; $y = RSin\theta Sin\Phi$; $z = RCos\theta$

where θ and Φ are intrinsic co ordinates .In figure (1) the relation $x^{\alpha}(y^{a})$ describes the curves contained in $\sum$.

4.2 NORMAL VECTOR:

The vector $\Phi_{,\alpha}$ is normal to the hypersurface because the value of Φ changes only in the direction orthogonal to $\sum$. A unit normal n_{α} can be introduced if the hypersurface is not null. This is defined by

$$n^{\alpha} n_{\alpha} = \in \equiv -1 \text{ ; if } \sum \text{ is space like.}$$

$$n^{\alpha} n_{\alpha} = \in \equiv 1 \text{ ; if the } \sum \text{ is time like.}$$

and it is demanded that n^{α} points in the direction of increasing

$$\Phi : n^{\alpha} \Phi_{,\alpha} \rangle 0$$

Also n_{α} is given by, $n_{\alpha} = \dfrac{\in \Phi_{,\alpha}}{\sqrt{\left| g^{\mu\gamma} \Phi_{,\mu} \Phi_{,\gamma} \right|}}$

if the hypersurface is either space like or time like.

If the hypersurface $\sum$ is null then $g^{\mu\gamma} \phi_{,\mu} \phi_{,\gamma}$ is zero .So in that case the unit normal is not defined and so in that case we let

$$K_{\alpha} = -\phi_{,\alpha}$$

be the normal vector. The sign is so chosen that k^{α} is future directed when Φ increases towards the future. Because k^{α} is orthogonal to itself $(k^{\alpha} k_{\alpha} = 0)$ this vector is also tangent to the null hypersurface $\sum$ as in figure (2).By computing $k^{\alpha}_{;\beta} k^{\beta}$ and showing that it is proportional to k^{α},we can prove that k^{α} is tangent to null geodesics contained in $\sum$.

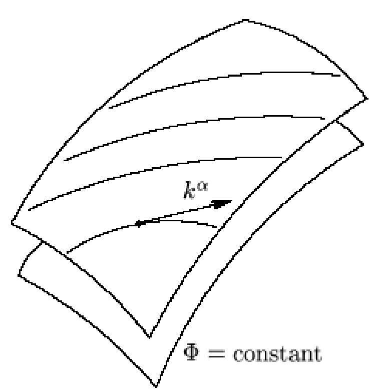

FIG: 2

We have

$$k_{\alpha;\beta} = \phi_{;\alpha\beta}\phi^{,\beta} = \phi_{;\beta\alpha}\phi^{,\beta} = \tfrac{1}{2}(\phi_{,\beta}\phi^{,\beta})_{;\alpha}$$

because $\phi_{,\beta}\phi^{\beta}$ is zero everywhere on $\sum$, its gradient must be directed along k_α and we have $(\phi_{,\beta}\phi^{,\beta})_{;\alpha} = 2lk_\alpha$ for some scalar l. We have found that the normal vector satisfies

$$k^{\alpha}_{;\beta}k^{\beta} = lk^{\alpha}$$

the general form of geodesics equation. The hypersurface is therefore generated by null geodesics and $k^{\alpha} = \frac{dx^{\alpha}}{d\lambda}$ is tangent to the generators. In general the parameters λ is not affine but in special situation in which the relation $\phi(x^{a})$ =constant describe a whole family of null hypersurface (so that $\phi_{,\beta}\phi^{,\beta}$ is not zero not only on $\sum$ but also in neighborhood around $\sum$.) l =0 and λ is an affine parameter. When the hypersurface is null it is advantageous to install on a coordinate system that is well adapted to the behaviors of the generators. We therefore let the parameter λ be one of the coordinator and we introduce two additional coordinators. $\theta^{A}(A = 2,3)$ to label the generators , these are constant on each generators , thus will shall adopt

$$y^{a} = (\lambda, \theta^{A}) \qquad 4.1$$

when $\sum$ is null ;varying λ while keeping θ^{A} constant produces a displacement along a single generator and changing θ^{A} produces a displacement across the generators.

4.3 INDUCED METRIC ON HYPERSURFACE:

The metric intrinsic to hypersurface $\sum$ is obtained by restricting the line elements to displacement confined to the hypersurface. Recalling the parametric equations $x^{\alpha} = x^{\alpha}(y^{a})$ .We have that the vectors

$$e^{\alpha}_{a} = \frac{\partial x^{\alpha}}{\partial y^{a}} \qquad 4.2$$

are tangent to curves contained in $\sum$.Thus implies that $e^{\alpha}_{a}n_{\alpha} = 0$ in the null case and $e^{\alpha}_{a}k_{\alpha} = 0$ in the null case. Now for displacement within $\sum$, we get

$$ds_{\Sigma}^{\;2} = g_{\alpha\beta}dx^{\alpha}dx^{\beta}$$

$$= g_{\alpha\beta}(\tfrac{\partial x^{\alpha}}{\partial y^{a}}dy^{a})(\tfrac{\partial x^{\beta}}{\partial y^{b}}dy^{b})$$

$$= h_{ab}dy^{a}dy^{b} \qquad 4.3$$

where $h_{ab} = g_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b}$ 4.4

is called the induced metric or first fundamental form of the hypersurface. It is scalar with respect to transformation $x^{\alpha} \to x'^{\alpha}$ of the space time co ordinates but it transforms as a tensor under transformation $y^{a} \to y'^{a}$ of the hypersurface co ordinates. Such objects are known as **three tensors**. These relation simplify when the hypersurface is null and we use the co ordinates of equation $y^{a} = (\lambda, \theta^{A})$; A=2,3.

Then $e^{\alpha}_{1} = (\frac{\partial x^{\alpha}}{\partial \lambda})_{\theta^{A}} \equiv k^{\alpha}$ and it follows that

$$h_{11} = g_{\alpha\beta} k^{\alpha} k^{\beta} = 0$$

and $$h_{1A} = g_{\alpha\beta} k^{\alpha} e^{\beta}_{A} = 0$$

Because by construction $e^{\alpha}_{A} \equiv (\frac{\partial x^{\alpha}}{\partial \theta^{A}})_{\lambda}$ is orthogonal to k^{α} .In the null case therefore

$$ds^{2} = \sigma_{AB}\, d\theta^{A}\, d\theta^{B}$$

where $\sigma_{AB} = g_{\alpha\beta}\, e^{\alpha}_{A}\, e^{\beta}_{B}$; $e^{\alpha}_{A} = (\frac{\partial x^{\alpha}}{\partial \theta^{A}})_{\lambda}$ .Here the induced metric is a two tensor .We conclude by writing down the completeness relation for the inverse metric. In the non null case

$$g^{\alpha\beta} = \in n^{\alpha} n^{\beta} + h^{ab}\, e^{\alpha}_{a}\, e^{\beta}_{b} \qquad 4.5$$

where h^{ab} is the inverse of the induced metric .Equation (4.5) is verified by computing all inner products between n^{α} *and* e^{α}_{a} .In the non null case we must introduce everywhere on $\sum$, an auxiliary null vector field N^{α} satisfying $N_{\alpha}\, k^{\alpha} = -1$ and $N_{\alpha}\, e^{\alpha}_{a} = 0$

where k^{α} is the tangent vector field, defined as $k^{\alpha} = \frac{dx^{\alpha}}{d\lambda}$.

The inverse metric can be expressed as

$$g^{\alpha\beta} = -k^{\alpha}\, N^{\beta} - N^{\alpha}\, k^{\beta} + \sigma^{AB}\, e^{\alpha}_{A}\, e^{\beta}_{B} \qquad 4.6$$

where σ^{AB} is the inverse of σ_{AB} .Equation (4.6) is verified by computing all inner product between k^{α} ,N^{α} *and* e^{α}_{A} .

4.4 LIGHT CONE IN FLAT SPACE TIME:

An example of a null hypersurface in flat space time is the future light cone of an event P, which we place at the origin of a Cartesian co-ordinate systems x^{α} . The defining relation for this hypersurface is $\Phi \equiv t - r = 0$ where $r^{2} = x^{2} + y^{2} + z^{2}$ .The normal vector is

$$k_\alpha = -\partial_\alpha (t-r) = (-1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}).$$

A suitable set of parametric equations $t = \lambda,\ x = \lambda\, Sin\theta\, Cos\varphi,\ y = \lambda\, Sin\theta\, Cos\varphi,\ z = \lambda Cos\theta$ in which $y^a = (\lambda, \theta, \varphi)$ are the intrinsic co-ordinates; λ is an affine parameter on the light cones null generators which moves constant values of $\theta^A = (\theta, \varphi)$ .From the parametric equation we compute the hypersurfaces tangent vectors [5][19]

$$e^\alpha_\lambda = \frac{\partial x^\alpha}{\partial \lambda} = (1, Sin\theta Cos\varphi, Sin\theta Sin\varphi, Cos\theta) = k^\alpha$$

$$e^\alpha_\theta = \frac{\partial x^\alpha}{\partial \theta} = (1, Cos\theta\, Cos\varphi, Cos\theta Sin\varphi, -\lambda Sin\theta)$$

$$e^\alpha_\varphi = \frac{\partial x^\alpha}{\partial \varphi} = (1, -\lambda Sin\theta Sin\varphi, \lambda Sin\theta Cos\varphi, 0)$$

We may check that this vectors are orthogonal to k^α . Inner product between e^α_θ *and* e^α_φ define the two metric σ_{AB} and we find

$$\sigma_{AB} d\theta^A\, d\theta^B = \lambda^2 (d\theta^2 + Sin^2\theta\, d\varphi^2)$$

Not surprisingly, the hypersurface has a spherical geometry and λ is a real radius of the two sphere.

4.5 DIFFERENTIATION OF TANGENT VECTOR FIELDS:

4.5(a) TANGENT TENSOR FIELD:

In this section we consider that the hypersurface Σ is either space like or time like .With a hypersurface Σ it is common situation to have tensor field $A^{\alpha\beta\ldots\ldots\ldots\ldots}$ that are defined only on Σ and which are purely tangent to the hypersurface .Such tensor admits the following decomposition:

$$A^{\alpha\beta} = A^{ab\ldots\ldots\ldots\ldots} e^\alpha_a e^\beta_b \ldots\ldots \qquad 4.7$$

where $e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$ are basis vectors on Σ .Equation (4.7) implies that

$$A^{\alpha\beta\ldots\ldots} n_\alpha = A^{\alpha\beta\ldots\ldots} n_\beta = 0$$

which confirms that $A^{\alpha\beta\cdots\cdots}$ is tangent to the hypersurface .We note that an arbitrary tensor $A^{\alpha\beta\cdots\cdots}$ can always be projected down to the hyper surface ,so that only its tangential components survive .The quantity that effects the projection is

$$h^{\alpha\beta} \equiv h^{ab} e^{\alpha}_{a} e^{\beta}_{b} = g^{\alpha\beta} - \in n^{\alpha} n^{\beta}$$

and $h^{\alpha}_{\mu} h^{\beta}_{\gamma} ......T^{\mu\gamma\cdots\cdots}$ is evidently tangent to the hypersurface. The projections

$$A_{\alpha\beta} ......e^{\alpha}_{a} e^{\beta}_{b} = A_{ab..........} \equiv h_{am} h_{bn} .....A^{mn.......} \qquad 4.8$$

give the three tensors $A^{ab..........}$ associated with the tangent tensor field $A^{\alpha\beta\cdots\cdots}$; Latin indices are lowered and raised with h_{ab} and h^{ab} respectively. Equation (4.7) and (4.8) shows that one can easily go back and forth between a tangent tensor field $A^{\alpha\beta\cdots\cdots}$ and its equivalent three tensor $A^{ab..........}$.It is emphasized that while $A^{ab..........}$ transform as a tensor under transformation $y^{a} \to y'^{a}$ of the co-ordinates intrinsic to $\sum$, it is scalar under transformation $x^{\alpha} \to x'^{\alpha}$ of the space time co-ordinates.

4.5 (b) INTRINSIC COVARIANT DERIVATIVES:

We will consider how tangent tensor field are differentiated .We want to relate the covariant derivatives of $A^{\alpha\beta\cdots\cdots}$(with respect to connection that is compatible with the space time metric $g_{\alpha\beta}$) to the co variant derivative of $A^{ab..........}$(defined in terms of connection that is compatible with the induced metric h_{ab} .

For simplicity we shall restrict our case to the case of tangent vector field A^{α} such that

$$A^{\alpha} = A^{a} e^{\alpha}_{a} ; \quad A^{\alpha} n_{\alpha} = 0 ; \quad A_{a} = A_{\alpha} e^{\alpha}_{a}$$

We define the intrinsic covariant derivative of a three vector A_{a} to be the projection of $A_{\alpha;\beta}$ onto the hypersurface:

$$A_{a|b} \equiv A_{\alpha;\beta} e^{\alpha}_{a} e^{\beta}_{b} \qquad 4.9$$

We will show that, $A_{a|b}$ as defined here is nothing but the covariant derivative of A_{a}

defined in the usual way in terms of connection Γ^{a}_{bc} that compatible with h_{ab}. Let us express R .H.S of equation (4.9) as

$$A_{\alpha;\beta}e^{\alpha}_{a}e^{\beta}_{b} = (A_{\alpha}e^{\alpha}_{a})_{;\beta}e^{\beta}_{b} - A_{\alpha}e^{\alpha}_{a\,;\beta}e^{\beta}_{b}$$

$$= A_{a,\beta}e^{\beta}_{b} - e_{a\gamma;\beta}\, e^{\beta}_{b} A^{c} e^{\gamma}_{c}$$

$$= \frac{\partial A_{a}}{\partial x^{\beta}}\frac{\partial x^{\beta}}{\partial y^{b}} - e^{\gamma}_{c}\, e_{a\gamma;\beta}\, e^{\beta}_{b} A^{c}$$

$$= A_{a,b} - \Gamma_{cab}A^{c} \qquad 4.10$$

where we have defined

$$\Gamma_{cab} = e^{\gamma}_{c}e_{a\gamma;\beta}e^{\beta}_{b} \qquad 4.11$$

Equation (4.10) becomes then

$$A_{a|b} = A_{a,b} - \Gamma^{c}_{ab}A_{c} \qquad 4.12$$

which is familiar expression for the covariant derivative. The connection used here is the one defined by (4.11) and it is compatible with induced metric. In other words Γ_{cab} as defined by (4.10) can also expressed as

$$\Gamma_{cab} = \tfrac{1}{2}(h_{ca,b} + h_{bc.a} - h_{ab,c}) \qquad 4.13$$

This could be easily done by directly working out the R .H.S. of (4.11).It is easier, however to show that the connection is such that:

$$h_{ab|c} \equiv h_{\alpha\beta;\gamma}e^{\alpha}_{a}e^{\beta}_{b}e^{\gamma}_{c} = 0$$

indeed

$$h_{\alpha\beta;\gamma}e^{\alpha}_{a}e^{\beta}_{b}e^{\gamma}_{c} = (g_{\alpha\beta} - \in n_{\alpha}n_{\beta})_{;\gamma}\, e^{\alpha}_{a}e^{\phi}_{b}e^{\gamma}_{c}$$

$$= -\in (n_{\alpha;\gamma}n_{\beta} + n_{\alpha}n_{\beta;\gamma})e^{\alpha}_{a}e^{\beta}_{b}e^{\gamma}_{c}$$

$$= 0$$

because $n_{\alpha}\, e^{\alpha}_{a} = 0$.

4.5(c) EXTRINSIC CURVATURE:

The quantities $A_{a|b} \equiv A_{\alpha;\beta}\, e_a^\alpha\, e_b^\beta$ are the tangential components of the vector $A^\alpha_{;\beta}\, e_b^\beta$ .We would like to investigate now whether this vector possess a normal component.

Let us express $A^\alpha{}_{;\beta} e_b^\beta$ and $g^\alpha_\mu A^\mu{}_{;\beta} e_b^\beta$ and decompose the metric into its normal and tangential part as below:

$$g^{\alpha\beta} = \in n^\alpha n^\beta + h^{ab} e_a^\alpha e_b^\beta$$

This gives

$$A^\alpha_{;\beta} e_b^\beta = (\in n^\alpha n_\mu + h^{am} e_a^\alpha e_{m\mu}) A^\mu_{;\beta}\, e_b^\beta$$

$$= \in (n_\mu A^\alpha_{;\beta}\, e_b^\beta) n^\alpha + h^{am} (A_{\mu;\beta} e_m^\mu\, e_b^\beta)\, e_a^\alpha$$

where we see that the second term is tangent to the hypersurface .Using the fact that $A_{a|b} = A_{\alpha;\beta} e_a^\alpha e_b^\beta$ and A^μ is orthogonal to n^μ, the above equation becomes

$$A^\alpha_{;\beta} e_b^\beta = - \in (n_{\mu;\beta} A^\mu e_b^\beta) n^\alpha + h^{am} A_{a|b}\, e_a^\alpha$$

$$= A^a{}_{|b} e_a^\alpha - \in A^a (n_{\mu;\beta} e_a^\mu e_b^\beta) n^\alpha$$

At this point we introduce the three vector

$$K_{ab} \equiv n_{\alpha;\beta} e_a^\alpha e_b^\beta \qquad 4.14$$

called the **extrinsic curvature** or second fundamental form of the hypersurface $\sum$. In terms of this we have

$$A^\alpha_{;\beta} = A^a{}_{|b} e_a^\alpha - \in A^a K_{ab} n^\alpha \qquad 4.15$$

and we see that $A^a{}_{|b}$ gives the purely tangential part of the vector field while $- \in A^a K_{ab}$ represent the normal components. This answers our requirements: the normal components vanish if only if the extrinsic curvature vanishes.

We note that if e_a^α is substituted in place of A^α then $A^c = \delta_a^c$ and $A_{a|b} = A_{a,b} - \Gamma^c_{ab} A_c$ and equation (4.14) imply

$$e^\alpha_{a\;;\beta} e_b^\beta = \Gamma^c_{ab} e_c^\alpha - \in K_{ab} n^\alpha$$

which is known as the **Gauss –Weingarten** equation.[5][19]

The extrinsic curvature is very important quantity and it is a symmetric tensor i.e. $K_{ab} = K_{ba}$ .The symmetry of the extrinsic curvature implies the relation

$$K_{ab} = n_{(\alpha;\beta)} e_a^{\alpha} e_b^{\beta} = \frac{1}{2}(L_n g_{\alpha\beta}) e_a^{\alpha} e_b^{\beta}$$

and the extrinsic curvature is therefore intimately related to the normal derivative of the metric tensor. We also note the relation

$$K \equiv h^{ab} K_{ab} = n^{\alpha}_{;\alpha}$$

which shows that K is equal the expansion of a congruence of geodesic that intersect the hyper surface orthogonally. (So that their tangent vector is equal to n^{α} on the hypersurface). From this result we conclude that the hypersurface is convex if $K \rangle 0$ (congruence is diverging) or concave if $K \langle 0$ (congruence is converging).

Thus we see that while h_{ab} is concerned with the purely intrinsic aspects of hypersurface geometry, K_{ab} is concerned with the extrinsic aspects – the embedding of the hypersurface in the enveloping space-time manifold. Taking together these tensors provide a virtually complete characterization of the hypersurface.

4.6 GAUSS CODAZZI EQUATION:

4.6(a) GENERAL FORM: We have introduced the induced metric h_{ab} and its associated intrinsic covariant derivative. A purely intrinsic curvature tensor can be defined by the relation;

$$A^{c}{}_{|ab} - A^{c}{}_{|ba} = -R^{c}_{d\,ab} \qquad 4.16$$

which of course implies

$$R^{c}_{d\,ab} = \Gamma^{c}_{db,a} - \Gamma^{c}_{da,b} + \Gamma^{c}_{am} \Gamma^{m}_{db} - \Gamma^{c}_{mb} \Gamma^{m}_{da} \qquad 4.17$$

We show that, whether this three dimensional Riemannian tensor can be expressed in terms of $R^{\gamma}_{\delta\alpha\beta}$ -the four dimensional version, evaluated on $\sum$.

We get

$$e^{\alpha}_{a;\beta} e^{\beta}_{b} = \Gamma^{d}_{ab} e^{\alpha}_{d} - \in K_{ab} n^{\alpha} .$$

Then we can write

$$(e^{\alpha}_{a;\beta} e^{\beta}_{b})_{;\gamma} e^{\gamma}_{c} = (\Gamma^{d}_{ab} e^{\alpha}_{d} - \in K_{ab} n^{\alpha} .)_{;\gamma} e^{\gamma}_{c} \qquad 4.18$$

Let us first develop the L.H.S of (4.17)

$$\text{L.H.S}= \quad (e^{\alpha}_{a;\beta}\, e^{\beta}_{b})_{;\beta}\, e^{\gamma}_{c}$$

$$= e^{\alpha}_{a;\beta\gamma}\, e^{\beta}_{b}\, e^{\gamma}_{c} + e^{\alpha}_{a;\beta}\, e^{\beta}_{b;\gamma}\, e^{\gamma}_{c}$$

$$= e^{\alpha}_{a;\beta\gamma}\, e^{\beta}_{b}\, e^{\gamma}_{c} + e^{\alpha}_{a;\beta}\, (\Gamma^{d}_{bc} e^{\beta}_{d} - \in K_{bc} n^{\beta})$$

$$= e^{\alpha}_{a;\beta\gamma}\, e^{\beta}_{b}\, e^{\gamma}_{c} + \Gamma^{d}_{bc} (\Gamma^{p}_{ad} e^{\alpha}_{p} - \in K_{ad} n^{\alpha}) - \in K_{bc} e^{\alpha}_{a;\beta} n^{\beta}$$

Let us develop the R.H.S of (4.17)

$$\text{R.H.S} = (\Gamma^{d}_{ab}\, e^{\alpha}_{d} - \in K_{ab} n^{\alpha}.)_{;\gamma}\, e^{\gamma}_{c}$$

$$= \Gamma^{d}_{ab,c} e^{\alpha}_{d} + \Gamma^{d}_{ab}\, e^{\alpha}_{d;\gamma} e^{\gamma}_{c} - \in K_{ab,c} n^{\alpha} - \in K_{ab} n^{\alpha}_{;\gamma}\, e^{\gamma}_{c}$$

$$= \Gamma^{d}_{ab,c} e^{\alpha}_{d} + \Gamma^{d}_{ab} (\Gamma^{p}_{dc}\, e^{\alpha}_{p} - \in K_{dc} n^{\alpha}) - \in K_{ab,c} n^{\alpha} - \in K_{ab} n^{\alpha}_{;\gamma}\, e^{\gamma}_{c}$$

From equation (4.17) we get,

$$e^{\alpha}_{a;\beta\gamma}\, e^{\beta}_{b}\, e^{\gamma}_{c} + \Gamma^{d}_{bc} (\Gamma^{p}_{ad} e^{\alpha}_{p} - \in K_{ad} n^{\alpha}) - \in K_{bc} e^{\alpha}_{a;\beta} n^{\beta} =$$
$$\Gamma^{d}_{ab,c} e^{\alpha}_{d} + \Gamma^{d}_{ab} (\Gamma^{p}_{dc}\, e^{\alpha}_{p} - \in K_{dc} n^{\alpha}) - \in K_{ab,c} n^{\alpha} - \in K_{ab} n^{\alpha}_{;\gamma}\, e^{\gamma}_{c}$$

$$or, e^{\alpha}_{a;\beta\gamma} e^{\beta}_{b}\, e^{\gamma}_{c} = \Gamma^{d}_{ab,c}\, e^{\alpha}_{d} + \Gamma^{d}_{ab}\, \Gamma^{m}_{dc}\, e^{\alpha}_{m} - \in \Gamma^{d}_{ab} K_{dc} n^{\alpha} - \in K_{ab,c} n^{\alpha} - \in K_{ab} n^{\alpha}_{;\gamma} e^{\gamma}_{c}$$
$$- \Gamma^{d}_{bc}\, \Gamma^{m}_{ad}\, e^{\alpha}_{m} + \in \Gamma^{d}_{bc} K_{ad} n^{\alpha} + \in K_{bc}\, e^{\alpha}_{a;\beta} n^{\beta}$$

$$or, e^{\alpha}_{a;\beta\gamma} e^{\beta}_{b}\, e^{\gamma}_{c} = \Gamma^{d}_{ab,c}\, e^{\alpha}_{d} + \Gamma^{d}_{ab}\, \Gamma^{m}_{dc}\, e^{\alpha}_{m} - \Gamma^{d}_{bc}\, \Gamma^{m}_{ad}\, e^{\alpha}_{m} + \in \Gamma^{d}_{bc} K_{ad} n^{\alpha} - \in \Gamma^{d}_{ab} K_{dc} n^{\alpha}$$
$$- \in K_{ab,c} n^{\alpha} - \in K_{ab} n^{\alpha}_{;\gamma} e^{\gamma}_{c} + \in K_{bc}\, e^{\alpha}_{a;\beta} n^{\beta} \qquad 4.19$$

Similarly we get , $e^{\alpha}_{a;\gamma\beta}\, e^{\gamma}_{c} e^{\beta}_{b} (\gamma \to \beta, \beta \to \gamma, b \to c, c \to b)$

$$e^{\alpha}_{a;\gamma\beta}\, e^{\gamma}_{c}\, e^{\beta}_{b} = \Gamma^{d}_{ac,b} e^{\alpha}_{d} + \Gamma^{\ d}_{ac}\, \Gamma^{m}_{db}\, e^{\alpha}_{m} - \Gamma^{d}_{cb}\ \Gamma^{m}_{ad}\, e^{\alpha}_{m} + \in \Gamma^{d}_{cb} K_{ad} n^{\alpha} - \in \Gamma^{d}_{ac} K_{db} n^{\alpha} -$$
$$\in K_{ac,b} n^{\alpha}\ - \in K_{ac}\, n^{\alpha}_{;\beta} e^{\beta}_{b} + \in K_{cb}\, e^{\alpha}_{a;\gamma} n^{\gamma} \qquad 4.20$$

Now subtracting (4.19) from (4.18) we get

$$(e^{\alpha}_{a;\beta\gamma} - e^{\alpha}_{a;\gamma\beta}) e^{\gamma}_{c}\, e^{\beta}_{b} = (\Gamma^{m}_{ab,c} + \Gamma^{m}_{dc} \Gamma^{d}_{ab} - \Gamma^{m}_{ac,b} - \Gamma^{m}_{db} \Gamma^{d}_{ac}) e^{\alpha}_{m} - \in K_{ab} n^{\alpha}_{;\gamma} e^{\gamma}_{c} + \in K_{ac} n^{\alpha}_{;\beta} e^{\beta}_{b}$$
$$- \in \Gamma^{d}_{ab} K_{dc}\, n^{\alpha} - \in K_{ab,c} n^{\alpha} + \in \Gamma^{d}_{ac} K_{db} n^{\alpha} + \in K_{ac,b} n^{\alpha}$$

$$or, -R^{\mu}_{\alpha\beta\gamma} e^{\alpha}_{a} e^{\beta}_{b} e^{\gamma}_{c} = R^{m}_{acb} e^{\mu}_{m} + \in (K_{ac,b} - \Gamma^{d}_{ab} K_{dc}) n^{\mu} - \in (K_{ab,c} - \Gamma^{d}_{ac} K_{db}) n^{\mu}$$
$$\in K_{ac} n^{\mu}_{;\beta} e^{\beta}_{b} - \in K_{ab} n^{\mu}_{;\gamma} e^{\gamma}_{c}$$
$$= R^{m}_{acb} e^{\mu}_{m} + \in (K_{ac|b} - K_{ab|c}) n^{\mu} + \in K_{ac} n^{\mu}_{;\beta} e^{\mu}_{b} - \in K_{ab} n^{\mu}_{;\gamma} e^{\gamma}_{c}$$

$$or, R^{\mu}_{\alpha\beta\gamma} e^{\alpha}_{a} e^{\beta}_{b} e^{\gamma}_{c} = R^{m}_{abc} e^{\mu}_{m} + \in (K_{ab|c} - K_{ac|b}) n^{\mu} + \in K_{ab} n^{\mu}_{;\gamma} e^{\mu}_{c} - \in K_{ac} n^{\mu}_{;\beta} e^{\beta}_{b}$$

Projecting along $e_{d\mu}$ we get,

$$R_{\alpha\beta\gamma\delta} e^{\alpha}_{a} e^{\beta}_{b} e^{\gamma}_{c} e^{\delta}_{d} = R_{abcd} + \in (K_{ad} K_{bc} - K_{ac} K_{bd}) \qquad 4.21$$

and this the desired relation between R_{abcd} and the full Riemannian tensor. Projecting instead along n_{μ} gives

$$R_{\mu\alpha\beta\gamma} n^{\mu} e^{\alpha}_{a} e^{\beta}_{b} e^{\gamma}_{c} = K_{ab|c} - K_{ac|b} \qquad 4.22$$

Equation (4.20) and (4.21) are known as **Gauss –Codazzi** equation. They reveal that space time curvature can be expressed in terms of the extrinsic and intrinsic curvature of a hypersurface.

4.6(b) CONTRACTED FORM OF GAUSS CODAZZI EQUATION:

We obtain that the general form of Gauss –Codazzi equation can be expressed as:

$$R_{\alpha\beta\gamma\delta} e^{\alpha}_{a} e^{\beta}_{b} e^{\gamma}_{c} e^{\delta}_{d} = R_{abcd} + \in (K_{ad} K_{bc} - K_{ac} K_{bd}) \qquad 4.23$$

and $$R_{\mu\alpha\beta\gamma} n^{\mu} e^{\alpha}_{a} e^{\beta}_{b} e^{\gamma}_{c} = K_{ab|c} - K_{ac|b} \qquad 4.24$$

The Gauss-Codazzi equation can be expressed in contracted form, in terms of the Einstein tensor

$$G_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R \; .$$

The space time Ricci tensor is given by

$$R_{\alpha\beta} = g^{\mu\gamma} R_{\mu\alpha\gamma\beta}$$
$$= (\in n^{\mu} n^{\gamma} + h^{mn} e^{\mu}_{m} e^{\gamma}_{n}) R_{\mu\alpha\beta\gamma}$$
$$= \in R_{\mu\alpha\beta\gamma} n^{\mu} n^{\gamma} + h^{mn} R_{\mu\alpha\beta\gamma} e^{\mu}_{m} e^{\gamma}_{n}$$

where h^{mn} is the inverse metric of $h_{mn} = g_{\alpha\beta}\, e^{\alpha}_{m}\, e^{\beta}_{n}$ i.e. called induced metric .Also the Ricci scalar is given by

$$\begin{aligned} R &= g^{\alpha\beta} R_{\alpha\beta} \\ &= (\in n^{\alpha} n^{\beta} + h^{ab} e^{\alpha}_{a} e^{\beta}_{b})\,(\in R_{\mu\alpha\gamma\beta} n^{\mu} n^{\gamma} + h^{mn} R_{\mu\alpha\gamma\beta} e^{\mu}_{m}\, e^{\gamma}_{n}) \\ &= \in^{2} R_{\mu\alpha\gamma\beta} n^{\alpha} n^{\beta} n^{\mu} n^{\gamma} + \in h^{mn} R_{\mu\alpha\gamma\beta}\, e^{\mu}_{m}\, e^{\gamma}_{n}\, n^{\alpha}\, n^{\beta} \\ &\quad + \in h^{ab} R_{\mu\alpha\gamma\beta}\, e^{\alpha}_{a}\, e^{\beta}_{b}\, n^{\mu} n^{\gamma} + h^{ab} h^{mn} R_{\mu\alpha\gamma\beta}\, e^{\mu}_{m} e^{\gamma}_{n} e^{\alpha}_{a} e^{\beta}_{b} \end{aligned}$$

$$or, \quad R = 2 \in h^{ab} R_{\mu\alpha\gamma\beta} n^{\mu}\, e^{\alpha}_{a}\; n^{\gamma}\, e^{\beta}_{b} + h^{ab} h^{mn} R_{\mu\alpha\gamma\beta}\, e^{\mu}_{m}\, e^{\alpha}_{a} e^{\gamma}_{n}\, e^{\beta}_{b}$$

where we use the fact that ,

$$R_{\mu\alpha\gamma\beta}\, n^{\mu} n^{\alpha} n^{\gamma} n^{\beta} = 0$$

and we obtain with the help of (4.22)

$$R = 2 \in R_{\alpha\beta} n^{\alpha} n^{\beta} + h^{ab} h^{mn} \{ R_{manb} + \in (K_{mb}\, K_{an} - K_{mn} K_{ab}) \} \qquad 4.25$$

To obtain the value of the first term in the above equation, let us consider the definition of Riemannian tensor $R^{\alpha}_{\beta\gamma\delta}$ given by :

$$\begin{aligned} n^{\alpha}_{;\gamma\delta} - n^{\alpha}_{;\delta\gamma} &= -R^{\alpha}_{\beta\gamma\delta}\, n^{\beta} \\ or, R_{\beta\delta} n^{\beta} &= n^{\alpha}_{;\delta\alpha} - n^{\alpha}_{;\alpha\delta} \\ or, R_{\beta\delta}\, n^{\beta}\, n^{\delta} &= n^{\alpha}_{;\delta\alpha} n^{\delta} - n^{\alpha}_{;\alpha\delta} n^{\delta} \end{aligned}$$

Thus we may write

$$\begin{aligned} R_{\alpha\beta}\, n^{\alpha}\, n^{\beta} &= n^{\alpha}_{;\beta\alpha} n^{\beta} - n^{\alpha}_{;\alpha\beta} n^{\beta} \\ &= -n^{\alpha}_{;\alpha\beta} n^{\beta} + n^{\alpha}_{;\beta\alpha} n^{\beta} \\ &= -(n^{\alpha}_{;\alpha} n^{\beta})_{;\beta} + n^{\alpha}_{;\alpha}\, n^{\beta}_{;\beta} + (n^{\alpha}_{;\beta} n^{\beta})_{;\alpha} - n^{\alpha}_{;\beta} n^{\beta}_{;\alpha} \\ &= -(n^{\alpha}_{;\alpha} n^{\beta})_{;\beta} + K^{2} + (n^{\alpha}_{;\beta} n^{\beta})_{;\alpha} - n^{\alpha}_{;\beta} n^{\beta}_{;\alpha} \end{aligned}$$

$$\begin{aligned} or, R_{\alpha\beta} n^{\alpha}\, n^{\beta} &= -(n^{\alpha}\, n^{\beta}_{;\beta})_{;\alpha} + (n^{\alpha}_{;\beta} n^{\beta})_{;\alpha} + K^{2} - n^{\alpha}_{;\beta}\, n^{\beta}_{;\alpha} \\ &= (n^{\alpha}_{;\beta} n^{\beta} - n^{\alpha}\, n^{\beta}_{;\beta})_{;\alpha} + K^{2} - n^{\alpha}_{;\beta}\, n^{\beta}_{;\alpha} \end{aligned}$$

where $K = n^{\alpha}_{;\alpha}$ is the trace of the extrinsic curvature .Let us calculate the 3^{rd} term of the above equation separately,

$$\begin{aligned} n^{\alpha}_{;\beta}\, n^{\beta}_{;\alpha} &= g^{\beta\mu} g^{\alpha\gamma}\, n_{\alpha;\beta}\, n_{\mu;\gamma} \\ &= (\in n^{\beta} n^{\mu} + h^{\beta\mu})(\in n^{\alpha}\, n^{\gamma} + h^{\alpha\gamma})\, n_{\alpha;\beta}\, n_{\mu;\gamma} \end{aligned}$$

$$or,\ n^{\alpha}_{;\beta}\, n^{\beta}_{;\alpha} = (\in n^{\beta} n^{\mu} + h^{\beta\mu}) h^{\alpha\gamma}\, n_{\alpha;\beta}\, n_{\mu;\gamma}$$
$$= h^{\beta\mu} h^{\alpha\gamma} n_{\alpha;\beta}\, n_{\mu;\gamma}$$
$$= h^{bm} h^{an}\, n_{\alpha;\beta}\, e^{\alpha}_{a}\, e^{\beta}_{b}\, n_{\mu;\gamma} e^{\mu}_{m}\, e^{\gamma}_{n}$$
$$= h^{bm} h^{an} K_{ab}\, K_{mn}$$
$$= K_{ab} K^{ab}$$
$$or,\ n^{\alpha}_{;\beta}\, n^{\beta}_{;\alpha} = K^{ab} K_{ab}$$

where we have used the fact that $n^{\alpha} n_{\alpha;\beta} = \frac{1}{2}(n^{\alpha} n_{\alpha})_{;\beta} = 0$.

Thus we can write now

$$R_{\alpha\beta} n^{\alpha}\, n^{\beta} = (n^{\alpha}_{;\beta} n^{\beta} - n^{\alpha}\, n^{\beta}_{;\beta})_{;\alpha} + K^{2} - K^{ab} K_{ab}$$

Putting the values in equation (4.24)

$$R = 2 \in \left\{ (n^{\alpha}_{;\beta} n^{\beta} - n^{\alpha}\, n^{\beta}_{;\beta})_{;\alpha} + K^{2} - K^{ab} K_{ab} \right\}$$
$$+ h^{ab} h^{mn} \left\{ R_{manb} + \in (K_{mb} K_{an} - K_{mn} K_{ab}) \right\}$$

$$or,\ R = 2 \in \left\{ K^{2} - K^{ab} K_{ab} + (n^{\alpha}_{;\beta} n^{\beta} - n^{\alpha}\, n^{\beta}_{;\beta})_{;\alpha} \right\}$$
$$+ {}^{3}R + \in (K^{ab} K_{ab} - K^{2})$$

$$or,\ R = {}^{3}R + \in (K^{2} - K^{ab} K_{ab}) + 2 \in (n^{\alpha}_{;\beta} n^{\beta} - n^{\alpha}\, n^{\beta}_{;\beta})_{;\beta} \qquad 4.26$$

where ${}^{3}R = h^{ab} R^{m}_{amb}$ is the three dimensional Ricci scalar. Equation (4.25) indicates the four dimensional Ricci scalar evaluated on the hypersurface.

We can write now the Einstein tensor

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

$$= \in R_{\mu\alpha\gamma\beta} n^{\mu}\, n^{\gamma} + h^{mn} R_{\mu\alpha\gamma\beta} e^{\mu}_{m}\, e^{\gamma}_{n}$$
$$-\frac{1}{2}(\in h_{\alpha\beta} + h_{ab} e^{a}_{\alpha} e^{b}_{\beta}) \left\{ {}^{3}R + \in (K^{2} - K^{ab} K_{ab}) + 2 \in (n^{\alpha}_{;\beta}\, n^{\beta} - n^{\alpha} n^{\beta}_{;\beta})_{;\alpha} \right\}$$

$$or,\ G_{\alpha\beta} n^{\alpha} n^{\beta} = \in R_{\mu\alpha\gamma\beta} n^{\mu}\, n^{\gamma} n^{\alpha} n^{\beta} + h^{mn} R_{\mu\alpha\gamma\beta} e^{\mu}_{m}\, e^{\gamma}_{n} n^{\alpha} n^{\beta}$$
$$-\frac{1}{2} \left\{ {}^{3}R + \in (K^{2} - K^{ab} K_{ab}) + 2 \in (n^{\alpha}_{;\beta}\, n^{\beta} - n^{\alpha} n^{\beta}_{;\beta})_{;\alpha} \right\}$$
$$(\in h_{\alpha\beta} + h_{ab} e^{a}_{\alpha} e^{b}_{\beta}) n^{\alpha} n^{\beta}$$

$$or, G_{\alpha\beta}n^{\alpha}n^{\beta} = 0 + h^{mn}h^{\mu\alpha}h^{\beta\gamma}R_{\mu\alpha\gamma\beta}e^{\mu}_{m}e^{\gamma}_{n}n_{\mu}n_{\gamma}$$
$$-\frac{1}{2}\in h_{\alpha\beta}n^{\alpha}n^{\beta}\left\{{}^{3}R+\in(K^{2}-K^{ab}K_{ab})+2\in(n^{\alpha}_{;\beta}n^{\beta}-n^{\alpha}n^{\beta}_{;\beta})_{;\alpha}\right\}$$

$$\text{since}, R_{\mu\alpha\gamma\beta}n^{\mu}n^{\gamma}n^{\alpha}n^{\beta} = 0 \ and\ e^{\alpha}_{a}n_{\alpha} = 0$$

$$or, G_{\alpha\beta}n^{\alpha}n^{\beta} = -\frac{1}{2}\in h_{\alpha\beta}n^{\alpha}n^{\beta}\left\{{}^{3}R+\in(K^{2}-K^{ab}K_{ab})+2\in(n^{\alpha}_{;\beta}n^{\beta}-n^{\alpha}n^{\beta}_{;\beta})_{;\alpha}\right\}$$
$$= -\frac{1}{2}\in n_{\beta}n^{\beta}\left\{{}^{3}R+\in(K^{2}-K^{ab}K_{ab})+2\in(n^{\alpha}_{;\beta}n^{\beta}-n^{\alpha}n^{\beta}_{;\beta})_{;\alpha}\right\}$$
$$= -\frac{1}{2}\in {}^{3}R+\tfrac{1}{2}\in(K^{2}-K^{ab}K_{ab})$$
$$= -\frac{1}{2}\in\{{}^{3}R+(K^{2}-K^{ab}K_{ab})\}$$
$$or, -2G_{\alpha\beta}n^{\alpha}n^{\beta} = \in {}^{3}R+\in(K^{2}-K^{ab}K_{ab})$$

$$or, -2\in G_{\alpha\beta}n^{\alpha}n^{\beta} = {}^{3}R+\in(K^{2}-K^{ab}K_{ab})$$

where $\in = n^{\alpha}n_{\alpha} = n^{\beta}n_{\beta} = 1$ *in case of time like hyper surface.*
Hence we write then,

$$-2\in G_{\alpha\beta}n^{\alpha}n^{\beta} = {}^{3}R+\in(K^{2}-K^{ab}K_{ab}) \qquad 4.27$$
$$and\ \ G_{\alpha\beta}e^{\alpha}_{a}n^{\beta} = K^{b}_{a1b} - K_{,a} \qquad 4.28$$

where ${}^{3}R = h^{ab}R_{amb}$ is the three dimensional Ricci scalar.

The importance of equation (4.26) and (4.27) lies with the fact that they form the part of the Einstein field equation on a hypersurface $\sum$. It is noted that $G_{\alpha\beta}e^{\alpha}_{a}n^{\beta}$ the remaining part of this Einstein tensor, can not be expressed solely in terms of h_{ab}, K_{ab} and related quantities.

4.7 CONSTRAINED IN INITIAL VALUE PROBLEM:

In Newtonian mechanics, a complete solution to the equation of motion requires the specification of initial values for the position and velocity of each moving body. But in field theories, a complete solution to the field equation requires the specification of field and its time derivatives at one instant of time. Since the Einstein field equation is 2^{nd} order partial differential equations, we would expect that a complete solution should require the specification of $g_{\alpha\beta}$ and $g_{\alpha\beta,t}$ at one instant of time. While this is correct, it is desirable to convert this decidedly non-covariant statement into something more geometrical.

The initial value problem in general relativity starts with the selection of a space like hyper-surface Σ which represent an instant of time. This hypersurface can be chosen freely. On this hypersurface we put some arbitrary co–ordinates y^a.

The space –time metric $g_{\alpha\beta}$, when evaluated on Σ, has components that characterizes the displacements away from the hypersurface. For example, g_{tt} is such a component if Σ is a surface of constant *t*. These components cannot be given meaning in terms of geometric properties of Σ alone. To provide meaningful initial values for the space-time metric, we must consider displacement within the hypersurface only. In other words the initial values for $g_{\alpha\beta}$ can only be the six components of the induced metric $h_{ab} = g_{\alpha\beta}\, e^{\alpha}_{a}\, e^{\beta}_{b}$; the remaining four components are arbitrary and this reflects the complete freedom in choosing the space time co–ordinates x^{α}.

Similarly the initial value for the time derivative of the metric must be described by a three tensor that carries information about the derivative of the metric in the direction normal to the hypersurface. Because $K_{ab} = \frac{1}{2}(L_n g_{\alpha\beta})\, e^{\alpha}_{a}\, e^{\beta}_{b}$, the extrinsic curvature is clearly an appropriate choice. (L_n stands for lie derivative)

The initial value problem of general relativity therefore consists in specifying two symmetric tensor fields h_{ab} and K_{ab} on a space like hypersurface Σ. In the complete space time, h_{ab} is interpreted as the induced metric on the hypersurface while K_{ab} is the extrinsic curvature. These tensors cannot be chosen freely; they satisfy the covariant equations of general relativity. They are given by

$$-2 \in G_{\alpha\beta}\, n^{\alpha}\, n^{\beta} = {}^{3}R + \in (K^{ab}K_{ab} - K^{2})$$

$$\text{and} \qquad G_{\alpha\beta} e^{\alpha}_{a}\, e^{\beta}_{b} = K^{b}_{a|b} - K_{,a}$$

together with the Einstein field equation

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

such that

$${}^{3}R + K^{2} - K^{ab}K_{ab} = 16\pi T_{\alpha\beta}\, n^{\alpha} n^{\beta} \equiv 16\pi\rho$$

$$\text{and} \quad K^{b}_{a|b} - K_{,a} = 8\pi T_{\alpha\beta}\, e^{\alpha}_{a}\, n^{\beta} \equiv 8\pi\, j_{a}$$

The remaining components of the Einstein field equations provide evolution equation for h_{ab} and K_{ab} .

4.8 JUNCTION CONDITION AND THIN SHELL:

A hypersurface $\sum$ partitions space time into two regions ν^{+} and ν^{-} as in figure (3). In ν^{+},the metric is $g^{+}_{\alpha\beta}$ and it is expressed as a system of co- ordinate x^{α}_{+} .Similarly in ν^{-} the metric is $g^{-}_{\alpha\beta}$ and it is expressed in co-ordinate x^{α}_{-} . Now what condition should be put on the metrics to ensure that ν^{+} and ν^{-} are joined smoothly at $\sum$ -so that the union of $g^{+}_{\alpha\beta}$ and $g^{-}_{\alpha\beta}$ forms a valid solution to the Einstein equation.

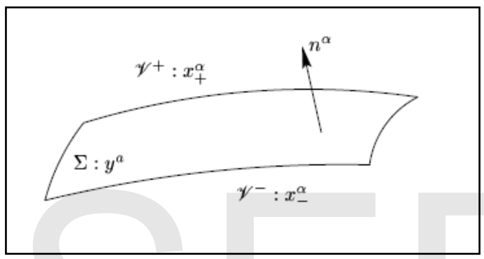

Fig: 3

The answer of this question is not entirely straightforward because in practical situation, the co-ordinate system $x^{\alpha}_{\pm}$ will often be different and it may not be possible to compare the metrics directly. To circumvent this difficulty we will endeavor to formulate junction conditions that involve only three tensor on $\sum$ .In that case we will assume that $\sum$ is either time like or space like.

4.9 NOTATION AND ASSUMPTION:

We assume that the same co-ordinates y^{α} can be installed on both sides of the hypersurface and we choose n^{α}, the unit normal to $\sum$ to point from ν^{-} to ν^{+}.We suppose that an overlapping co-ordinate system x^{α} distinct from $x^{\alpha}_{\pm}$ can be introduced in the neighborhood of the $\sum$.(This is for our short term convenience ;the formulation of the junction condition will not involve this co-ordinate system).We imagine the hypersurface $\sum$ to be pierced by a congruence of geodesics that intersect it orthogonally. We take l to denote proper distance (or proper time) along the geodesics and we will adjust the parameterization so that l =0 when geodesic cross the hypersurface. Our convention is that l is -ve in ν^- and l is +ve in ν^+.We can think of l as a scalar field: The point P characterizes by the co-ordinates x^α is linked to $\sum$ by a member of congruence and $l\,(x^a)$ is the proper distance from $\sum$ to P along the geodesic.

Our construction is that n^α is equal to $\frac{dx^\alpha}{dl}$ at the hypersurface and that

$$n_\alpha = \in \partial_\alpha l$$

We also have $n^\alpha n_\alpha = \in$.We will also use language of distribution. We introduce the Heaviside distribution $\Theta(l)$ [5] and is

equal to +1 if $l \rangle 0$

or equal to –1 if $l \langle 0$

and intermediate if $l = 0$

We note the following properties:

$$\Theta^2(l) = \Theta(l)\ , \Theta(l)\Theta(-l) = 0\ , \tfrac{d}{dt}\Theta(l) = \delta(l)$$

where $\delta(l)$ is the Dirac distribution .We note that the product $\Theta(l)\delta(l)$ is not defined as distribution. The following notation will be useful:

$$[A] \equiv A(\nu^+)\ \Big|_\Sigma - A(\nu^-)\ \Big|_\Sigma \qquad 4.29$$

where A is any tensorial quantity defined on both sides of the hypersurface; $[A]$ is therefore the jump of a across $\sum$.We note the relation

$$[n^\alpha] = [e^\alpha_a] = 0 \qquad 4.30$$

where $e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$.The first follows from the relation $n^\alpha = \frac{dx^\alpha}{dl}$ and the continuity of both l and x^α across $\sum$,the second follows from the fact that the co-ordinates y^a are the same on the both sides of the hypersurface.

4.10 FIRST JUNCTION CONDITION:

Let us begin by expressing the metric $g_{\alpha\beta}$ in the co-ordinate x^α as a distribution valued tensor:

$$g_{\alpha\beta} = \Theta(l)\, g^+_{\alpha\beta} + \Theta(-l) g^-_{\alpha\beta} \qquad 4.31$$

where $g^\pm_{\alpha\beta}$ is the metric in ν^+ expressed in the co-ordinates x^α.We want to know whether the metric (4.24) makes a valid distributional solution to the Einstein field equation. To deduce we must verify that geometrical quantities constructed from $g_{\alpha\beta}$ such that Riemannian tensor are properly defined as distribution. We must then try to eliminate or at least give an interpretation to, singular terms that might arise in these geometrical quantities. Differentiating (4.31) we obtain [5]

$$g_{\alpha\beta,\gamma} = \Theta(l) g^{+}_{\alpha\beta,\gamma} + \Theta(l) g^{-}_{\alpha\beta,\gamma} + \in \delta(l)[g_{\alpha\beta}]n_{\gamma} \qquad 4.32$$

The last term is singular and it causes problem when we compute the Christoffel symbols, because it generates terms proportional to $\Theta(l)\delta(l)$.If the last term is allowed to survive, therefore the connection would not be defined as a distribution .To eliminate this term, we impose continuity of the metric across the hyper surface: $\left[g_{\varepsilon\beta}\right] = 0$.This statement holds in this co-ordinate system x^{α} only. However we can easily turn this into a co-ordinate invariant statement; $0 = \left[g_{\alpha\beta}\right] e^{\alpha}_{a}\, e^{\beta}_{b} = \left[g_{\alpha\beta} e^{\alpha}_{a}\, e^{\beta}_{b}\right]$; the last step is followed by (4.23).We have obtained then

$$\left[g_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b}\right] = 0 \Rightarrow \left[h_{ab}\right] = 0 \qquad 4.33$$

The statement that the induced metric must be the same on the both sides of $\sum$.This is clearly required if the hypersurface is to have a well defined geometry. Equation (4.33) will be our first junction condition and it is expressed independently of the co-ordinates $x^{\alpha}\ or, x^{\alpha}_{\pm}$.

4.11 RIEMANNIAN TENSOR:

To find the second junction condition, more works are required. We must calculate the distribution valued Riemannian tensor. Using the result of (4.26), we can write the Christoffel symbol as:

$$\Gamma^{\alpha}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\delta} (g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma\,\delta})$$

$$\begin{aligned} or, \Gamma^{\alpha}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\delta} \ [\ &\{\Theta(l)\, g^{+}_{\delta\beta,\gamma} + \Theta(-l)\, g^{-}_{\delta\beta,\gamma} + \in \delta(l)[g_{\delta\beta}]\, n_{\gamma}\} \\ &+ \{\Theta(l)\, g^{+}_{\delta\gamma,\beta} + \Theta(-l)\, g^{-}_{\delta\gamma,\beta} + \in \delta(l)[g_{\delta\gamma}] n_{\beta}\} \\ &- \{\Theta(l)\, g^{+}_{\beta\gamma,\delta} + \Theta(-l)\, g^{-}_{\beta\gamma,\delta} + \in \delta(l)[g_{\beta\gamma}]\, n_{\delta}\}\ ] \end{aligned}$$

$$\begin{aligned} or,\ \Gamma^{\alpha}_{\beta\gamma} = \tfrac{1}{2}\ \Theta(l)\ g^{\alpha\delta} (g^{+}_{\delta\beta,\gamma} + g^{+}_{\gamma\delta,\beta} - g^{+}_{\beta\gamma,\delta}) \\ + \tfrac{1}{2} \Theta(-l) g^{\alpha\delta} (g^{-}_{\delta\gamma,\beta} + g^{-}_{\delta\beta,\gamma} - g_{\beta\gamma,\delta}) \end{aligned}$$

$$or,\ \Gamma^{\alpha}_{\beta\gamma} = \Theta(l)\Gamma^{+\alpha}_{\beta\gamma} + \Theta(-l)\Gamma^{-\alpha}_{\beta\gamma}$$

where $\Gamma^{\pm\alpha}_{\beta\gamma}$ are the Christoffel symbol constructed from $g^{\pm}_{\alpha\beta}$. A straightforward calculation then reveals

$$\Gamma^{\alpha}_{\beta\gamma,\delta} = \Theta(l)\,\Gamma^{+}_{\beta\gamma,\delta} + \Theta(-l)\,\Gamma^{-}_{\beta\gamma,\delta} + \in \delta(l)[\Gamma^{\alpha}_{\beta\gamma}]\; n_{\delta}$$

and from this we get the Riemannian tensor :

$$R^{\alpha}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\alpha}_{\gamma\mu}\Gamma^{\mu}_{\beta\delta} - \Gamma^{\alpha}_{\mu\delta}\Gamma^{\mu}_{\beta\gamma}$$

$$\begin{aligned} or, R^{\alpha}_{\beta\gamma\delta} &= \Theta(l)\Gamma^{+\alpha}_{\beta\delta,\gamma} + \Theta(-l)\Gamma^{-\alpha}_{\beta\delta,\gamma} + \in \delta(l)[\Gamma^{\alpha}_{\beta\delta}]n_{\gamma} - \Theta(l)\Gamma^{+\alpha}_{\beta\gamma,\delta} - \Theta(-l)\Gamma^{-\alpha}_{\beta\gamma,\delta} \\ &- \in \delta(l)[\Gamma^{\alpha}_{\beta\gamma}]n_{\delta} - \{\Theta(l)\,\Gamma^{+\alpha}_{\gamma\mu} + \Theta(-l)\,\Gamma^{-\alpha}_{\gamma\mu}\}\{\Theta(l)\,\Gamma^{+\mu}_{\beta\delta} + \Theta(-l)\,\Gamma^{-\mu}_{\beta\delta}\} \\ &- \{\Theta(l)\,\Gamma^{+\alpha}_{\mu\delta} + \Theta(-l)\,\Gamma^{-\alpha}_{\mu\delta}\}\{\Theta(l)\,\Gamma^{+\mu}_{\beta\gamma} + \Theta(-l)\,\Gamma^{-\mu}_{\beta\gamma}\} \end{aligned}$$

$$\begin{aligned} &= \Theta(l)[\Gamma^{+\alpha}_{\beta\delta,\gamma} - \Gamma^{+\alpha}_{\beta\gamma,\delta} + \Gamma^{+\alpha}_{\gamma\mu}\,\Gamma^{+\mu}_{\beta\delta} - \Gamma^{+\alpha}_{\mu\delta}\Gamma^{+\mu}_{\beta\gamma}] \\ &+ \Theta(-l)[\Gamma^{-\alpha}_{\beta\delta,\gamma} - \Gamma^{-\alpha}_{\beta\gamma,\delta} + \Gamma^{-\alpha}_{\gamma\mu}\Gamma^{-\mu}_{\beta\delta} - \Gamma^{-\alpha}_{\mu\delta}\Gamma^{-\mu}_{\beta\gamma}] \\ &+ \in \delta(l)\;\{\,[\Gamma^{\alpha}_{\beta\delta}]n_{\gamma} - [\Gamma^{\alpha}_{\beta\gamma}]n_{\delta}\,\} \end{aligned}$$

$$or, R^{\alpha}_{\beta\gamma\delta} = \Theta(l)\; R^{+\alpha}_{\beta\gamma\delta} + \Theta(-l)\; R^{-\alpha}_{\beta\gamma\delta} + \delta(l)\,A^{\alpha}_{\beta\gamma\delta} \qquad 4.34$$

where $A^{\alpha}_{\beta\gamma\delta} = \in \{[\Gamma^{\alpha}_{\beta\delta}]n_{\gamma} - [\Gamma^{\alpha}_{\beta\gamma}]n_{\delta}\}$

We see that the Riemannian tensor is properly defined as a distribution but the δ function term represent a curvature singularity at $\sum$ .The second junction condition will seek to eliminate this term. Failing this, we see that a physical interpretation can nevertheless be given to the singularity which is our next topic.

Although they are constructed from Christoffel symbol, the quantities $A^{\alpha}_{\beta\gamma\delta}$ form a tensor because the difference between two sets of Christoffel symbol is a tensorial quantity. We must find now an explicit expression for this tensor.

The fact that the metric is continuous across $\sum$ in the co-ordinates x^{α} implies that its tangential derivative must also be continuous. This means that if $g_{\alpha\beta,\gamma}$ is to be discontinuous, the discontinuity must be directed along the normal vector n^{α} . Therefore there must exist a tensor field k_{ab} such that

$$[g_{\alpha\beta,\,\gamma}] = k_{\alpha\beta}\, n_{\gamma} \qquad 4.35$$

Also this tensor is given by

$$k_{\alpha\beta} = \in [g_{\alpha\beta,\gamma}] n^{\gamma} \qquad 4.36$$

with the help of (4.30) we get

$$[\Gamma^{\alpha}_{\beta\gamma}] = [\tfrac{1}{2} g^{\alpha\delta} (g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} + - g_{\beta\gamma,\delta})]$$

$$= \tfrac{1}{2}[g^{\alpha\delta}(k_{\delta\beta} n_{\gamma} + k_{\delta\gamma} n_{\beta} - k_{\beta\lambda} n_{\delta})]$$

$$= \tfrac{1}{2}(k^{\alpha}_{\beta} n_{\gamma} + k^{\alpha}_{\gamma} n_{\beta} - k_{\beta\gamma} n^{\alpha}) \qquad 4.37$$

Putting the value of (4.36) and (4.37) we obtain

$$A^{\alpha}_{\beta\gamma\delta} = \in ([\Gamma^{\alpha}_{\beta\delta}] n_{\gamma} - [\Gamma^{\alpha}_{\beta\gamma}] n_{\delta})$$

$$= \frac{\in}{2}(k^{\alpha}_{\delta} n_{\beta} n_{\gamma} - k^{\alpha}_{\gamma} n_{\beta} n_{\delta} - k_{\beta\delta} n^{\alpha} n_{\gamma} + k_{\beta\gamma} n^{\alpha} n_{\delta})$$

This is the δ function part of the Riemannian tensor. Contracting over the first and third indices, we get the δ function part of the Ricci tensor.

$$A_{\alpha\beta} = A^{\mu}_{\alpha\mu\beta} = \frac{\in}{2}(k_{\mu\alpha} n^{\mu} n_{\beta} + k_{\mu\beta} n^{\mu} n_{\alpha} - k n_{\alpha} n_{\beta} - \in k_{\alpha\beta}) \qquad 4.38$$

where $k \equiv k^{\alpha}_{\alpha}$. After an additional contraction, we obtain the δ function part of the Ricci scalar

$$A \equiv A^{\alpha}_{\alpha} = \in (k_{\mu\gamma} n^{\mu} n^{\gamma} - \in k) \qquad 4.39$$

Using (4.33) and (4.34) we calculate the Einstein tensor

$$G_{\alpha\beta} = A_{\alpha\beta} - \frac{1}{2} A g_{\alpha\beta}$$

where $A_{\alpha\beta}$ and A are the Ricci tensor and Ricci scalar for δ function respectively.

4.12 SECOND JUNCTION CONDITION:

The surface stress energy tensor is given by [5]

$$S_{\alpha\beta} = \frac{1}{8\pi}(A_{\alpha\beta} - \frac{1}{2} A g_{\alpha\beta})$$

$$or, 8\pi S_{\alpha\beta} = \frac{\in}{2}(k_{\mu\alpha} n^{\mu} n_{\beta} + k_{\mu\beta} n^{\beta} n_{\alpha} - k n_{\alpha} n_{\beta} - \in k_{\alpha\beta})$$
$$- \frac{1}{2} g_{\alpha\beta} \in (k_{\mu\gamma} n^{\mu} n^{\gamma} - \in k)$$

$$or,\ 16\pi \in S_{\alpha\beta} = k_{\mu\alpha}\, n^{\mu} n_{\beta} + k_{\mu\beta}\, n^{\beta}\, n_{\alpha} - k\, n_{\alpha}\, n_{\beta} - \in k_{\alpha\beta}$$
$$- (k_{\mu\gamma}\, n^{\mu}\, n^{\gamma} - \in k) g_{\alpha\beta}$$

From this we notice that $S_{\alpha\beta}$ is tangent to the hypersurface: $S_{\alpha\beta}\, n^{\beta} = 0$ .It therefore admits the decomposition

$$S^{\alpha\beta} = S^{ab} e_a^{\alpha}\, e_b^{\beta}$$

where $S_{ab} = S_{\alpha\beta}\, e_a^{\alpha}\, e_b^{\beta}$ is a symmetric three tensor. This is evaluated as follows:

$$16\pi\, S_{ab} = -k_{\alpha\beta}\, e_a^{\alpha}\, e_b^{\beta} - \in (k_{\mu\gamma}\, n^{\mu}\, n^{\gamma} - \in k)\, h_{ab}$$
$$= -k_{\alpha\beta}\, e_a^{\alpha}\, e_b^{\beta} - k_{\mu\gamma}\, (g^{\mu\gamma} - h^{mn}\, e_m^{\mu}\, e_n^{\gamma}) h_{ab} + k\, h_{ab}$$
$$= -k_{\alpha\beta}\, e_a^{\alpha}\, e_b^{\beta} + h^{mn}\, k_{\mu\gamma}\, e_m^{\mu}\, e_n^{\gamma} h_{ab}$$

On the other hand we have,

$$[\, n_{\alpha,\beta}] = -[\Gamma^{\gamma}_{\alpha\beta}] n_{\gamma}$$
$$= -\frac{1}{2}(k_{\gamma\alpha} n_{\beta} + k_{\gamma\beta} n_{\alpha} - k_{\alpha\beta} n_{\gamma}) n^{\gamma}$$
$$= \frac{1}{2}(\in k_{\alpha\beta} - k_{\gamma\alpha}\, n_{\beta}\, n^{\gamma} - k_{\gamma\beta}\, n_{\ \alpha} n^{\gamma}$$

which allows us to write

$$[K_{ab}] = [n_{\alpha\,;\beta}] e_a^{\alpha}\, e_b^{\beta} = \frac{\in}{2} k_{\alpha\beta}\, e_a^{\alpha}\, e_b^{\beta}$$

Combining these we have,

$$S_{ab} = -\frac{\in}{8\pi}(\ [K_{ab}] - [K] h_{ab}\ )$$

which relates the surface stress energy tensor to the jump in extrinsic curvature from one side of $\sum$ to the other. We conclude that a smooth transition across $\sum$ requires $[K_{ab} = 0]$,the extrinsic curvature must be the same on the both sides of the hypersurface. This requirement does more than just remove the δ function term from the Einstein tensor.

Again $[K_{ab} = 0]$ implies $A^{\alpha}_{\beta\gamma\delta} = 0$, which means that the full Riemannian tensor is then non-singular at $\sum$.

The condition $[K_{ab} = 0]$ is our second junction condition and it is expressed independently of the co-ordinates x^{α} and $x^{\alpha}_{\pm}$ .If this condition is violated, then the space time is singular at $\sum$.

Chapter five

EMBEDDING OF SPACE TIME IN FIVE DIMENSIONAL WEYL SPACES

5.0 INTRODUCTION:

In this chapter we review the Weyl geometry in the context of higher dimensional space time. In recent year our ordinary space time may be viewed as a hyper surface embedded in a higher dimensional manifold often referred as the bulk. As far as the geometry of this hyper surface is concerned, it has been generally assumed that it has a Riemanian geometrical structure. After introducing the Weyl theory in a modern geometrical language we present some results that represent extensions of Riemannian theorems. We consider the theory of local embeddings and sub manifolds in the context of Weyl geometry and show how a Riemannian space time may be locally and isometric ally embedded in a Weyl bulk.

An important class of higher dimensional models in the brane world scenario share the following properties : (a) Our space time is viewed as 4D Riemannian hyper surface (brane) embedded in a 5D Riemannian manifold (bulk) (b) The geometry of the bulk space is characterized by a warped product space. (c) Fermionic matter is confined to the brane by means of an interaction of the fermions with a scalar field which depends only on the extra dimension.

In this chapter we discuss the problem of classical confinement and the stability of motion of particles and photons in the neighborhood of brane for the case when the bulk has the geometry of warped product space .We studied confinement and stabilities properties of geodesics near the brane that may be affected by Weyl field.

This chapter is mainly a review work of the article "On the embedding of space time in five dimensional Weyl spaces" of F.Dahia, G.A.T.Gomez and C.Romero, published in the journal of mathematical physics 49, 102501(2008) .

EMBEDDING OF SPACE TIME IN FIVE DIMENSIONAL WEYL SPACE.

5.1 WEYL TENSOR: The Einstein equation given as

$$G_{\alpha\beta} = kT_{\alpha\beta}$$

can be regarded as ten algebraic equations for certain traces of the Riemannian tensor $R_{\mu\gamma\rho\sigma}$ .But $R_{\mu\gamma\rho\sigma}$ has twenty independent components .The reason is that we solve the Einstein equation for the metric $g_{\mu\gamma}$ and then calculate the Riemannian curvature tensor for that metric .However this reason does not really provide an explanation of how the information about the other components are encoded in the Einstein equation. It is interesting to understand this because it is precisely these components of the Riemannian tensor which represent the effects of gravity in vacuum i.e where $T_{\mu\gamma} = 0$,like tidal forces and gravitational waves.

The more insightful answer is that the information is encoded in the Bianchi identity

$$\nabla_{[\lambda} R_{\rho\sigma]} = 0$$

which serves as propagation equation for the trace parts of the Riemannian tensor away from the region where $T_{\mu\gamma} \neq 0$.

To see this, first of all let decompose the Riemannian tensor into its trace part $R_{\mu\gamma}$ (Ricci part) and R (Ricci scalar) and its traceless part $C_{\mu\gamma\rho\sigma}$ -is called the Weyl tensor which is basically the Riemannian tensor with all of its contractions removed.

In any $n \geq 4$,the Weyl tensor is defined [15] [21] by :

$$C_{\mu\gamma\rho\sigma} = R_{\mu\gamma\rho\sigma} - \frac{1}{n-2}(g_{\mu\rho}R_{\gamma\sigma} + R_{\mu\rho}g_{\gamma\sigma} - g_{\gamma\rho}R_{\mu\sigma} - R_{\gamma\rho}g_{\mu\sigma}) + \frac{1}{(n-1)(n-2)}R(g_{\mu\rho}g_{\gamma\sigma} - g_{\gamma\rho}g_{\mu\sigma})$$

This definition is such that $C_{\mu\gamma\rho\sigma}$ has all the symmetries of the Riemanian tensor i.e

$$C_{\mu\gamma\rho\sigma} = C_{[\mu\gamma]}C_{[\rho\sigma]}$$

$$C_{\mu\gamma\rho\sigma} = C_{\rho\sigma\mu\gamma}$$

$$C_{\mu[\gamma\rho\sigma]} = 0$$

and that of all of its traces are zero i.e

$$C^{\mu}_{\gamma\mu\sigma} = 0 .$$

In the vacuum $R_{\mu\gamma} = 0$ and therefore

$$T_{\mu\gamma}(x) = 0 \Rightarrow R_{\mu\gamma\rho\sigma} = C_{\mu\gamma\rho\alpha}(x)$$

and as anticipated ,the Weyl tensor encodes the information about the gravitational field in vacuum.

Again the Weyl tensor is also useful in other context as it is conformally invariant i.e $C_{\mu\gamma\rho\sigma}$ is invariant under conformal rescaling of the metric

$$g_{\mu\gamma}(x) \to e^{f(x)} g_{\mu\gamma}(x)$$

in particular the Weyl tensor is zero if the metric is conformally flat i.e. related by a conformal transformation to the flat metric and conversely vanishing of the Weyl tensor is also a sufficient condition for a metric to be conformal to the flat metric.

5.2 WEYL GEOMETRY:

Now we will review some basic definition and results, which are valid in Riemannian and Weyl geometries. Again Weyl geometry may be viewed as a kind of generalization of a Riemannian geometry and some theorems that will be presented here are straightforward extensions of corresponding theorems of the former. These extensions have a different and new flavor especially when they are applied to the study of geodesic motion. Let us start with the definition of affine connection.

Let M be a differentiable manifold and T(M) , the set of all differentiable vector fields on M. An affine connection is a mapping $\nabla : T(M) \times T(M) \to T(M)$, which is denoted by $(U\, V) \to \nabla_U V$, satisfying the following condition or properties:

$$(a) \qquad \nabla_{fV+gU} W = f\nabla_V W + g\nabla_U W$$

$$(b) \qquad \nabla_V (U+W) = \nabla_V U + \nabla_V W$$

$$(c) \qquad \nabla_V (fU) = V(f)U + f\nabla_V U$$

where U,V,W $\in$ T(M) and f, g are C^∞ scalar function defined on M. From the above results an important result comes which help us to define a co-variant derivative along a differentiable curve.

PROPOSITION:

Let M be a differentiable manifold endowed with an affine connection ∇ and V is a vector field defined along a differentiable curve $\alpha : (a,b) \subset R \to M$ Then there exist unique rule which associates another vector field $\frac{DV}{d\lambda}$ along the curve α with V such that [17]:

$$\frac{D(V+U)}{d\lambda} = \frac{DV}{d\lambda} + \frac{DU}{d\lambda}$$

and $$\frac{D(fV)}{d\lambda} = \frac{df}{d\lambda}V + f\,\frac{Dv}{d\lambda}$$

where $\alpha = \alpha(\lambda)$ and $\lambda \in (a,b)$. If the vector field $U(\lambda)$ is induced by a vector field $\hat{U} \in T(M)$, then $\frac{DU}{d\lambda} = \nabla_V U$, where V is the tangent vector field to the curve α i.e. $V = \frac{d}{d\lambda}$.

Now we define he concept of parallel transport along a curve. Let M be a differentiable manifold with an affine connection ∇ and $\alpha(a,b) \subseteq R \to M$ be a differentiable curve on M and V is a vector field defined along $\alpha = \alpha(\lambda)$. The vector field V is said to be parallel if $\frac{DV}{d\lambda} = 0$ i.e its co-variant derivative vanishes for any value of the parameter $\lambda \in (a,b)$.

Among all admissible affine connection defined on a manifold, an important role in Riemannian and also in Weyl theory is played by a special class of connection –namely the torsion less connection defined as below:

An affine connection ∇ defined on a manifold M is torsion less (symmetric) if for any U, V$\in$ T(M) the following condition is hold.

$$\nabla_V U - \nabla_U V = [V,U]$$

Now we introduce the concept of Weyl manifold throughout the following definition:

Let M be a differentiable manifold endowed with an affine connection ∇, a metric tensor g and a one form field σ-called Weyl field, globally defined on M. We say that ∇ is Weyl compatible (W-compatible) with g if for any vector fields U,V,W$\in$ T(M) the following condition is satisfied:

$$V[g(U,W)] = g(\nabla_V U, W) + g(U, \nabla_V W) + \sigma(V) g(U,W) \qquad 5.1$$

This is of course a generalization of the idea of Riemannian compatibility between ∇ and g. If the one form σ vanishes throughout M, we recover the Riemanian compatibility condition. It is natural to expect that a generalized version of the Levi-Civita theorem hold if we restrict ourselves to torsion less connections. Indeed we have the following result:

THEOREM: In a given differentiable manifold M endowed with a metric g and a differentiable one-form field σ defined on M, there exist a only one affine connection ∇ such that (a) ∇ is torsion less.

(b) ∇ is W-compatible.

Proof: Let us first suppose that ∇ exist. Then from (1) we have the following three equations:

$$V[g(U,W)] = g(\nabla_V U,W) + g(U,\nabla_V W) + \sigma(V)g(U,W) \qquad 5.2$$

$$W[g(V,U)] = g(\nabla_W V,U) + g(V,\nabla_W U) + \sigma(W)g(V,U) \qquad 5.3$$

$$U[g(W,V)] = g(\nabla_U W,V) + g(W,\nabla_U V) + \sigma(U)g(W,V) \qquad 5.4$$

Adding (5.2) and (5.3) and then subtracting (5.4) we get

$$\begin{aligned} V[g(U,W)]+W[g(V,U)]-U[g(W,V)] &= g(\nabla_V U,W) + g(U,\nabla_V W) + \sigma(V)g(U,W) \\ &\quad + g(\nabla_W V,U) + g(V,\nabla_W U) + \sigma(W)g(V,U) \\ &\quad - g(\nabla_U W,V) - g(W,\nabla_U V) - \sigma(U)g(W,V) \end{aligned}$$

$$\begin{aligned} \text{or,} \quad g(\nabla_W V,U) &= V[g(U,W)]+W[g(V,U)] - U[g(W,V)] \\ &\quad - g(\nabla_V U,W) - g(U,\nabla_V W) - \sigma(V)g(U,W) \\ &\quad - g(V,\nabla_W U) - \sigma(W)g(V,U) + g(\nabla_U W,V) \\ &\quad + g(W,\nabla_U V) + \sigma(U)g(W,V). \end{aligned}$$

$$\begin{aligned} \text{or,} \quad g(\nabla_W V,U) + g(\nabla_W V,U) &= V[g(U,W)]+W[g(V,U)]-U[g(W,V)] \\ &\quad - g(\nabla_V U - \nabla_U V,W) - g(\nabla_W U - \nabla_U W,V) \\ &\quad - g(\nabla_V W - \nabla_W V,U) + \sigma(U)g(W,V) - \sigma(W)g(V,U) \\ &\quad - \sigma(V)g(U,W). \end{aligned}$$

$$\begin{aligned} \text{or,} \quad 2g(\nabla_W V,U) &= V[g(U,W)]+W[g(V,U)]-U[g(W,V)] - g([V,U],W) \\ &\quad - g([W,U],V) - g([V,W],U) + \sigma(U)g(W,V) - \sigma(W)g(V,U) \\ &\quad - \sigma(V)g(U,W). \end{aligned} \qquad 5.5$$

The above equation shows that the affine connection ∇ ,if it exist –is uniquely determined from the metric g and the Weyl field of one form σ. Now to prove the existence of such connection we just define $\nabla_U V$ by means of (4.5). Now choose that $U = e_a, V = e_b, and\ W = e_c$ and in a local co-ordinate system $\{x^a\}$;a=1,2,3----n. the terms in (5.5) having commutator vanishes.

So we obtain then

$$or,\ 2g(\nabla_c e_b, e_a) = e_b[g(e_a,e_c)] + e_c[g(e_b,e_a)] - e_a[g(e_c,e_b)]$$

$$+\sigma(e_a)\,g(e_c,e_b) - \sigma(e_c)\,g(e_c,e_a) - \sigma(e_b)\,g(e_a,e_c)$$

$$\text{or, } 2g(\Gamma^k_{bc} e_k, e_a) = e_b(g_{ac}) + e_c(g_{ba}) - e_a(g_{cb}) + \sigma_a g_{cb} - \sigma_c g_{ba} - \sigma_b g_{ac}$$

$$\text{or, } 2\Gamma^k_{bc}\, g_{ak} = e_b(g_{ac}) + e_c(g_{ba}) - e_a(g_{cb}) + \sigma_a g_{cb} - \sigma_c g_{ba} - \sigma_b g_{ac}$$

$$\text{or, } \Gamma^k_{bc} = \frac{1}{2} g^{ak}(g_{ac,b} + g_{ba,c} - g_{cb,a}) - \frac{1}{2} g^{ak}(g_{ba}\sigma_c + g_{ac}\sigma_b - g_{bc}\sigma_a)$$

$$\text{or, } \Gamma^k_{bc} = \begin{Bmatrix} k \\ bc \end{Bmatrix} - \frac{1}{2} g^{ak}(g_{ba}\sigma_c + g_{ac}\sigma_b - g_{bc}\sigma_a)$$

Thus the components of connection is completely determined in terms of the components of g and σ.

We say that the Weyl compatibility condition (5.1) may be interpreted as requiring that the co-variant derivative of the metric tensor g in the direction of a vector field $V \in T(M)$ does not vanish-as in Riemannian geometry but in stead that it be regulated by the Weyl field σ defined in the manifold M. Thus we have

$$\nabla g = \sigma \otimes g$$

where $\nabla g = \sigma \otimes g$ is the direct product of g and σ.

Let us discuss a geometrical property of Weyl parallel transport, which is given by the following corollary.

Corollary: Let M be differential manifold with an affine connection ∇, a metric g and a field of one form σ .If ∇ Weyl compatible, then for any smooth curve $\alpha = \alpha(\lambda)$ and any pair of two parallel vectors V and U along α ,we have

$$\frac{d}{d\lambda} g(V,U) = \sigma\left(\frac{d}{d\lambda}\right) g(V,U) \qquad 5.6$$

where $\frac{d}{d\lambda}$ denotes the vector tangent to the curve α .Let us integrate the above equation along α , starting from the point $p_0 = \alpha(\lambda_0)$.

$$\frac{\frac{d}{d\lambda} g(V,U)}{g(V,U)} = \sigma\left(\frac{d}{d\lambda}\right)$$

$$\Rightarrow \log\ g(V,U) = \int \sigma(\frac{d}{d\lambda}) d\lambda + \log C$$

$$\Rightarrow g\{V(\lambda)U(\lambda_0)\} = C\, e^{\int_{\lambda_0}^{\lambda} \sigma(\frac{d}{d\lambda}) d\lambda}$$

Applying initial condition we get

$$g\{U(\lambda_0)V(\lambda_0)\} = C$$

Thus we obtain

$$g\{V(\lambda)U(\lambda_0)\} = g\{U(\lambda_0)V(\lambda_0)\}\ e^{\int_{\lambda_0}^{\lambda} \sigma(\frac{d}{d\mu}) d\mu} \qquad 5.7$$

Putting V=U and denoting $L(\lambda)$, the length of the vector $V(\lambda)$ at an arbitrary point $p = \alpha(\lambda)$ of the curve,

$$|V| = \sqrt{|g(V,V)|}$$

$$= \sqrt{g_{ab} V^a V^b}$$

$$= \sqrt{V^a V_a}$$

Now in local co-ordinates $\{x^a\}$. Equation (4.6) leads to

$$\frac{d}{d\lambda} L^2 = \sigma_a\, dx^\alpha\, (\frac{d}{d\lambda})\ L^2$$

$$\Rightarrow 2L \frac{dL}{d\lambda} = \sigma_a \frac{dx^a}{d\lambda} L^2$$

$$\Rightarrow \frac{dL}{d\lambda} = \frac{\sigma_a}{2} \frac{dx^a}{d\lambda} L$$

Consider a set of all closed curves $\alpha : [a,b] \in R \to M$ i.e. with $\alpha(a) = \alpha(b)$. Then the equation

$$g\ \{V(b)U(b)\} = g\{U(a)V(a)\}\ e^{\int_a^b \sigma(\frac{d}{d\lambda}) d\lambda}$$

defines a holonomy group whose elements are in general ,a composition of homothetic transformation and an isometry. The elements of this group correspond to an isometry only when

$$\oint \sigma(\frac{d}{d\lambda}) d\lambda = 0 \quad ; \text{ for every loop .}$$

It follows from the Stokes theorem that σ must be exact form i.e there exist a scalar function ϕ such that $\sigma = d\phi$.Thus in that case we have Weyl integrable manifold.

Weyl manifold are completely characterized by the triple (M, g, σ), which is known as Weyl frame. It is noted that the compatibility condition (5.6) remains unchanged when transformed into another Weyl frame $(\overline{M}, \overline{g}, \overline{\sigma})$ by performing the following simultaneous transformation on g and σ

$$\overline{g} = e^{-\phi} g \qquad 5.8$$

and $$\overline{\sigma} = \sigma - d\phi \qquad 5.9$$

where ϕ is the scalar function defined on M. The conformal map (5.8) and the Gauge transformation (5.9) define classes of equivalence in the set of Weyl frames. The compatibility condition (5.6) led Weyl to his attempts at unifying gravity and electromagnetism –extending the concept of space time to that of collection of manifolds equipped with a conformal structure i.e the space time would be viewed as a class [g] of conformally equivalent Lorentzian metrices.

5.3 ISOMETRIES IN RIEMANNIAN SPACE:

Let M^n *and* N^n be two smooth manifolds with Riemannian structure g and χ respectively. The mapping $f : M^n \to N^n$ is called isometry if f is diffeomorphism and the reciprocal image $f^* \chi$ of χ is equal to g [11] i.e.

$$f^* \chi = g.$$

The induced metric $f^* \chi$ is sometimes called the first fundamental form of M^n.Two manifolds are said to be isometric if there exist an isometry of one onto another. The mapping $f : M^n \to N^n$ is called local isometry if for each $x \in M^n$ there exists a neighborhood U of x and V of $f(x)$ such that f is an isometry of U onto V. The isometry of M^n onto itself form a group.

5.4 SUBMANIFOLDS AND ISOMETRIC EMBEDDING IN WEYL GEOMETRY:

Let (M, g, σ) and $(\overline{M}, \overline{g}, \overline{\sigma})$ be differentiable Weyl manifolds of dimensions m and n =m+k respectively. A differentiable map $f : M \to \overline{M}$ is called an immersion if the following conditions are hold:

(a) the differential $f_* : T_P(M) \to T_{f(P)}\overline{M}$ is injective for any P∈M.

(b) $\sigma(V) = \overline{\sigma}(f_* V)$ for any $V \in T_P(M)$.

The number k is called the co-dimension of f .

The immersion $f : M \to \overline{M}$ is called isometric at a point P$\in$M if $g(U,V) = \overline{g}(f_*U, f_*V)$ for every U, V in the tangent space $T_P(M)$.If in addition, f is a homoeomorphism onto $f(M)$ then f is an embedding. If $M \subset \overline{M}$ and the inclusion $i : M \subset \overline{M} \to \overline{M}$ is an embedding then M is called a sub manifold of $\overline{M}$.It is important to note that locally any immersion is an embedding. Indeed $f : M \to \overline{M}$ be an immersion,then around each P$\in$M, there is a neighborhood U$\in$M such that the restriction of f to U is an embedding onto f (U).We may therefore identify U with its image under f, so that locally we can regard M as a sub-manifold embedded in $\overline{M}$ with f actually being the inclusion map. Thus we shall identify each vector $V \in T_P(M)$ with $f_* V \in T_{f(P)}(\overline{M})$ and consider $T_P(M)$ as a sub-space of $T_{f(P)}(\overline{M})$.

Now in the vector space $T_P(\overline{M})$, the metric $\overline{g}$ allows to make a decomposition $T_P(\overline{M}) = T_P(M) + T_P(M)^{\perp}$; where $T_P(M)^{\perp}$ is the orthogonal complement of $T_P(M) \subset T_P(\overline{M})$.That is for any vector $\overline{V} \in T_P(\overline{M})$ with P$\in$M, we can decompose $\overline{V}$ into $\overline{V} = V + V^{\perp}$, where $V \in T_P(M)$ and $V^{\perp} \in T_P(M)$.Let us denote Weyl connection on $\overline{M}$ by $\overline{\nabla}$ and prove the following theorem.

Theorem: If V and U are local vector fields on M and $\overline{V}$ and $\overline{U}$ are local extensions of these fields to $\overline{M}$, then the Weyl connection will be given by

$$\nabla_V U = (\overline{\nabla}_{\overline{V}} \overline{U})^T \qquad 5.10$$

where $(\overline{\nabla}_{\overline{V}} \overline{U})^T$ is the tangential component of $(\overline{\nabla}_{\overline{V}} \overline{U})$.

Proof: Let start with the Weyl compatibility condition

$$\overline{V}[\ g(\overline{U}, \overline{W})] = g(\nabla_{\overline{V}} \overline{U}, \overline{W}) + g(\overline{U}, \nabla_{\overline{V}} \overline{W}) + \overline{\sigma}(\overline{V})\, g(\overline{U}, \overline{W}) \qquad 5.11$$

where $\overline{V}, \overline{U}, \overline{W} \in T(\overline{M})$.Now suppose that $\overline{V}, \overline{U}, \overline{W}$ are local extension of the vector fields $V, U, W\ to\ M$. Clearly at a point P$\in$ M we have

$$\overline{V}[\overline{g}(\overline{U}, \overline{W})] = V[\overline{g}(\overline{U}, \overline{W})] = V[g(U, W)] \qquad 5.12$$

where we have taken into account that the inclusion of M into $\overline{M}$ is isometric. On the other hand evaluating separately each term of R.H.S of (5.12) at P yields

$$\overline{g}(\overline{\nabla}_{\overline{V}} \overline{U}, \overline{W}) = \ \{\overline{g}(\overline{\nabla}_{\overline{V}} \overline{U})^T + (\overline{\nabla}_{\overline{V}} \overline{U})^{\perp}, \overline{W}\}$$

$$= \overline{g}\{(\overline{\nabla}_{\overline{V}}\overline{U})^T, \overline{W}) = g\{(\overline{\nabla}_{\overline{V}}\vec{U})^T, \overline{W} \quad 5.13$$

with an analogous expression for $\overline{g}(\overline{\nabla}_{\overline{V}}\overline{U}, \overline{W})$. From the above equations and the fact that $\overline{\sigma}(\overline{V}) = \sigma(V)$ we finally obtain

$$V[\ g(U,W)] = g\{(\nabla_{\overline{V}}\overline{U})^T, \overline{W}\} + g\{\overline{U}, (\nabla_{\overline{V}}\overline{W})^T) + \sigma(V)\, g(U, \overline{W})$$

From the Levi-Civita theorem extended to Weyl manifold which asserts the uniqueness of affine connection ∇ in a Weyl manifold we conclude that (5.10) holds in other words the tangential components of the co-variant derivative $\overline{\nabla}_{\overline{V}}\overline{U}$ evaluated at points of M –is nothing more than the co-variant derivative of the induced Weyl connection from the metric g on M by

$$g(V,U) = \overline{g}(f_* V, f_* U)$$

5.5 EMBEDDING THE SPACE TIME IN WEYL BULK [17]:

It is possible to have a Riemannian sub manifold embedded in a Weyl ambient space, since a Riemannian manifold is a particular type of Weyl manifold in which the Weyl field σ vanishes. Therefore a sub manifold M embedded in a Weyl space $\overline{M}$ will be Riemannian if only if the field of one forms σ induced by pullback from $\overline{\sigma}$ vanishes throughout M. that is the necessary and sufficient condition for M to be an embedded manifold is that $\sigma(V) = 0$ for any $V \in T(M)$.

To illustrate the above, let us consider the case in which the manifold $\overline{M}$ is foliated by a family of sub manifolds defined by k equations $y^A = y_0^A$=constant, with the space time M corresponding to one of these manifolds $y^A = y_0^A$=constant. In local co ordinates $\{y^a\}$ of $\overline{M}$ adapted to the embedding the condition $\sigma(V) = 0$ reads $\sigma_\alpha V^\alpha = 0$ where $\sigma = \sigma_a dx^a$ *and* $V = V^\beta \partial_\beta$. In case of Weyl integrable manifold $\sigma = d\phi$. In this case $\sigma(V) = 0$ for any $V \in T(M)$ if only if $\frac{\partial \phi}{\partial x^\alpha} = 0$. Therefore in a Weyl integrable manifold if the scalar field is a function of the extra co- ordinate only, then the space time Sub- manifold M embedded in the bulk $\overline{M}$ is Riemannian.

The fact that Riemannian space-time M embedded in a Weyl bulk $\overline{M}$ does not mean that physical or geometrical effects coming from the extra co ordinate /dimensions should be absent. A nice interesting illustration of this point is given by the behavior of geodesic motion near the M. Thus a Weyl field may affect the geodesic motion in the case of bulk with warped product geometry.

5.6 GEODESIC MOTION IN A RIEMANNIAN WARPED PRODUCT SPACE:

Let us consider the case where the geometry of the bulk contains two special ingredients.

(a) It is Riemannian manifold

(b) Its metric has the structure of the warped product space.

The importance of warped product geometry is closely related to the so-called brain-world scenario. Let us consider the matter of geodesics in warped product spaces, first considering the Riemannian case. A warped product space is defined in the following way:

Let (M,g) and (N,h) be two Riemannian manifolds of dimension m and r with matrices g and h respectively. Consider a smooth function $f : N \to R$ which will be called warping function. Then we can construct a Riemanian manifold by setting $\overline{M} = M \times N$ and defining a metric $\overline{g} = e^{2f} g \oplus K$ where K is the tensor of type like as g. For simplicity let $M = M^4$ and N=R, where M^4 denotes four dimensional Lorentzian manifold with signature (+ - - -) referred to as space time.

In local co ordinates $\{y^a = (x^\alpha, y^4)\}$ the line element corresponding to this metric will be written as:

$$ds^2 = \overline{g}_{ab}\, dy^a\, dy^b$$

The equation of geodesic in five dimensional space time $\overline{M}$ will be given by :

$$\frac{d^2 y^a}{d\lambda^2} + {}^{(5)}\Gamma^a_{bc} \frac{dy^b}{d\lambda} \frac{dy^c}{d\lambda} = 0 \qquad 5.14$$

where λ is an affine parameter and ${}^{(5)}\Gamma^a_{bc}$ denotes the five dimensional Christoffel symbol of second kind defined as :

$${}^{(5)}\Gamma^a_{bc} = \frac{1}{2} \overline{g}^{ad} (\overline{g}_{db,c} + \overline{g}_{dc,b} - \overline{g}_{bc,d})$$

Denoting the fifth coordinate y^4 by y and the remaining coordinate y^μ (space time coordinate) by x^μ i.e. $y^a = (x^\mu, y)$ .We can separate the 4D – part of the geodesic equation (5.14) is as follows:

$$\frac{d^2 x^\mu}{d\lambda^2} + {}^{(4)}\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} + 2\, {}^{(5)}\Gamma^\mu_{\alpha 4} \frac{dx^\alpha}{d\lambda} \frac{dy}{d\lambda} + {}^{(5)}\Gamma^\mu_{44} (\frac{dy}{d\lambda})^2 = 0$$

or, $$\frac{d^2 x^\mu}{d\lambda^2} + \frac{1}{2} \overline{g}^{\mu k} (\overline{g}_{k\alpha,\beta} + \overline{g}_{k\beta,\alpha} - \overline{g}_{\alpha\beta,k}) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = - {}^{(5)}\Gamma^\mu_{44} (\frac{dy}{d\lambda})^2 - 2\, {}^{(5)}\Gamma^\mu_{4\alpha} \frac{dy}{d\lambda} \frac{dx^\alpha}{d\lambda}$$

$$\text{or,} \frac{d^2x^\mu}{d\lambda^2} + \frac{1}{2}\bar{g}^{\mu\nu}(\bar{g}_{\nu\alpha,\beta} + \bar{g}_{\nu\beta,\alpha} - \bar{g}_{\alpha\beta,\nu})\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} + \frac{1}{2}\bar{g}^{\mu 4}(\bar{g}_{4\alpha,\beta} + \bar{g}_{4\beta,\alpha} - \bar{g}_{\alpha\beta,4})\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$$
$$= -\,{}^{(5)}\Gamma^\mu_{44}\left(\frac{dy}{d\lambda}\right)^2 - 2\,{}^{(5)}\Gamma^\mu_{4\alpha}\frac{dy}{d\lambda}\frac{dx^\alpha}{d\lambda}$$

$$\text{or,} \quad \frac{d^2x^\mu}{d\lambda^2} + {}^{(4)}\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = -\,{}^{(5)}\Gamma^\mu_{44}\left(\frac{dy}{d\lambda}\right)^2 - 2\,{}^{(5)}\Gamma^\mu_{4\alpha}\frac{dy}{d\lambda}\frac{dx^\alpha}{d\lambda}$$
$$-\frac{1}{2}\bar{g}^{\mu 4}(\bar{g}_{4\alpha,\beta} + \bar{g}_{4\beta,\alpha} - \bar{g}_{\alpha\beta,4})\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}.$$

$$\text{or,} \quad \frac{d^2x^\mu}{d\lambda^2} + {}^{(4)}\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = \xi^\mu \qquad 5.15$$

where we define ξ^μ as

$$\xi^\mu = -\,{}^{(5)}\Gamma^\mu_{44}\left(\frac{dy}{d\lambda}\right)^2 - 2\,{}^{(5)}\Gamma^\mu_{4\alpha}\frac{dy}{d\lambda}\frac{dx^\alpha}{d\lambda} - \frac{1}{2}\bar{g}^{\mu 4}(\bar{g}_{4\alpha,\beta} + \bar{g}_{4\beta,\alpha} - \bar{g}_{\alpha\beta,4})\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$$

Let us turn our attention to the 5D brane world scenario where the bulk correspond to the 5D manifold $\bar{M}$ which was supposed to be foliated by a family of sub manifolds (in the case of hypersurface) defined by the equation y=constant.

It turns out that the geometry of a generic hypersurface, say y= y_0, will be determined by the induced metric $g_{\alpha\beta}(x) = \bar{g}_{\alpha\beta}(x, y_0)$. Thus on the hypersurface we have

$$ds^2 = \bar{g}_{\alpha\beta}(x, y_0)\,dx^\alpha dx^\beta$$

The quantities which appear on the L.H.S of (5.15) are to be identified with the Christoffel symbol associated with the metric in the leaves of the foliation above.

Let us consider the class of warped geometries given by the following line element:

$$ds^2 = e^{2f} g_{\alpha\beta}\,dx^\alpha dx^\beta - dy^2 \qquad 5.16$$

where $f = f(y)$ and $g_{\alpha\beta} = g_{\alpha\beta}(x)$.

For this metric we get ${}^{(5)}\Gamma^\mu_{44}=0$ and ${}^{(5)}\Gamma^\mu_{4\nu} = \frac{1}{2}\bar{g}^{\mu\beta}\,\bar{g}_{\beta\nu,4} = f'\delta^\mu_\nu$ where the prime sign denotes the derivative with respect to y .Thus in case of warped product space the R.H.S of (5.15) reduces to

$$\xi^{\mu} = -2f'(\frac{dx^{\mu}}{d\lambda})(\frac{dy}{d\lambda})$$

and hence the 4D –geodesic equation becomes

$$\frac{d^2x^{\mu}}{d\lambda^2} + {}^{(4)}\Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{d\lambda}\frac{dx^{\beta}}{d\lambda} = -2f'(\frac{dx^{\mu}}{d\lambda})(\frac{dy}{d\lambda}) \qquad 5.17$$

Again the geodesic equation for the fifth co-ordinate is given by

$$\frac{d^2y}{d\lambda^2} + {}^{(5)}\Gamma^{4}_{\alpha\beta}\frac{dx^{\alpha}}{d\lambda}\frac{dx^{\beta}}{d\lambda} + 2\,{}^{(5)}\Gamma^{4}_{4\alpha}\frac{dy}{d\lambda}\frac{dx^{\alpha}}{d\lambda} + {}^{(5)}\Gamma^{4}_{44}(\frac{dy}{d\lambda})^2 = 0$$

where,

$$\Gamma^{4}_{44} = \frac{1}{2}\bar{g}^{4\nu}(\bar{g}_{4\nu,4} + \bar{g}_{4\nu,4} - \bar{g}_{4\nu,4}) = 0$$

$$\Gamma^{4}_{\alpha 4} = \frac{1}{2}\bar{g}^{4\beta}(\bar{g}_{\beta\alpha,4} + \bar{g}_{\beta 4,\alpha} - \bar{g}_{\alpha 4,\beta}) = 0$$

And $\Gamma^{4}_{\alpha\beta} = \frac{1}{2}\bar{g}^{44}(\bar{g}_{\alpha 4,\beta} + \bar{g}_{\beta 4,\alpha} - \bar{g}_{\alpha\beta,4})$

$$= -\frac{1}{2}\bar{g}^{44}\bar{g}_{\alpha\beta,4}$$

$$= \frac{1}{2}.1.(e^{2f}g_{\alpha\beta})_{,4} \qquad \text{as } \bar{g}^{44} = -1$$

$$= f'e^{2f}g_{\alpha\beta}.$$

Hence the geodesic equation for the fifth co-ordinate become

$$\frac{d^2y}{d\lambda^2} + f'e^{2f}g_{\alpha\beta}\frac{dx^{\alpha}}{d\lambda}\frac{dx^{\beta}}{d\lambda} = 0$$

or, $$\frac{d^2y}{d\lambda^2} + f'[\bar{g}_{ab}\frac{dx^{a}}{d\lambda}\frac{dx^{b}}{d\lambda} - g_{44}(\frac{dy}{d\lambda})^2] = 0 \qquad 5.18$$

as $\bar{M} = M^4 \times R$. But for time like and null like geodesic we get,

$$\bar{g}_{ab}\frac{dx^{a}}{d\lambda}\frac{dx^{b}}{d\lambda} = 1$$

and $$\bar{g}_{ab}\frac{dx^{a}}{d\lambda}\frac{dx^{b}}{d\lambda} = 0$$ respectively.

Hence equation (5.17) becomes for time like geodesic

$$\frac{d^2y}{d\lambda^2} + f'[1 + (\frac{dy}{d\lambda})^2] = 0 \qquad 5.19$$

and for null like geodesic ,

$$\frac{d^2y}{d\lambda^2} + f'(\frac{dy}{d\lambda})^2 = 0 \qquad 5.20$$

equation (5.18) and (5.19) are ordinary differential equation of second order in principle – can be solved if $f' = f'(y)$ is known. A qualitative picture of the motion in the fifth dimension may be obtained by defining the variable $q = \frac{dy}{d\lambda}$ and then investigating the autonomous dynamical system:

$$q = \frac{dy}{d\lambda} \quad \text{and} \quad \frac{dq}{d\lambda} = P(q, y) \qquad 5.21$$

with P(q,y) $= -f'(\in + q^2)$ where $\in = 1$ in case of (5.19) and $\in = 0$ in case of (5.20).In the investigation of dynamical system a crucial role is played by their equilibrium points which in case of (5.21) are given by

$$\frac{dy}{d\lambda} = 0 \quad \text{and} \quad \frac{dq}{d\lambda} = 0 \qquad 5.22$$

the knowledge of these points together with their stability properties provides a huge information on the types of behavior allowed by the system.

A. The case of massive particles: In case of non zero rest mass particles the motion in the fifth dimension is governed by the dynamical system

$$\frac{dy}{d\lambda} = q \qquad 5.23$$

and $$\frac{dq}{d\lambda} + f'(1+q^2) = 0$$

$$\Rightarrow \frac{dq}{d\lambda} = -f'(1+q^2) \qquad 5.24$$

The critical points of (5.23) are given by $q = 0$ and the zeros of the function $f'(y)$ (if they exist) which we generally denote by y_0.These solution pictured as isolated points in the phase plane, correspond to curves which lie entirely on a hypersurface M of our foliation (since for them y=constant).It turns out these curves are time like geodesics with respect to the hyper - surface induced geometry.

To obtain information about the possible modes of behavior of particles and light rays in such hypersurface, we test the nature and stability of the corresponding equilibrium points. This can be done by linearising equation (5.23) and studying the eigen values of the Jacobean matrix about the equilibrium points .Assuming that the function $f'(y)$ vanishes at least at one points y_0. it can readily be shown that the corresponding eigen values are determined by the sign of second derivative $f''(y_0)$ at the equilibrium points and some possibilities arise for the equilibrium points of the dynamical system (5.23).

We shall discuss only the following three cases:

Case 1: If $f''(y_0) \rangle 0$, then the equilibrium point $(q = 0, y = y_0)$ is a center. Thus correspond to the case in which the massive particles oscillate about the hypersurface $\mathrm{M}(y = y_0)$. Such cyclic motion is independent of the ordinary 4D-space time dimensions and except for the conditions $f'(y_0) = 0$ and $f'(y_0) \rangle 0$, the warping function completely remains arbitrary.

Case 2: If $f''(y_0) \ \langle\ 0$, then the point $(q = 0, y = y_0)$ is a saddle point. In this case the solution corresponding to the equilibrium point is highly unstable and the smallest transversal perturbation in the motion of particles along the brain will cause them to be expelled into the extra dimension. An example of this unstable “confinement” at the hypersurface y=0 is provided by Gremms warping function

$$f(y) = -b|\ln Cosh(cy) \qquad 5.25$$

where b and c are positive constant.

Case 3: There are no equilibrium points at all. The warping function $f(y)$ does not have any turning points for any value of y. This implies that in this we can’t have confinements of classical particles to hypersurfaces solely due to gravitational effects.

An example of this situation is illustrated by warping function

$$f(y) = \frac{1}{2}\ln(\Lambda\frac{y^2}{3}) \qquad 5.26$$

Similarly for a large value of y the warping function (L) approaches that of the Randall-Sundrum metric

$$ds^2 = e^{-2k|y|}\ \eta_{\alpha\beta}dx^{\alpha}\ dx^{\beta} - dy^2 \qquad 5.27$$

where k is constant. In this case $f'(y) = \mp k$ according to whether y is positive or negative. Again there exist no equilibrium points and therefore no confinement of particles is possible due only to gravity.

B .The case of Photon: The motion of photon is governed by the dynamical system

$$\frac{dy}{d\lambda} = q$$

and $$\frac{dq}{d\lambda} = -f'q^2 \qquad 5.28$$

The equilibrium points in this case are given by $q = 0$, so they consist of a line of equilibrium points along the Y-axis with eigen values both equal to zero.

Any point along the Y axis is an equilibrium point and correspond to a 5D null geodesics in the hyper surface y = constant. The existence of photons confined to hyper surfaces does not depend on the warping factor.

As well known, in the brain world scenario the stability of the confinement of matter fields at the quantum level is made possible by assuming an interaction of matter with a scalar field. An example of how this mechanism works is clearly illustrated by a field theoretical model devised by Rubakov in which fermions may be trapped to a brain by interacting with a scalar field that depend only on the extra dimension. On the other hand the kind of confinement we are concerned which is purely geometrical and that means the only force acting on the particle is the gravitational force. In a purely classical picture (non-quantum) one would like to have effective mechanisms other then a quantum scalar field in order to constrain massive particle to move on a hypersurface in a stable way. At this point two possibilities arises. One is to assume direct interaction between the particle and a physical scalar field. Following this approach it has been shown that stable confinement in a thick brane is possible by means of direct interaction of the particles with a scalar field through a modification of the Lagrangian of the particle. Another approach would appeal to pure geometry: for instance modeling the bulk with a Weyl geometrical structure. As we shall see in this case the Weyl field may provide the mechanism necessary for confinement and stabilization of the motion of particles in the brane.

5.7 GEODESIC MOTION IN PRESENCE OF WEYL FIELD:

We shall discuss the geodesic motion pictured in a Weyl field. Let us consider the case the warped product bulk is an integrable Weyl manifold $(\overline{M}, \overline{g}, \phi)$. If the Weyl scalar depends only on the extra co-ordinate, then Weyl field of one forms $\sigma = d\phi$ induced on the hyper surface of the foliation vanishes. Indeed any tangent vector V of a given leaf has the form

$$V = V^{\alpha}\partial_{\alpha}$$

Thus we have $\sigma(V) = V^{\alpha}(\frac{\partial\phi}{\partial x^{\alpha}}) = 0$. Therefore if M represent our space time embedded in an integrable Weyl bulk $\overline{M}$ with $\phi = \phi(y)$ then we can sure that M has a Riemannian structure. In a Weyl manifold the co efficient of Weyl connection Γ^{a}_{bc} are related to the Christoffel symbol as:

$$\Gamma^{a}_{bc} = \{^{\,a}_{bc}\} - \frac{1}{2}\overline{g}^{ad}(\overline{g}_{db}\sigma_c + \overline{g}_{dc}\sigma_b - \overline{g}_{bc}\sigma_d)$$

Now the geodesic equation of the fifth co-ordinate y in the warped product space for a massive particle is obtained from the definition given above. The geodesic equation in the fifth co-ordinate is given by

$$\frac{d^2y}{d\lambda^2} + {}^{(5)}\Gamma^{4}_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = 0$$

or, $$\frac{d^2y}{d\lambda^2} + \left[\{^{\,4}_{bc}\} - \frac{1}{2}\overline{g}^{4d}(\overline{g}_{db}\sigma_c + \overline{g}_{dc}\sigma_b - \overline{g}_{bc}\sigma_d)\right]\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = 0$$

or, $$\frac{d^2y}{d\lambda^2} + \left[\frac{1}{2}\overline{g}^{4a}(\overline{g}_{ab,c} + \overline{g}_{ac,b} - \overline{g}_{bc,a}) - \frac{1}{2}\overline{g}^{4d}(\overline{g}_{db}\sigma_c + \overline{g}_{dc}\sigma_b - \overline{g}_{bc}\sigma_d)\right]\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = 0$$

or, $$\frac{d^2y}{d\lambda^2} + \left[\frac{1}{2}\overline{g}^{44}(\overline{g}_{4b,c} + \overline{g}_{4c,b} - \overline{g}_{bc,4}) - \frac{1}{2}\overline{g}^{44}(\overline{g}_{4b}\sigma_c + \overline{g}_{4c}\sigma_b - \overline{g}_{bc}\sigma_4)\right]\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = 0$$

or, $$\frac{d^2y}{d\lambda^2} + \left[\frac{1}{2}\overline{g}^{44}(-\overline{g}_{bc,4}) - \frac{1}{2}\overline{g}^{44}(-\overline{g}_{bc}\sigma_4)\right]\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = 0$$

as $\overline{g}_{b4} = \overline{g}_{4c} = 0 \ (b, c = 0,1,2,3)$

or, $$\frac{d^2y}{d\lambda^2} + \frac{1}{2}\left[\overline{g}_{bc,4} - \overline{g}_{bc}\sigma_4\right]\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = 0$$

or, $$\frac{d^2y}{d\lambda^2} + \frac{1}{2}\overline{g}_{bc,4}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} - \frac{1}{2}\overline{g}_{bc}\sigma_4\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = 0$$

$$\text{or, } \frac{d^2y}{d\lambda^2} + \frac{1}{2}\left[(e^{2f}\, g_{\alpha\beta})_{,4} \frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} - (g_{44})_{,4}(\frac{dy}{d\lambda})^2 \right]$$

$$-\frac{1}{2}\left[(e^{2f}\, g_{\alpha\beta})\sigma_4 \frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} - g_{44}\,\sigma_4\,(\frac{dy}{d\lambda})^2 \right] = 0$$

$$\text{or, } \frac{d^2y}{d\lambda^2} + f'e^{2f}g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} - \frac{1}{2}\,\sigma_4\left[e^{2f}\, g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} + (\frac{dy}{d\lambda})^2 \right] = 0$$

$$\text{or, } \frac{d^2y}{d\lambda^2} + f'\left[\bar{g}_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda} - \bar{g}_{44}\,(\frac{dy}{d\lambda})^2 \right] - \frac{1}{2}\sigma_4\left[\bar{g}_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda} - \bar{g}_{44}(\frac{dy}{d\lambda})^2 \right]$$

$$-\frac{1}{2}\sigma_4(\frac{dy}{d\lambda})^2 = 0. \qquad 5.29$$

$$\text{or, } \frac{d^2y}{d\lambda^2} + f'\left[1+(\frac{dy}{d\lambda})^2\right] - \frac{1}{2}\sigma_4\left[1+(\frac{dy}{d\lambda})^2\right] - \frac{1}{2}\sigma_4(\frac{dy}{d\lambda})^2 = 0$$

$$\text{or, } \frac{d^2y}{d\lambda^2} + f'\left[1+(\frac{dy}{d\lambda})^2\right] - \sigma_4\left[\frac{1}{2}+(\frac{dy}{d\lambda})^2\right] = 0$$

$$\text{or, } \frac{d^2y}{d\lambda^2} + f'\left[1+(\frac{dy}{d\lambda})^2\right] - \sigma(e_4)\left[\frac{1}{2}+(\frac{dy}{d\lambda})^2\right] = 0$$

$$\text{or, } \frac{d^2y}{d\lambda^2} + f'\left[1+(\frac{dy}{d\lambda})^2\right] - d\phi\,(\frac{\partial}{\partial x^4})\left[\frac{1}{2}+(\frac{dy}{d\lambda})^2\right] = 0$$

$$\text{or, } \frac{d^2y}{d\lambda^2} + f'\left[1+(\frac{dy}{d\lambda})^2\right] - (\frac{d\phi}{\partial x^4})\left[\frac{1}{2}+(\frac{dy}{d\lambda})^2\right] = 0$$

$$\text{or, } \frac{d^2y}{d\lambda^2} + f'\left[1+(\frac{dy}{d\lambda})^2\right] - \phi'\left[\frac{1}{2}+(\frac{dy}{d\lambda})^2\right] = 0 \qquad 5.30$$

where $\phi' = \frac{d\phi}{dy}$. On the other hand for photon ,putting $\bar{g}_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda} = 0$ in equation (5.29) we obtain

$$\frac{d^2 y}{d\lambda^2} + f'(\frac{dy}{d\lambda})^2 - \phi'(\frac{dy}{d\lambda})^2 = 0$$

or, $$\frac{d^2 y}{d\lambda^2} + (f' - \phi')(\frac{dy}{d\lambda})^2 = 0 \qquad 5.31$$

Equation (5.30) and (5.31) respectively define the following dynamical system.

$$\frac{dy}{d\lambda} = q$$

$$\frac{dq}{d\lambda} = (\phi' - f')q^2 + \frac{\phi'}{2} - f'$$

and

$$\frac{dy}{d\lambda} = q$$

$$\frac{dq}{d\lambda} = (\phi' - f')q^2$$

the presence of the derivative of the Weyl scalar in the above equation may completely change the picture of the solution of determined by the dynamical system considered before. This is due to the existence of equilibrium points. There topology and stability properties now depend not only on the values of the derivative of the warping function take at the brane but also on the derivative of the Weyl scalar field $\phi(y)$.

Finally, in case of photon the Weyl scalar field ϕ has no influence on the confinement. The presence of scalar Weyl is equivalent to perform a conformal transformation in the Riemannian metric $\overline{g} = e^{2f} g \oplus k$. This essentially result in change the warping function from f to $f - \phi/2$. Because the existance of confined photon in the hypersurface is independent of the warping function, the Weyl scalar has no effect in the confinement.

Chapter six

SOLUTION OF EINSTEIN FIELD EQUATION IN ROTATING FRAME

6.0 INTRODUCTION:

The discovery of a class of stationary solution of Einstein vacuum field equation i.e. the Kerr metric (1963) and the proof of its unique role in the physics of black hole have made an immense impact on the development of general relativity and astrophysics. This can hardly be be more eloquently demonstrated than by an emotional text from Chandrasekhar:

" In my entire scientific life –extending to forty five years ,the most shattering expression has been the realization that an exact solution of Einstein field equation of general relativity discovered by the New zeland mathematician Roy Kerr , provides the absolutely exact representation of untold number of massive black hole that populate the universe……."

In this chapter we derive an axially symmetric metric based on two physical assumptions i.e. steady rotation of star and the field around it is axially symmetric. Then we found the Einstein vacuum field equation for that metric .From these equations we derive the Ernst form of Einstein equation and also express it in terms of spheroid al co-ordinates. A systematic mathematical formulation of this Ernst form of Einstein equation led to the required Kerr solution in Boyer-Lindquist form.

This chapter is mainly quoted from the book [7].Beside this, the following books are used as references : [9] ,[21].

SOLUTION OF EINSTEIN FIELD EQUATION IN ROTATING FRAME

6.1 AXIALLY SYMMETRIC STATIONARY METRIC:

To derive an axially symmetric stationary metric consider a suitable co-ordinate system and some physical assumptions. The first assumption is that, the field is generated by time independent (steady) rotation of a star made of perfect fluid and its energy momentum tensor is given by

$$T^{\mu\nu} = (\rho + P)U^{\mu}U^{\nu} - Pg^{\mu\nu} \qquad 6.1$$

Again the second is that, the star and the field around it posses axial symmetry about the axis of rotation which passes through the center of star. This center of star will be treated here as an origin of co-ordinate system and the axis of rotation is the Z-axis. Due to time independence and axial symmetry it is reasonable to assume the existence of time like co-ordinate $x^0 = t$ and an angular co-ordinate $x^3 = \varphi$ respectively of which the metric co-efficient and all the matter variables are independent. Hence consider a co-ordinate system (x^0, x^1, x^2, φ) such that

$$\frac{\partial g_{bc}}{\partial x^0} = \frac{\partial g_{bc}}{\partial \varphi} = 0 \qquad 6.2$$

as φ is the angular co-ordinate about the axis of rotation, the co-ordinate values $(t.x^1, x^2, \varphi)$ and $(t.x^1, x^2, \varphi + 2\pi)$ correspond to the same point in the space time :

$$(t, x^1, x^2, \varphi) = (t, x^1, x^2, \varphi + 2\pi)$$

The metric as well as the field generated by the rotation of star is not invariant under the transformation $t \to -t$,since such a transformation would reverse the sense of rotation resulting in a different space time or invariant under the transformation $\varphi \to -\varphi$ since such transformation would also reverse the sense of rotation. But invariant under a simultaneous reversal of t and φ .i.e. $(t, \varphi) \to (-t, -\varphi)$ .Thus we get

$$g_{01} = g_{02} = g_{13} = g_{23} = 0$$

The most general such metric can be written as:

$$ds^2 = g_{00}dt^2 + 2g_{03}dt\,d\varphi + g_{33}d\varphi^2 + g_{AB}dx^A dx^B \qquad 6.3$$

Let us write the metric (3) in the following way,

$$ds^2 = fdt^2 - 2kdtd\varphi - ld\varphi^2 - Ad\rho^2 - 2Bd\rho\,dz - Cdz^2 \qquad 6.4$$

where f, k, l, A, B, C are function of ρ and z. Let us consider an arbitrary co-ordinate transformation (ρ, z) to (ρ', z') as follows :

$$\rho' = F(\rho, z) \quad \text{and} \quad z' = G(\rho, z) \tag{6.5}$$

Now taking differential we can write

$$d\rho' = F_1 d\rho + F_2 dz$$
$$\Rightarrow d\rho = \frac{1}{F_1}(d\rho' - F_2 dz) \tag{6.6}$$

And, $dz' = G_1 d\rho + G_2 dz$

$$\Rightarrow dz = \frac{1}{G_2}(dz' - G_1 d\rho) \tag{6.7}$$

where $\frac{\partial F}{\partial \rho} \equiv F_1, \frac{\partial F}{\partial z} \equiv F_2$ and $\frac{\partial G}{\partial \rho} \equiv G_1, \frac{\partial G}{\partial z} \equiv G_2$. 6.8

From equation (6.6) we get,

$$d\rho = \frac{1}{F_1}\left[d\rho' - \frac{F_2}{G_2}dz' + \frac{F_2}{G_2}G_1 d\rho\right]$$

$$\Rightarrow d\rho = \frac{1}{F_1 G_2}\left[G_2 d\rho' - G_2 dz' + F_2 G_1 d\rho\right]$$

$$\Rightarrow d\rho = \frac{1}{F_1 G_2 - F_2 G_1}(G_2 d\rho' - F_2 dz')$$

We get the Jacobean of transformation as

$$J = \frac{\partial(F, G)}{\partial(\rho, z)} = F_1 G_2 - F_2 G_1$$

Hence we obtain

$$d\rho = J^{-1}(G_2 d\rho' - F_2 dz') \tag{6.9}$$

And similarly,

$$dz = J^{-1}(-G_1 d\rho' + F_1 dz') \tag{6.10}$$

Now substituting the value of $d\rho$ and dz in the equation (6.4)

$$\begin{aligned} ds^2 = \; & fdt^2 - 2kdtd\varphi - ld\varphi^2 - Ad\rho^2 - J^{-2}[(AG_2^2 - 2BG_1G_2 + CG_1^2)d\rho^2 \\ & + 2\{-AG_2F_2 + B(G_2F_1 + G_1F_2) - CG_1F_1\}d\rho' dz' \\ & + (AF_2^2 - 2BF_1F_2 + CF_1^2)dz'^2] \end{aligned} \tag{6.11}$$

The functions F and G are so far arbitrary and are required to satisfy the following two coupled non linear partial differential equations as a function of ρ and z.

$$AG_2^2 - 2BG_1G_2 + CG_2^2 = AF_2^2 - 2BF_1F_2 + CF_1^2 \qquad 6.12$$

and $$-AG_2F_2 + B(G_2F_1 + G_1F_2) - CG_1F_1 = 0 \qquad 6.13$$

where we assume that for given *A*, *B*, *C* the system of equation (6.12) and (6.13) has a non trival solution with $J \neq 0$ .Then in the co-ordinate system (ρ', z') the metric (6.11) has its co-efficient of $d\rho'^2$ equal to its co-efficient of dz'^2 and the co-efficient of $d\rho' dz'$ vanishes. Now write the metric (6.11) by dropping out prime sign as follows [7]:

$$ds^2 = fdt^2 - 2kdtd\varphi - ld\varphi^2 - e^{\mu}(d\rho^2 + dz^2) \qquad 6.14$$

where f, k, l are not same as in (5.11), but f, k, l, μ in (6.14) are function of ρ and z.

Equation (6.14) is known as the Weyl Papapetrou form where $(x^1 = \rho, x^2 = z)$ and let $a = e^{\mu}$.

From the metric (6.14) we get the following metric components:

$$g_{00} = f,\ g_{03} = g_{30} = -k,\quad g_{11} = g_{22} = -a,\ g_{33} = -l$$

and $g^{00} = \dfrac{l}{(lf + k^2)},\ g^{11} = g^{22} = \dfrac{-1}{a},\ g^{03} = g^{30} = \dfrac{-k}{(lf + k^2)},\ g^{33} = \dfrac{-l}{(lf + k^2)}$

Also $\mathbf{det}(g_{bc}) = g = -a^2(lf + k^2)$.

From definition we get $\Gamma^a_{bc} = \dfrac{1}{2} g^{ap}(g_{pb,c} + g_{pc,b} - g_{bc,p})$

Then we can calculate the non-zero Γ 's for the metric (6.14)

$$\Gamma^0_{01} = \frac{(lf_\rho + kk_\rho)}{2(lf + k^2)}$$

$$\Gamma^0_{02} = \frac{(lf_z + kk_z)}{2(lf + k^2)}$$

$$\Gamma^0_{13} = \frac{(-lk_\rho + kl_\rho)}{2(lf + k^2)}$$

$$\Gamma^0_{23} = \frac{(-lk_z + kl_z)}{2(lf + k^2)}$$

$$\Gamma^1_{00} = \frac{f_\rho}{2a}$$

$$\Gamma^1_{11} = \frac{a_\rho}{2a}$$

$$\Gamma^1_{12} = \frac{a_z}{2a}$$

$$\Gamma^2_{00} = \frac{f_z}{2a}$$

$$\Gamma^2_{03} = \frac{-k_z}{2a}$$

$$\Gamma^2_{11} = \frac{-a_z}{2a}$$

$$\Gamma^2_{12} = \frac{a_\rho}{2a}$$

$$\Gamma^2_{22} = \frac{a_z}{2a}$$

$$\Gamma^2_{33} = \frac{-l_z}{2a}$$

$$\Gamma^3_{10} = \frac{(fk_\rho - kf_\rho)}{2(lf + k^2)}$$

$$\Gamma^1_{03} = \frac{-k_\rho}{2a} \qquad \Gamma^3_{20} = \frac{(fk_z - kf_z)}{2(lf + k^2)}$$

$$\Gamma^1_{22} = \frac{-a_\rho}{2a} \qquad \Gamma^3_{13} = \frac{(kk_\rho + fl_\rho)}{2(lf + k^2)}$$

$$\Gamma^1_{33} = \frac{-l_\rho}{2a} \qquad \Gamma^3_{23} = \frac{(kk_z + fl_z)}{2(lf + k^2)}$$

6.2 EINSTEIN EQUATION FOR ROTATING METRIC:

Now we are interested to find out the Einstein vacuum equation for the metric (6.14) i.e. for the rotating metric.

The vacuum Einstein equations are given by

$$R_{\mu\nu} = 0 \qquad 6.15$$

Let us level the co-ordinates as $(x^0, x^1, x^2, x^3) = (t, \rho, z, \varphi)$. The Riemannian Christoffel curvature tensor is given by:

$$R^p_{bcd} = \Gamma^p_{bd,c} - \Gamma^p_{bc,d} + \Gamma^h_{bd}\Gamma^p_{hc} - \Gamma^h_{bc}\Gamma^p_{hd}$$

Then contracting on p and d we obtain the Ricci tensor.

$$R_{bc} \equiv \Gamma^p_{bp,c} - \Gamma^p_{bc,p} + \Gamma^h_{bp}\Gamma^p_{hc} - \Gamma^h_{bc}\Gamma^p_{hp} \qquad 6.16$$

$$\therefore\ R_{00} = -\Gamma^1_{00,1} - \Gamma^2_{00,2} + \{\Gamma^1_{00}\Gamma^0_{10} + \Gamma^2_{00}\Gamma^0_{20} + \Gamma^0_{01}\Gamma^1_{00} + \Gamma^3_{01}\Gamma^1_{30} + \Gamma^0_{02}\Gamma^2_{00} + \Gamma^3_{02}\Gamma^2_{30}$$

$$+ \Gamma^1_{03}\Gamma^3_{10} + \Gamma^2_{03}\Gamma^3_{20}\} - \{\Gamma^1_{00}\Gamma^0_{10} + \Gamma^2_{00}\Gamma^0_{20} + \Gamma^1_{00}\Gamma^1_{11} + \Gamma^2_{00}\Gamma^1_{21} + \Gamma^1_{00}\Gamma^2_{12} + \Gamma^2_{00}\Gamma^2_{22}$$

$$= -\left(\frac{f_\rho}{2a}\right)_\rho - \left(\frac{f_z}{2a}\right)_z + \frac{f_\rho}{2a}\frac{(lf_\rho + kk_\rho)}{2(lf + k^2)} + \frac{f_z}{2a}\frac{(lf_z + kk_z)}{2(lf + k^2)} + \frac{f_\rho}{2a}\frac{(lf_\rho + kk_\rho)}{2(lf + k^2)}$$

$$+ \frac{(fk_\rho - kf_\rho)}{2(lf + k^2)}\frac{(-k_\rho)}{2a} + \frac{f_z}{2a}\frac{(lf_z + kk_z)}{2(lf + k^2)} + \frac{(fk_z - kf_z)}{2(lf + k^2)}\frac{(-k_z)}{2a} + \frac{(fk_\rho - kf_\rho)}{2(lf + k^2)}\frac{(-k_\rho)}{2a}$$

$$+ \frac{(fk_z - kf_z)}{2(lf + k^2)}\frac{(-k_z)}{2a} - \{\ \frac{f_\rho}{2a}\frac{(lf_\rho + kk_\rho)}{2(lf + k^2)} + \frac{f_z}{2a}\frac{(lf_z + kk_z)}{2(lf + k^2)} + \frac{f_\rho}{2a}\frac{a_\rho}{2a} + \frac{f_z}{2a}\frac{a_z}{2a} + \frac{f_\rho}{2a}\frac{a_\rho}{2a}$$

$$+ \frac{f_z}{2a}\frac{a_z}{2a} + \frac{f_\rho}{2a}\frac{(fl_\rho + kk_\rho)}{2(lf + k^2)} + \frac{f_z}{2a}\frac{(fl_z + kk_z)}{2(lf + k^2)}.$$

$$or, R_{00} = \frac{-f_{\rho\rho}}{2a} - \frac{f_{zz}}{2a} + \frac{a_\rho f_\rho}{2a^2} + \frac{a_z f_z}{2a^2} - \frac{fk_\rho^2 - kk_\rho f_\rho}{2a(lf+k^2)} - \frac{f k_z^2 - kk_z f_z}{2a(lf+k^2)} + \frac{lf_\rho^2 + kk_\rho f_\rho}{4a(lf+k^2)} + \frac{lf_z^2 + kk_z f_z}{4a(lf+k^2)} - \frac{ff_z l_z + kk_z f_z}{4a(lf+k^2)} - \frac{ff_\rho l_\rho + kk_\rho f_\rho}{4a(lf+k^2)} - \frac{a_\rho f_\rho}{2a^2} - \frac{a_z f_z}{2a^2}$$

$$or,\ R_{00} = \frac{-f_{\rho\rho}}{2a} - \frac{f_{zz}}{2a} + \frac{1}{4a(lf+k^2)}\{lf_\rho^2 + lf_z^2 + 2kk_z f_z + 2kk_\rho f_\rho - 2fk_\rho^2 - 2fk_z^2 - fl_\rho f_\rho - fl_z f_z\}$$

$$or, -2aD^{-1}R_{00} = (D^{-1}f_\rho)_\rho + (D^{-1}f_z)_z + D^{-3}f(l_\rho f_\rho + l_z f_z + k_\rho^2 + k_z^2)$$

where $D^2 = lf + k^2;\ D^{-1} = 1/\sqrt{lf+k^2}$

$$or, -2e^\mu D^{-1}R_{00} = (D^{-1}f_\rho)_\rho + (D^{-1}f_z)_z + D^{-3}f(l_\rho f_\rho + l_z f_z + k_\rho^2 + k_z^2) = 0 \qquad 6.17$$

with the help of (6.15)

Again,

$$R_{03} = \frac{k_{\rho\rho}}{2a} - \frac{k_\rho a_\rho}{2a^2} + \frac{k_{zz}}{2a} - \frac{k_z a_z}{2a^2} + \frac{f_\rho}{2a}\frac{(kl_\rho - lk_\rho)}{2(lf+k^2)} + \frac{f_z}{2a}\frac{(kl_z - lk_z)}{2(lf+k^2)} + \frac{(lf_\rho + kk_\rho)}{2(lf+k^2)}\frac{(-k_\rho)}{2a} + \frac{(fk_\rho - kf_\rho)}{2(lf+k^2)}\frac{(-l_\rho)}{2a} + \frac{(lf_z + kk_z)}{2(lf+k^2)}\frac{(-k_z)}{2a} + \frac{(fk_z - kf_z)}{2(lf+k^2)}\frac{(-l_z)}{2a} + \frac{(kk_\rho + fl_\rho)}{2(lf+k^2)}\frac{(-k_\rho)}{2a} + \frac{(kk_z + fl_z)}{2(lf+k^2)}\frac{(-k_z)}{2a} - \{\frac{(lf_\rho + kk_\rho)}{2(lf+k^2)}\frac{(-k_\rho)}{2a} - \frac{k_\rho}{2a}\frac{a_\rho}{2a} - \frac{k_z}{2a}\frac{a_z}{2a} - \frac{k_\rho}{2a}\frac{a_\rho}{2a} - \frac{k_z}{2a}\frac{a_z}{2a} - \frac{k_\rho}{2a}\frac{(kk_\rho + fl_\rho)}{2(lf+k^2)} - \frac{k_z}{2a}\frac{(kk_z + fl_z)}{2(lf+k^2)} - \frac{k_z}{2a}\frac{(lf_z + kk_z)}{2(lf+k^2)}\}$$

$$or,\ R_{03} = \frac{k_{\rho\rho}}{2a} + \frac{k_{zz}}{2a} + \frac{f_\rho}{2a}\frac{kl_\rho - lk_\rho}{2(lf+k^2)} + \frac{f_z}{2a}\frac{kl_z - lk_z}{2(lf+k^2)} - \frac{l_\rho}{2a}\frac{fk_\rho - kf_\rho}{2(lf+k^2)} - \frac{l_z}{2a}\frac{fk_z - kf_z}{2(lf+k^2)}$$

$$or, R_{03} = \frac{k_{\rho\rho}}{2a} + \frac{k_{zz}}{2a} + \frac{1}{4a(lf+k^2)}\{2kl_\rho f_\rho - lf_\rho k_\rho + 2kf_z l_z - lk_z f_z - fl_\rho k_\rho - fk_z l_z\}$$

$$or, 2aD^{-1}R_{03} = (D^{-1}k_\rho)_\rho + (D^{-1}k_z)_z + D^{-3}k(l_\rho f_\rho + f_z l_z + k_\rho^2 + k_z^2)$$

$$or, 2e^\mu D^{-1}R_{03} = (D^{-1}k_\rho)_\rho + (D^{-1}k_z)_z + D^{-3}k(l_\rho f_\rho + f_z l_z + k_\rho^2 + k_z^2) = 0 \qquad 6.18$$

Similarly,

$$-2e^\mu D^{-1}R_{33} = (D^{-1}l_\rho)_\rho + (D^{-1}l_z)_z + D^{-3}l(l_\rho f_\rho + f_z l_z + k_\rho^2 + k_z^2) = 0 \qquad 6.19$$

We have $D^2 = lf + k^2$; where D can be considered as the real part of an analytic function **analytic** $(\rho + iz)$ of $(\rho + iz)$ .Let E be the imaginary part of **analytic** $(\rho + iz)$. i.e.

analytic $(\rho + iz) = \quad D(\rho, z) + iE(\rho, z)$

Now,

$$D_\rho = \frac{1}{2\sqrt{lf + k^2}}(lf_\rho + fl_\rho + 2kk_\rho)$$

$$\therefore D_{\rho\rho} = \frac{1}{2\sqrt{lf + k^2}}(l_{\rho\rho} f + 2l_\rho f_\rho + lf_{\rho\rho} + 2k_\rho^2 + 2kk_{\rho\rho}) - \frac{1}{4(lf + k^2)^{3/2}}(lf_\rho + fl_\rho + 2kk_\rho)^2$$

And, $D_{zz} = \frac{1}{2\sqrt{(lf + k^2)}}(l_{zz} f + 2l_z f_z + lf_{zz} + 2k_z^2 + 2kk_{zz}) - \frac{1}{4(lf + k^2)^{3/2}}(lf_z + fl_z + 2kk_z)^2$

Then we can calculate,

$$\begin{aligned} D_{\rho\rho} + D_{zz} &= \frac{1}{2\sqrt{lf + k^2}}(l_{\rho\rho} f + 2l_\rho f_\rho + lf_{\rho\rho} + 2k_\rho^2 + 2kk_{\rho\rho}) \\ &- \frac{1}{4(lf + k^2)^{3/2}}(lf_\rho + fl_\rho + 2kk_\rho)^2 - \frac{1}{4(lf + k^2)^{3/2}}(lf_z + fl_z + 2kk_z)^2 \\ &+ \frac{1}{2\sqrt{(lf + k^2)}}(l_{zz} f + 2l_z f_z + lf_{zz} + 2k_z^2 + 2kk_{zz}) \end{aligned}$$

$$\begin{aligned} &= a(lf + k^2)^{-1/2}[\, -l\{-\frac{f_{\rho\rho}}{2a} - \frac{f_{zz}}{2a} + \frac{1}{4a(lf + k^2)}(lf_\rho^2 + lf_z^2 + 2kk_z f_z \\ &+ 2kk_\rho f_\rho - 2fk_\rho^2 - 2fk_z^2 - fl_\rho f_\rho - fl_z f_z)\} + 2k\ \{\frac{k_{\rho\rho}}{2a} + \frac{k_{zz}}{2a} \\ &+ \frac{1}{4a(lf + k^2)}(2kl_\rho f_\rho - lf_\rho k_\rho + 2kf_z l_z - lk_z f_z - fl_\rho k_\rho - fl_z k_z)\} \\ &+ f\{\frac{l_{\rho\rho}}{2a} + \frac{l_{zz}}{2a} + \frac{1}{4a(lf + k^2)}(2lk_\rho^2 + 2lk_z^2 - fl_\rho^2 - fl_z^2 - 2kk_\rho l_\rho - 2kk_z l_z \\ &+ ll_z f_z + lf_\rho l_\rho)\}\,] \end{aligned}$$

Using (6.17) (6.18) and (6.19) We can write

$$D_{\rho\rho} + D_{zz} = aD^{-1}\{-lR_{00} + 2kR_{03} + fR_{33}\} = 0$$

$$or, e^\mu D^{-1}(lR_{00} - 2kR_{03} - fR_{33}) = -(D_{\rho\rho} + D_{zz}) = 0 \qquad 6.20$$

Thus the function D satisfy the two dimensional Laplace equation in the variables ρ and z. Let us consider the transformation $(\rho, z) \to (\rho', z')$ given by :

$$\bar{\rho} = D(\rho, z) \quad : \quad \bar{z} = E(\rho, z) \qquad 6.21$$

where E is the conjugate function of D. Since D is the real part and E is the corresponding imaginary part of analytic function of $\rho + iz$, then by Caucy-Riemann equation

$$D_\rho = E_z \quad and \quad D_z = -E_\rho \qquad 6.22$$

Because of (6.22) ,we can write from (6.21)

$$(d\bar{\rho})^2 + (d\bar{z})^2 = (D_\rho^2 + E_\rho^2)(d\rho^2 + dz^2) \quad 6.23$$

From the equation (6.23) we see that the form of metric (6.14) is unaltered by the transformation (6.21) since we can define a new function $\bar{\mu}$ given by

$$e^{\bar{\mu}} = e^{\mu}(D_\rho^2 + E_\rho^2)^{-1} \quad 6.24$$

Now expressing all functions $f, l, k, \bar{\mu}$ in terms of $(\bar{\rho}, \bar{z})$ and after transformation omitting the prime sign we obtain the algebraic relation in f, k, l [7] such as

$$D^2 = fl + k^2 = \rho^2 \quad 6.25$$

$$\therefore D = \sqrt{fl + k^2} = \rho$$

Now to get the non-trival Einstein field equation for the metric (6.14) we proceed as follows

$$\begin{aligned} R_{11} &= \Gamma^0_{10,1} + \Gamma^1_{11,1} + \Gamma^2_{12,1} + \Gamma^3_{13,1} - \Gamma^1_{11,1} - \Gamma^2_{11,2} + \{\Gamma^0_{10}\Gamma^0_{01} + \Gamma^3_{10}\Gamma^0_{31} + \Gamma^1_{11}\Gamma^1_{11} + \Gamma^2_{11}\Gamma^1_{21} \\ &+ \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{21} + \Gamma^0_{13}\Gamma^3_{01} + \Gamma^3_{13}\Gamma^3_{31}\} - \{\Gamma^1_{11}\Gamma^0_{10} + \Gamma^2_{11}\Gamma^0_{20} + \Gamma^1_{11}\Gamma^1_{11} + \Gamma^2_{11}\Gamma^1_{21} \\ &+ \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} + \Gamma^1_{11}\Gamma^3_{13} + \Gamma^2_{11}\Gamma^3_{23}\} \end{aligned}$$

$$\begin{aligned} or, R_{11} &= \left\{\frac{(lf_\rho + kk_\rho)}{2(lf + k^2)}\right\}_\rho + \left(\frac{a_\rho}{2a}\right)_\rho + \left\{\frac{fl_\rho + kk_\rho}{2(lf + k^2)}\right\}_\rho + \left(\frac{a_z}{2a}\right)_z + \left\{\frac{lf_\rho + kk_\rho}{2(lf + k^2)}\right\}^2 \\ &+ \frac{(fk_\rho - kf_\rho)}{2(lf + k^2)}\frac{(kl_\rho - lk_\rho)}{2(lf + k^2)} + \left(\frac{a_\rho}{2a}\right)^2 - \left(\frac{a_z}{2a}\right)^2 - \left(\frac{a_z}{2a}\right)^2 \\ &- \left(\frac{a_\rho}{2a}\right)^2 + \frac{(kl_\rho - lk_\rho)}{2(lf + k^2)}\frac{(fk_\rho - kf_\rho)}{2(lf + k^2)} + \left\{\frac{kk_\rho + fl_\rho}{2(lf + k^2)}\right\}^2 \\ &- \left\{\frac{a_\rho}{2a}\frac{(lf_\rho + kk_\rho)}{2(lf + k^2)} - \frac{a_z}{2a}\frac{(lf_z + kk_z)}{2(lf + k^2)} + \left(\frac{a_\rho}{2a}\right)^2 - \left(\frac{a_z}{2a}\right)^2 + \left(\frac{a_\rho}{2a}\right)^2 - \left(\frac{a_z}{2a}\right)^2 - \frac{(kk_z + fl_z)}{2(lf + k^2)}\frac{a_z}{2a}\right\} \end{aligned}$$

By using $D^2 = fl + k^2 = \rho^2$, the above equation become after simplification

$$\begin{aligned} R_{11} = &\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\right) + \frac{\partial}{\partial\rho}\left(\frac{a_z}{2a}\right) + \frac{\partial}{\partial z}\left(\frac{a_z}{2a}\right) + \frac{1}{4\rho^4}\{l^2 f_\rho^2 + 2lkf_\rho k_\rho + k^2k_\rho^2 + 2(f\, kk_\rho l_\rho - lf\, k_\rho^2 \\ &- k^2 f_\rho l_\rho + lkf_\rho k_\rho) + k^2k_\rho^2 + 2k\, f\, k_\rho l_\rho + l_\rho^2 f^2\} - \frac{1}{4a\rho^2}\{a_\rho(lf_\rho + fl_\rho + 2kk_\rho) \\ &- (lf_\rho + fl_z + 2kk_z)\} \end{aligned}$$

Putting $a = e^\mu$ and after simplification we obtain

$$R_{11} = \frac{1}{2}(\mu_{\rho\rho} + \mu_{zz}) - \frac{\mu_\rho}{2\rho} - \frac{1}{2\rho^2}(f_\rho l_\rho + k_\rho^2)$$

$$or,\ 2R_{11} = -\{-\mu_{\rho\rho} - \mu_{zz} + \mu_\rho\rho^{-1} + \rho^{-2}(f_\rho l_\rho + k_\rho^2)\}$$

Hence

$$2R_{11} = -\mu_{\rho\rho} - \mu_{zz} + \mu_\rho \rho^{-1} + \rho^{-2}(f_\rho l_\rho + k_\rho^2) \qquad 6.26$$

Similarly we get

$$R_{12} = \left\{\frac{(lf_\rho + kk_\rho)}{2(lf+k^2)}\right\}_z + \left(\frac{a_\rho}{2a}\right)_z - \left(\frac{a_\rho}{2a}\right)_z + \left\{\frac{fl_\rho + kk_\rho}{2(lf+k^2)}\right\}_z - \left(\frac{a_z}{2a}\right)_\rho + \left(\frac{a_\rho}{2a}\right)_z + \frac{(lf_\rho + kk_\rho)}{2(lf+k^2)}\frac{(lf_z + kk_z)}{2(lf+k^2)} + \frac{(fk_\rho - kf_\rho)}{2(lf+k^2)}\frac{(kl_z - lk_z)}{2(lf+k^2)}$$

$$+\left(\frac{a_\rho}{2a}\right)\left(\frac{a_z}{2a}\right) - \left(\frac{a_z}{2a}\right)\left(-\frac{a_\rho}{2a}\right) + \left(\frac{a_z}{2a}\right)\left(\frac{a_\rho}{2a}\right) + \left(\frac{a_z}{2a}\right)\left(\frac{a_\rho}{2a}\right) + \frac{(kl_\rho - lk_\rho)}{2(lf+k^2)}\frac{(fk_z - kf_z)}{2(lf+k^2)} + \frac{(lf_\rho + kk_\rho)}{2(lf+k^2)}\frac{(fl_z + kk_z)}{2(lf+k^2)}$$

$$-\{\frac{a_z}{2a}\frac{(lf_\rho + kk_\rho)}{2(lf+k^2)} + \frac{a_\rho}{2a}\frac{(lf_z + kk_z)}{2(lf+k^2)} + \left(\frac{a_z}{2a}\right)\left(\frac{a_\rho}{2a}\right) + \left(\frac{a_\rho}{2a}\right)\left(\frac{a_z}{2a}\right)$$

$$+\left(\frac{a_z}{2a}\right)\left(\frac{a_\rho}{2a}\right) + \left(\frac{a_\rho}{2a}\right)\left(\frac{a_z}{2a}\right) + \left(\frac{a_z}{2a}\right)\frac{(kk_\rho + fl_\rho)}{2(lf+k^2)} + \frac{a_z}{2a}\frac{(kk_z + fl_z)}{2(lf+k^2)}\}$$

By using $\rho^2 = fl + k^2$ and $a = e^\mu$ and after simplification

$$R_{12} = -\frac{1}{2}\{\frac{\mu_z}{\rho} + \frac{1}{2\rho^2}(l_\rho f_z + l_z f_\rho + 2k_\rho k_z)\}$$

Hence

$$2R_{12} \equiv \mu_z \rho^{-1} + \frac{1}{2}\rho^{-2}(l_z f_\rho + l_\rho f_z + 2k_\rho k_z) = 0 \qquad 6.27$$

Similarly,

$$2R_{22} \equiv -\mu_{\rho\rho} - \mu_{zz} - \rho^{-1}\mu_\rho + \rho^{-2}(f_z l_z + k_z{}^2) = 0 \qquad 6.28$$

For simplicity let us write the non- trival Einstein equations for rotating metric at a time

$$2R_{11} \equiv -\mu_{\rho\rho} - \mu_{zz} + \mu_\rho \rho^{-1} + \rho^{-2}(f_\rho l_\rho + k_\rho^2) = 0$$

$$2R_{12} \equiv \mu_z \rho^{-1} + \frac{1}{2}\rho^{-2}(l_z f_\rho + l_\rho f_z + 2k_\rho k_z) = 0$$

$$2R_{22} \equiv -\mu_{\rho\rho} - \mu_{zz} - \rho^{-1}\mu_\rho + \rho^{-2}(f_z l_z + k_z{}^2) = 0$$

Because $D^2 = fl + k^2 = \rho^2$ only two of the above three equations are independent Let use a function w in lieu of k defined by

$$w = f^{-1}k$$

Let us eliminate k and l from the equations (6.17) and (6.18) and so compute the following quantities:

We get,

$$k = fw$$

$$\therefore k_\rho = f_\rho w + fw_\rho$$

and $\quad k_z = f_z w + f w_z$

Also, $l = \frac{\rho^2}{f} - \frac{k^2}{f}$ and $l = \frac{\rho^2}{f} - f w^2$

$$\therefore l_\rho = \frac{2\rho}{f} - \frac{\rho^2 f_\rho}{f^2} - 2w\, f\, w_\rho - f_\rho w^2$$

and $\quad l_z = -\frac{\rho^2 f_z}{f^2} - f_z w^2 - 2wfw_z$

From equation (6.17) we get

$$(D^{-1} f_\rho)_\rho + (D^{-1} f_z)_z + D^{-3} f(l_\rho f_\rho + l_z f_z + k_\rho^2 + k_z^2) = 0$$

Putting the values of derivatives of k and l in the above equation

$$(\rho^{-1} f_\rho)_\rho + (\rho^{-1} f_z)_z + \rho^{-3} f(l_\rho f_\rho + l_z f_z + k_\rho^2 + k_z^2) = 0$$

$$or,\ -\rho^{-2} f_\rho + \rho^{-1} f_{\rho\rho} + \rho^{-1} f_{zz} + \rho^{-3} f\ [\, f_\rho(\frac{2\rho}{f} - \frac{\rho^2 f_\rho}{f^2} - 2ww_\rho f - f_\rho w^2)$$

$$+ f_z(-\frac{\rho^2 f_z}{f^2} - f_z w^2 - 2ww_z f) + (f_\rho w + fw_\rho)^2 + (f_z w + fw_z)^2\,] = 0$$

$$or,\ -\rho^{-2} f_\rho + \rho^{-1} f_{\rho\rho} + \rho^{-1} f_{zz} + 2\rho^{-2} f_\rho - \rho^{-1}\frac{f_\rho^2}{f} - \rho^{-1}\frac{f_z^2}{f} + \rho^{-3} f\ f_\rho^2 w^2$$

$$+ \rho^{-3} f^3 w_\rho^2 - 2\rho^{-3} ww_\rho f^2\ f_\rho - \rho^{-3} f\ f_\rho^2 w^2 + 2\rho^{-3} ww_\rho f^2\ f_\rho + \rho^{-3} f^3 w_z^2 = 0$$

$$or,\ f(f_{\rho\rho} + f_{zz} + \rho^{-1} f_\rho) - f_\rho^2 - f_z^2 + \rho^{-2} f^4 (w_\rho^2 + w_z^2) = 0$$

$$\therefore\ f(f_{\rho\rho} + f_{zz} + \rho^{-1} f_\rho) - f_\rho^2 - f_z^2 + \rho^{-2} f^4 (w_\rho^2 + w_z^2) = 0 \qquad 6.29$$

Again from (6.18) we get

$$(D^{-1} k_\rho)_\rho + (D^{-1} k_z)_z + D^{-3} k(l_\rho f_\rho + f_z l_z + k_\rho^2 + k_z^2) = 0$$

$$or, (\rho^{-1} f\, w_\rho + \rho^{-1} w\, f_\rho)_\rho + (\rho^{-1} f\, w_z + \rho^{-1} w\, f_z)_z + \rho^{-3} fw\,[\, f_\rho(\frac{2\rho}{f} - \frac{\rho^2 f_\rho}{f^2} - f_\rho w^2 - 2ww_\rho f)$$

$$+ f_z(-\frac{\rho^2}{f^2} f_z - f_z w^2 - 2ww_z f) + (f_\rho w + fw_\rho)^2 + (f_z w + fw_z)^2\,] = 0$$

$$or,\ -\rho^{-2} f\, w_\rho + 2\rho^{-1} w_\rho f_\rho + \rho^{-1} f\, w_{\rho\rho} - \rho^{-2} w\, f_\rho + \rho^{-1} w\, f_{\rho\rho} + 2\rho^{-1} w_z f_z + \rho^{-1} f\, w_{zz}$$
$$+\rho^{-1} f_{zz} w + 2\rho^{-2} w\, f_\rho - \frac{\rho^{-1}}{f} f_\rho^2 w - \rho^{-3} f\, f_\rho^2 w^3 - 2\rho^{-3} f^2 w^2 f_\rho w_\rho - \frac{\rho^{-1}}{f} f_z^2 w$$
$$-\rho^{-3} f\, f_z^2 w^3 - 2\rho^{-3} f^2 w^2 f_z w_z + \rho^{-3} f\, f_\rho^2 w^3 + \rho^{-3} f^3 w w_\rho^2 + 2\rho^{-3} f^2 w^2 f_\rho w_\rho$$
$$+\rho^{-3} f\, f_z^2 w^3 + \rho^{-3} f^3 w w_z^2 + 2\rho^{-3} f^2 w^2 f_z w_z = 0$$

$$or,\ -\rho^{-2} f\, w_\rho + 2\rho^{-1} w_\rho f_\rho + \rho^{-1} f\, w_{\rho\rho} + \rho^{-1} w\, f_{\rho\rho} + 2\rho^{-1} w_z f_z + \rho^{-1} f\, w_{zz}$$
$$+\rho^{-1} f_{zz} w + \rho^{-2} f w_\rho - \frac{\rho^{-1}}{f} f_\rho^2 w - \frac{\rho^{-1}}{f} f_z^2 w + \rho^{-3} f^3 w w_\rho^2 + \rho^{-3} f^3 w w_z^2 = 0$$

$$or,\ \rho^{-1}[\ \{\ f\,(w_{\rho\rho} + w_{zz} - \rho^{-1} w_\rho) + 2 w_\rho f_\rho + 2 w_z f_z\}$$
$$+\frac{w}{f}\{\ f\,(f_{\rho\rho} + f_{zz} + \rho^{-1} f_\rho) - f_\rho^2 - f_z^2 + \rho^{-2} f^4 (w_\rho^2 + w_z^2)] = 0$$

$$or,\ f\,(w_{\rho\rho} + w_{zz} - \rho^{-1} w_\rho) + 2 w_\rho f_\rho + 2 w_z f_z = 0 \qquad 6.30$$

By the help of equation (6.29).

Now subtracting (6.28) from (6.26) we get

$$\mu_\rho = -\frac{1}{2}\rho^{-1}(f_\rho l_\rho - f_z l_z + k_\rho^2 - k_z^2)$$

$$or, \mu_\rho = -\frac{1}{2}\rho^{-1}[\ \frac{2\rho}{f} f_\rho - \frac{\rho^2}{f^2} f_\rho^2 - 2w w_\rho f\, f_\rho - f_\rho^2 w^2 + \frac{\rho^2}{f^2} f_z^2 + f_z^2 w^2 + 2w w_z f\, f_z$$
$$+ f_\rho^2 w^2 + f^2 w_\rho^2 + 2w w_\rho f\, f_\rho - f_z^2 w^2 - f^2 w_z^2 - 2w w_z f\, f_z\ ]$$

$$\therefore\ \mu_\rho = -f^{-1} f_\rho + \frac{1}{2}\rho f^{-2}(f_\rho^2 - f_z^2) - \frac{1}{2}\rho^{-1} f^2 (w_\rho^2 - w_z^2) \qquad 6.31$$

Again from (6.27) we get

$$\mu_z = -\frac{1}{2}\rho^{-1}(l_z f_\rho + l_\rho f_z + 2k_\rho k_z)$$
$$= -\frac{1}{2}\rho^{-1}[\ -\frac{\rho^2}{f^2} f_\rho f_z - f_\rho f_z w^2 - 2w\, f\, f_\rho w_z + \frac{2\rho}{f} f_z - \frac{\rho^2}{f^2} f_\rho f_z$$
$$-2w w_\rho f f_z - f_\rho f_z w^2 + 2 f_\rho f_z w^2 + 2 f\, f_\rho w w_z + 2 f\, f_z w w_\rho + 2 f^2 w_\rho w_z\ ]$$

$$\therefore\quad \mu_z = -f^{-1} f_z + \rho\, f^{-2} f_\rho f_z - \rho^{-1} f^2 w_\rho w_z \qquad 6.32$$

The gravitational field of a uniformly rotating bounded source must depend on at least two variables. Finding any solution of Einstein equation depending on two or more variables is quite difficult and physically interesting. The first exact solution of Einstein equations to be found which could represent the exterior field of a bounded rotating source was that of Kerr (1963). An essential property of such solution is that it should be asymptotically flat, since the gravitational field tends to zero as one move further and further away from the source. The Kerr solution was the first known rotating solution, which was asymptotically flat with source having non-zero mass. No interior solution has yet been found which matches smoothly onto the Kerr solution. It is believed that Kerr solution represents the exterior gravitational field of a highly collapsed rotating star –a rotating black hole.

Although many stationary axially symmetric exact solutions of Einstein equations (6.29) and (6.30) are known, very few of these are asymptotically flat and so their physical interpretation is uncertain. The first rotating asymptotically flat solutions to be found after the Kerr solution were the Tomimatsu-Sato solution (1972.73). These solutions differ from the Kerr solution in one important respect. The Kerr solution has the property that when the angular momentum of the source producing the field tends to zero, the solution tends to the Schwarzschild solution – representing the exterior field of a spherically symmetric source. This behavior is what one would expect for realistic star, because for the latter departure from spherical symmetry is usually caused by rotation and if the rotation vanishes one would get a spherically symmetric star –whose exterior field is the Schwarzschild solution. However the Tomimatsu-Sato solutions do not tends to the Schwarzschild solution when the angular momentum parameter of the source tends to zero. Though there is no easy way to derive Kerr solution, we will proceed to find the solution through the Ernst's form of Einstein equations for the axially stationary symmetric metric derived before.

6.3 ERNST FORM OF EINSTEIN EQUATION:

From equation (6.30) we have

$$f\,(w_{\rho\rho} + w_{zz} - \rho^{-1} w_\rho) + 2 w_\rho f_\rho + 2 w_z f_z = 0$$

$$or,\quad (-\rho^{-1} w_\rho f + 2 f_\rho w_\rho + f\, w_{\rho\rho} + 2 w_z f_z + f\, w_{zz}) = 0$$

$$or,\quad f\rho^{-1}(-\rho^{-1} w_\rho f + 2 f_\rho w_\rho + f\, w_{\rho\rho} + 2 w_z f_z + f\, w_{zz}) = 0$$

$$or,\ -\rho^{-2} w_\rho f^2 + 2\rho^{-1} f\, f_\rho w_\rho + \rho^{-1} f^2 w_{\rho\rho} + 2\rho^{-1} f\, w_z f_z + \rho^{-1} f^2 w_{zz} = 0$$

$$or,\quad (\rho^{-1} f^2 w_\rho)_\rho + (\rho^{-1} f^2 w_z)_z = 0$$

Thus equation (6.30) can be written as;

$$(\rho^{-1} f^2 w_\rho)_\rho + (\rho^{-1} f^2 w_z)_z = 0 \qquad 6.33$$

which implies the existence of a function U such that

$$U_\rho = -\rho^{-1} f^2 w_z \quad ; \quad U_z = \rho^{-1} f^2 w_\rho \qquad 6.34$$

$$\Rightarrow \; U_\rho^2 = \rho^{-2} f^4 w_z^2 \quad ; \quad U_z^2 = \rho^{-2} f^4 w_\rho^2$$

let us express the equation (6.29) in terms of f and U

$$f(f_{\rho\rho} + f_{zz} + \rho^{-1} f_\rho) - f_\rho^2 - f_z^2 + \rho^{-2} f^4 (w_\rho^2 + w_z^2) = 0$$

$$or, \; f(f_{\rho\rho} + f_{zz} + \rho^{-1} f_\rho) - f_\rho^2 - f_z^2 + \rho^{-2} f^4 (\rho^2 f^{-4} U_z^2 + \rho^2 f^{-4} U_\rho^2) = 0$$

$$or, \; f \; \nabla^2 f = f_\rho^2 + f_z^2 - U_\rho^2 - U_z^2 . \qquad 6.35$$

By eliminating w from the equation (6.34) we get the following equation;

$$f \nabla^2 U = f(U_{\rho\rho} + U_{zz} + \rho^{-1} U_\rho)$$

$$= f(-\rho^{-1} U_\rho + 2f^{-1} f_\rho U_\rho + 2f^{-1} f_z U_z + -\rho^{-1} U_\rho)$$

$$= f(2f^{-1} f_\rho U_\rho + 2f^{-1} f_z U_z)$$

$$\therefore \quad f \; \nabla^2 U = 2 f_\rho U_\rho + 2 f_z U_z \qquad 6.36$$

Putting the value of w_ρ and w_z from (6.34), equation (6.31) and (6.32) can be written as follows:

$$\mu'_\rho = \frac{1}{2} \rho f^{-2} (f_\rho^2 - f_z^2) + \frac{1}{2} \rho f^{-2} (U_\rho^2 - U_z^2) \qquad 6.37$$

and $\mu'_z = \rho f^{-2} \; f_\rho \; f_z + \rho f^{-2} U_\rho U_z$ 6.38

where $\mu' = \mu + \log f$. Now define a complex function E as follows:

$$E = f + iU \qquad 6.39$$

Let us consider a single complex equation

$$(\mathrm{Re}\; E) \nabla^2 E = E_\rho^2 + E_z^2 \qquad 6.40$$

$$or, \; f \; (E_{\rho\rho} + E_{zz} + \rho^{-1} E_\rho) = (f_\rho + iU_\rho)^2 + (f_z + iU_z)^2$$

$$or, \; f \{ f_{\rho\rho} + iU_{\rho\rho} + f_{zz} + iU_{zz} + \rho^{-1} f_\rho + i\rho^{-1} U_\rho \}$$

$$= \{ f_\rho^2 - U_\rho^2 + f_z^2 - U_z^2 + 2i(f_\rho U_\rho + f_z U_z) \}$$

$$or, \; f(f_{\rho\rho} + f_{zz} + if_\rho) + if(U_{\rho\rho} + U_{zz} + \rho^{-1} U_\rho)$$

$$= (f_\rho^2 + f_z^2 - U_\rho^2 - U_z^2) + 2i(f_\rho U_\rho + f_z U_z)$$

From the above equation it is seen that (6.35) and (6.36) are the real and imaginary part of equation (6.40).Let us consider a new unknown function ξ in stead of E defined by

$$E = \frac{(\xi - 1)}{(\xi + 1)} \qquad 6.41$$

Let ξ^* be the complex conjugate of ξ .

Then we get

$$E=\frac{(\xi-1)(\xi^*+1)}{(\xi+1)(\xi^*+1)}=\frac{(\xi\xi^*-1)}{(\xi\xi^*+\xi+\xi^*+1)}+\frac{(\xi-\xi^*)}{(\xi\xi^*+\xi+\xi^*+1)}$$

$$(real\ part)\qquad (imaginary\ part)$$

$$Now\ \ E_\rho=\frac{2\xi_\rho}{(\xi+1)^2}\qquad\qquad And\ \ E_z=\frac{2\xi_z}{(\xi+1)^2}$$

$$E_{\rho\rho}=\frac{2(\xi+1)\xi_{\rho\rho}-4\xi_\rho^2}{(\xi+1)^3}\qquad\qquad E_{zz}=\frac{2(\xi+1)\xi_{zz}-4\xi_z^2}{(\xi+1)^3}$$

Now calculate,

$$\begin{aligned}\nabla^2E&=E_{\rho\rho}+E_{zz}+\rho^{-1}E_\rho\\&=\frac{2(\xi+1)\xi_{\rho\rho}-4\xi_\rho^2+2(\xi+1)\xi_{zz}-4\xi_z^2}{(\xi+1)^3}+\frac{1}{\rho}\frac{2\xi_\rho}{(\xi+1)^2}\\&=\frac{2(\xi+1)\ (\xi_{\rho\rho}+\xi_{zz}+\rho^{-1}\xi_\rho)-4\ (\xi_\rho^2+\xi_z^2)}{(\xi+1)^3}\\&=\frac{2(\xi+1)\ \nabla^2\xi-4\ (\xi_\rho^2+\xi_z^2)}{(\xi+1)^3}\end{aligned}$$

$$\therefore\nabla^2E=\frac{2\nabla^2\xi}{(\xi+1)^2}-\frac{4(\xi_\rho^2+\xi_z^2)}{(\xi+1)^3}$$

Putting the value of ∇^2E, E_ρ *and* E_z in equation (6.40)

$$\frac{(\xi\xi^*-1)}{(\xi\xi^*+\xi+\xi^*+1)}\left[\frac{2\nabla^2\xi}{(\xi+1)^2}-\frac{4(\xi_\rho^2+\xi_z^2)}{(\xi+1)^3}\right]=\frac{4(\xi_\rho^2+\xi_z^2)}{(\xi+1)^4}$$

$$or,\ \frac{(\xi\xi^*-1)}{(\xi+1)(\xi^*+1)}\ \frac{2\nabla^2\xi}{(\xi+1)^2}=\frac{4(\xi_\rho^2+\xi_z^2)}{(\xi+1)^4}+\frac{4(\xi\xi^*-1)(\xi_\rho^2+\xi_z^2)}{(\xi+1)^4(\xi^*+1)}$$

$$or,\ \frac{(\xi\xi^*-1)}{(\xi+1)(\xi^*+1)}\ \frac{2\nabla^2\xi}{(\xi+1)^2}=\frac{2(\xi^*+1)(\xi_\rho^2+\xi_z^2)+2(\xi\xi^*-1)(\xi_\rho^2+\xi_z^2)}{(\xi+1)^4(\xi^*+1)}$$

$$or,\ (\xi\xi^*-1)\nabla^2\xi=\frac{2(\xi_\rho^2+\xi_z^2)\{\xi^*+1+\xi\xi^*-1\}}{(\xi+1)}$$

$$or,\ (\xi\xi^*-1)\nabla^2\xi=\frac{2\xi^*(\xi_\rho^2+\xi_z^2)(\xi+1)}{(\xi+1)}$$

$$or,\ (\xi\xi^*-1)\nabla^2\xi=2\xi^*(\xi_\rho^2+\xi_z^2)\qquad 6.42$$

which is known as the **Ernst** form of Einstein equation.

Let us introduce prolate spheroidal co- ordinate (x,y) instead of variables $(\rho.z)$ as follows:

$$\rho=(x^2-1)^{1/2}\ (1-y^2)^{1/2}\quad;\quad z=xy\qquad 6.43$$

which can be solved for x and y as follows:

$$x = \frac{1}{2}(D+E) \quad where \quad D = \{\rho^2 + (z+1)^2\}^{1/2}$$

And $$y = \frac{1}{2}(D-E) \quad where \quad E = \{\rho^2 + (z-1)^2\}^{1/2}$$

Now we express the equation (6.41) in terms of spheroidal co-ordinates and so compute the followings:

$$\xi_\rho = \frac{\partial \xi}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial \xi}{\partial y}\frac{\partial y}{\partial \rho}$$

$$\therefore \xi_\rho = \frac{\rho}{2}\left\{\xi_x\left(\frac{1}{D}+\frac{1}{E}\right) + \xi_y\left(\frac{1}{D}-\frac{1}{E}\right)\right\}$$

And

$$\xi_{\rho\rho} = \frac{1}{2}\left\{\xi_x\left(\frac{1}{D}+\frac{1}{E}\right) + \xi_y\left(\frac{1}{D}-\frac{1}{E}\right)\right\} + \frac{\rho^2}{2}\left[\frac{1}{2}\xi_{xx}\left(\frac{1}{D}+\frac{1}{E}\right)^2 + \frac{1}{2}\xi_{yy}\left(\frac{1}{D}-\frac{1}{E}\right)^2\right]$$
$$+\frac{\rho^2}{2}\left[\xi_{xy}\left(\frac{1}{D^2}-\frac{1}{E^2}\right) + \xi_y\left(\frac{1}{E^3}-\frac{1}{D^3}\right) - \xi_x\left(\frac{1}{D^3}+\frac{1}{E^3}\right)\right]$$

Again

$$\xi_z = \frac{\partial \xi}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial \xi}{\partial y}\frac{\partial y}{\partial z}$$
$$= \frac{1}{2}\xi_x\left\{\frac{z+1}{D}+\frac{z-1}{E}\right\} + \frac{1}{2}\xi_y\left\{\frac{z+1}{D}-\frac{z-1}{E}\right\}$$

And

$$\xi_{zz} = \xi_{xx}\left(\frac{\partial x}{\partial z}\right)^2 + 2\xi_{xy}\frac{\partial x}{\partial z}\frac{\partial y}{\partial z} + \xi_x\frac{\partial^2 x}{\partial z^2} + \xi_{yy}\left(\frac{\partial x}{\partial z}\right)^2 + \xi_y\frac{\partial^2 y}{\partial z^2}$$

$$or,\ \xi_{zz} = \frac{1}{4}\xi_{xx}\left\{\frac{z+1}{D}+\frac{z-1}{E}\right\}^2 + \frac{1}{2}\xi_{xy}\left\{\frac{(z+1)^2}{D^2}-\frac{(z-1)^2}{E^2}\right\}$$
$$+\frac{1}{2}\xi_x\left\{\frac{1}{D}+\frac{1}{E}-\frac{(z+1)^2}{D^3}-\frac{(z-1)^2}{E3}\right\} + \frac{1}{4}\xi_{yy}\left\{\frac{z+1}{D}-\frac{z-1}{E}\right\}^2$$
$$+\frac{1}{2}\xi_y\left\{\frac{1}{D}-\frac{1}{E}-\frac{(z+1)^2}{D^3}+\frac{(z-1)^2}{E^3}\right\}$$

But we get

$$\therefore \nabla^2\xi = \xi_{\rho\rho} + \xi_{zz} + \rho^{-1}\xi_{\rho}$$

$$= \frac{1}{2}\left\{\xi_x\left(\frac{1}{D}+\frac{1}{E}\right)+\xi_y\left(\frac{1}{D}-\frac{1}{E}\right)\right\}+\frac{\rho^2}{2}\left[\frac{1}{2}\,\xi_{xx}\left(\frac{1}{D}+\frac{1}{E}\right)^2+\xi_{yy}\,\frac{1}{2}\left(\frac{1}{D}-\frac{1}{E}\right)^2\right]$$
$$+\frac{\rho^2}{2}\left[\xi_{xy}\left(\frac{1}{D^2}-\frac{1}{E^2}\right)+\xi_y\left(\frac{1}{E^3}-\frac{1}{D^3}\right)-\xi_x\left(\frac{1}{D^3}+\frac{1}{E^3}\right)\right]+\frac{1}{4}\xi_{xx}\left\{\frac{z+1}{D}+\frac{z-1}{E}\right\}^2$$
$$+\frac{1}{2}\xi_{xy}\left\{\frac{(z+1)^2}{D^2}-\frac{(z-1)^2}{E^2}\right\}+\frac{1}{2}\xi_x\left\{\frac{1}{D}+\frac{1}{E}-\frac{(z+1)^2}{D^3}-\frac{(z-1)^2}{E^3}\right\}+\frac{1}{4}\xi_{yy}\left\{\frac{z+1}{D}-\frac{z-1}{E}\right\}^2$$
$$+\frac{1}{2}\xi_y\left\{\frac{1}{D}-\frac{1}{E}-\frac{(z+1)^2}{D^3}+\frac{(z-1)^2}{E^3}\right\}+\frac{1}{2}\left\{\xi_x\left(\frac{1}{D}+\frac{1}{E}\right)+\xi_y\left(\frac{1}{D}-\frac{1}{E}\right)\right\}$$

$$or, \nabla^2\xi = \xi_x\left[\frac{1}{D}+\frac{1}{E}-\frac{\rho^2}{2}\left(\frac{1}{D^3}+\frac{1}{E^3}\right)+\frac{1}{2}\left\{\frac{1}{D}+\frac{1}{E}-\frac{(z+1)^2}{D^3}-\frac{(z-1)^2}{E^3}\right\}\right]$$
$$+\xi_y\left[\frac{1}{D}-\frac{1}{E}+\frac{1}{2}\left\{\frac{1}{D}-\frac{1}{E}-\frac{(z+1)^2}{D^3}+\frac{(z-1)^2}{E^3}\right\}+\frac{\rho^2}{2}\left(\frac{1}{E^3}-\frac{1}{D^3}\right)\right]$$
$$+\xi_{xy}\left[\frac{\rho^2}{2}\left(\frac{1}{D^2}-\frac{1}{E^2}\right)+\frac{1}{2}\left\{\frac{(z+1)^2}{D^2}-\frac{(z-1)^2}{E^2}\right\}\right]$$
$$+\xi_{xx}\left[\frac{\rho^2}{4}\left(\frac{1}{D}+\frac{1}{E}\right)^2+\frac{1}{4}\left\{\frac{z+1}{D}+\frac{z-1}{E}\right\}^2\right]$$
$$+\xi_{yy}\left[\frac{\rho^2}{4}\left(\frac{1}{D}-\frac{1}{E}\right)^2+\frac{1}{4}\left\{\frac{z+1}{D}-\frac{z-1}{E}\right\}^2\right]$$

$$or, \nabla^2\xi = \xi_x\left[\frac{1}{D}+\frac{1}{E}-\frac{\rho^2}{2}\left(\frac{1}{D^3}+\frac{1}{E^3}\right)+\frac{1}{2}\left\{\frac{1}{D}+\frac{1}{E}-\frac{(z+1)^2}{D^3}-\frac{(z-1)^2}{E^3}\right\}\right]$$
$$+\xi_y\left[\frac{1}{D}-\frac{1}{E}+\frac{1}{2}\left\{\frac{1}{D}-\frac{1}{E}-\frac{(z+1)^2}{D^3}+\frac{(z-1)^2}{E^3}\right\}+\frac{\rho^2}{2}\left(\frac{1}{E^3}-\frac{1}{D^3}\right)\right]$$
$$+\xi_{xy}\left[\frac{\rho^2}{2}\left(\frac{1}{D^2}-\frac{1}{E^2}\right)+\frac{1}{2}\left\{\frac{(z+1)^2}{D^2}-\frac{(z-1)^2}{E^2}\right\}\right]$$
$$+\xi_{xx}\left[\frac{\rho^2}{4}\left(\frac{1}{D}+\frac{1}{E}\right)^2+\frac{1}{4}\left\{\frac{z+1}{D}+\frac{z-1}{E}\right\}^2\right]$$
$$+\xi_{yy}\left[\frac{\rho^2}{4}\left(\frac{1}{D}-\frac{1}{E}\right)^2+\frac{1}{4}\left\{\frac{z+1}{D}-\frac{z-1}{E}\right\}^2\right]$$

To calculate the co-efficient of $\xi_{xx}, \xi_{yy,} \xi_{xy,} \xi_x$ *and* ξ_y compute the followings:

$$x = \frac{1}{2}(D+E) \qquad y = \frac{1}{2}(D-E)$$

$$or, x^2 = \frac{D^2 + 2ED + E^2}{4} \qquad or, y^2 = \frac{D^2 - 2ED + E^2}{4}$$

And

$$\begin{aligned} E^2 - D^2 &= \rho^2 + (z-1)^2 - \rho^2 - (z+1)^2 \\ &= z^2 - 2z + 1 - z^2 - 2z - 1 \\ &= -4z \\ &= 4xy \end{aligned}$$

$$\begin{aligned} E^2 + D^2 &= \rho^2 + (z-1)^2 + \rho^2 + (z+1)^2 \\ &= 2\rho^2 + 2z^2 + 2 \\ &= 2(x^2 - x^2y^2 - 1 + y^2) + 2x^2y^2 + 2 \\ &= 2(x^2 + y^2) \end{aligned}$$

Now the co-efficient of ξ_{xx} :

$$\frac{\rho^2}{4}(\frac{1}{D} + \frac{1}{E})^2 + \frac{1}{4}\left\{\frac{z+1}{D} + \frac{z-1}{E}\right\}^2$$

$$= \frac{\rho^2}{4}\frac{D^2 + 2ED + E^2}{D^2E^2} + \frac{1}{4}\frac{E^2(z^2+2z+1) + 2ED(z^2-1) + D^2(z^2-2z+1)}{D^2E^2}$$

$$= \frac{\rho^2 x^2}{D^2E^2} + \frac{z^2(D^2 + 2ED + E^2) + 2z(E^2 - D^2) + (E^2 - 2ED + D^2)}{4D^2E^2}$$

$$= \frac{\rho^2 x^2}{D^2E^2} + \frac{z^2x^2}{D^2E^2} + \frac{-8z^2}{4D^2E^2} + \frac{y^2}{D^2E^2}$$

$$= \frac{\rho^2x^2 + x^4y^2 - 2x^2y^2 + y^2}{D^2E^2}$$

$$= \frac{x^4 - x^4y^2 - x^2 + x^2y^2 + x^4y^2 - 2x^2y^2 + y^2}{D^2E^2}$$

$$= \frac{x^4 - x^2 - x^2y^2 + y^2}{D^2E^2}$$

$$= \frac{(x^2 - y^2)(x^2 - 1)}{D^2E^2}$$

The co-efficient of ξ_{yy}

$$\frac{\rho^2}{4}(\frac{1}{D} - \frac{1}{E})^2 + \frac{1}{4}\left\{\frac{z+1}{D} - \frac{z-1}{E}\right\}^2$$

$$= \frac{\rho^2}{4}\frac{E^2 - 2ED + D^2}{D^2E^2} + \frac{1}{4}\frac{E^2(z^2+2z+1) - 2ED(z^2-1) + D^2(z^2-2z+1)}{D^2E^2}$$

$$= \frac{\rho^2 y^2}{D^2E^2} + \frac{z^2(E^2 - 2ED + D^2) + 2z(E^2 - D^2) + (E^2 + 2ED + D^2)}{4D^2E^2}$$

$$=\frac{\rho^2 y^2}{D^2E^2}+\frac{z^2y^2}{D^2E^2}+\frac{-8z^2}{4D^2E^2}+\frac{x^2}{D^2E^2}$$

$$=\frac{y^2(x^2-x^2y^2-1+y^2)+x^2y^4-2x^2y^2+x^2}{D^2E^2}$$

$$=\frac{y^4-y^2-x^2y^2+x^2}{D^2E^2}$$

$$=\frac{(x^2-y^2)(1-y^2)}{D^2E^2}$$

The co-efficient of ξ_{xy}

$$\frac{\rho^2}{2}\left(\frac{1}{D^2}-\frac{1}{E^2}\right)+\frac{1}{2}\left\{\frac{(z+1)^2}{D^2}-\frac{(z-1)^2}{E^2}\right\}$$

$$=\frac{\rho^2}{2}\;\frac{E^2-D^2}{E^2D^2}+\frac{1}{2}\frac{E^2(z^2+2z+1)-D^2(z^2-2z+1)}{E^2D^2}$$

$$=\frac{\rho^2(-4z)+z^2(E^2-D^2)+2z(E^2+D^2)+(E^2-D^2)}{2E^2D^2}$$

$$=\frac{-4xy(x^2-x^2y^2-1+y^2)+x^2y^2(-4xy)+4xy(x^2-x^2y^2-1+y^2+x^2y^2+1)-4xy}{2E^2D^2}$$

$$=0$$

The co-efficient of ξ_x

$$\frac{1}{D}+\frac{1}{E}-\frac{\rho^2}{2}\left(\frac{1}{D^3}+\frac{1}{E^3}\right)+\frac{1}{2}\left\{\frac{1}{D}+\frac{1}{E}-\frac{(z+1)^2}{D^3}-\frac{(z-1)^2}{E^3}\right\}$$

$$=\frac{D+E}{DE}-\frac{1}{D^3}\left\{\frac{\rho^2}{2}+\frac{(z+1)^2}{2}\right\}-\frac{1}{E^3}\left\{\frac{\rho^2}{2}+\frac{(z-1)^2}{2}\right\}+\frac{E+D}{2ED}$$

$$=\frac{3E+3D}{2ED}-\frac{1}{2D^3}(x^2-x^2y^2-1+y^2+x^2y^2+2xy+1)$$
$$-\frac{1}{2E^3}(x^2-x^2y^2-1+y^2+x^2y^2-2xy+1)$$

$$=\frac{3E+3D}{2ED}-\frac{1}{2D^3}(x+y)(x+y)-\frac{1}{2E^3}(x-y)(x-y)$$

$$= \frac{3E+3D}{2ED} - \frac{1}{2D^2}(x+y) - \frac{1}{2E^2}(x-y) \qquad \sin ce\, E = x-y \;\; and \;\; D = x+y$$

$$= \frac{6x(x^2-y^2)-(x-y)^2(x+y)-(x+y)^2(x-y)}{2D^2E^2}$$

$$= \frac{(x^2-y^2)(6x-x+y-x-y)}{2D^2E^2}$$

$$= \frac{2x(x^2-y^2)}{D^2E^2}$$

The co-efficient of ξ_y

$$\frac{1}{D} - \frac{1}{E} + \frac{1}{2}\left\{\frac{1}{D} - \frac{1}{E} - \frac{(z+1)^2}{D^3} + \frac{(z-1)^2}{E^3}\right\} + \frac{\rho^2}{2}\left(\frac{1}{E^3} - \frac{1}{D^3}\right)$$

$$= \frac{E-D}{ED} + \frac{1}{E^3}\left\{\frac{\rho^2}{2} + \frac{(z-1)^2}{2}\right\} - \frac{1}{D^3}\left\{\frac{\rho^2}{2} + \frac{(z+1)^2}{2}\right\} + \frac{E-D}{2ED}$$

$$= \frac{3E-3D}{2ED} + \frac{1}{2E^3}(x^2 - x^2y^2 - 1 + y^2 + x^2y^2 - 2xy + 1)$$

$$- \frac{1}{2D^3}(x^2 - x^2y^2 - 1 + y^2 + x^2y^2 + 2xy + 1)$$

$$= \frac{3E-3D}{2ED} + \frac{1}{2E^3}(x-y)(x-y) - \frac{1}{2D^3}(x+y)(x+y)$$

$$= \frac{3E-3D}{2ED} + \frac{1}{2E^2}(x-y) - \frac{1}{2D^2}(x+y)$$

$$= \frac{3(-2y)(x^2-y^2)+(x+y)^2(x-y)-(x-y)^2(x+y)}{2E^2D^2}$$

$$= \frac{(x^2-y^2)(-6y+x+y-x+y)}{2E^2D^2}$$

$$= \frac{-2y(x^2-y^2)}{E^2D^2}$$

Therefore we can write

$$\nabla^2\xi = \frac{(x^2-y^2)(x^2-1)}{D^2E^2}\xi_{xx} + \frac{(x^2-y^2)(1-y^2)}{D^2E^2}\xi_{yy} + \frac{2x(x^2-y^2)}{D^2E^2}\xi_x - \frac{2y(x^2-y^2)}{E^2D^2}\xi_y$$

Now calculate,

$$\xi_\rho^2 + \xi_z^2 = \frac{\rho^2}{4}\left\{\xi_x\left(\frac{1}{D} + \frac{1}{E}\right) + \xi_y\left(\frac{1}{D} - \frac{1}{E}\right)\right\}^2 + \frac{1}{4}\left\{\xi_x\left(\frac{z+1}{D} + \frac{z-1}{E}\right) + \xi_y\left(\frac{z+1}{D} - \frac{z-1}{E}\right)\right\}^2$$

$$or,\ \xi_\rho^2+\xi_z^2 = \frac{\rho^2}{4E^2D^2}\{\xi_x(E+D)+\xi_y(E-D)\}^2+\frac{1}{4E^2D^2}[\xi_x\{E(z+1)+D(z+1)\} + \xi_y\{E(z+1)-D(z-1)\}]^2$$

$$or,\ \xi_\rho^2+\xi_z^2 = \frac{\rho^2}{4E^2D^2}\{\xi_x.2x+\xi_y(-2y)\}^2+\frac{1}{4E^2D^2}[\xi_x\{(x-y)(xy+1)+(x+y)(xy+1)\} + \xi_y\{(x-y)(xy+1)-(x+y)(xy-1)\}]^2$$

$$or,\ \xi_\rho^2+\xi_z^2 = \frac{\rho^2}{4E^2D^2}(4x^2\xi_x^2+4y^2\xi_y^2-8xy\xi_x\xi_y)+\frac{1}{4E^2D^2}[\xi_x(2x^2y-2y)+\xi_y(2x-2xy^2)]^2$$

$$or,\ \xi_\rho^2+\xi_z^2 = \frac{1}{4E^2D^2}[\xi_x^2(4x^4-4x^4y^2-4x^2+4x^2y^2+4x^4y^2-8x^2y^2+4y^2) + \xi_y^2(4x^2y^2-4x^2y^4-4y^2+4y^4+4x^2-8x^2y^2+4x^2y^4)+\xi_x\xi_y(-8x^3y+8x^3y^3 + 8xy-8xy^3+8x^3y-8x^3y^3-8xy+8xy^3)] \quad (putting\ the\ value\ of\ \rho)$$

$$or,\ \xi_\rho^2+\xi_z^2 = \frac{4}{4E^2D^2}[\xi_x^2(x^4-x^2-x^2y^2+y^2)+\xi_y^2(x^4-x^2y^2+y^4-y^2)$$

$$\therefore\ \xi_\rho^2+\xi_z^2 = \frac{1}{E^2D^2}[\xi_x^2(x^2-y^2)(x^2-1)+\xi_y^2(x^2-y^2)(1-y^2)]$$

Putting the value of $\nabla^2\xi$ *and* $(\xi_\rho^2+\xi_z^2)$ in equation (6.42)

$$(\xi\xi^*-1)[\frac{(x^2-y^2)(x^2-1)}{D^2E^2}\xi_{xx}+\frac{(x^2-y^2)(1-y^2)}{D^2E^2}\xi_{yy}+\frac{2x(x^2-y^2)}{D^2E^2}\xi_x-\frac{2y(x^2-y^2)}{E^2D^2}\xi_y] = 2\xi^*\frac{1}{E^2D^2}[\xi_x^2(x^2-y^2)(x^2-1)+\xi_y^2(x^2-y^2)(1-y^2)]$$

$$or,\ (\xi\xi^*-1)[(x^2-1)\xi_{xx}+(1-y^2)\xi_{yy}+2x\xi_x-2y\xi_y] = 2\xi^*[(x^2-1)\xi_x^2+(1-y^2)\xi_y^2] \qquad 6.44$$

Equation (6.44) is also an Ernst form of Einstein equation. Let us write the equation (6.34) in terms of x *and* y.We get

$$w_z=-\rho f^{-2}U_\rho$$

$$or,\ w_x\frac{\partial x}{\partial z}+w_y\frac{\partial y}{\partial z}=-\rho f^{-2}(U_x\frac{\partial x}{\partial\rho}+U_y\frac{\partial y}{\partial\rho})$$

$$or,\ \frac{1}{2}w_x\{\frac{z+1}{D}+\frac{z-1}{E}\}+\frac{1}{2}w_y\{\frac{z+1}{D}-\frac{z-1}{E}\} = -\rho f^{-2}[U_x\frac{\rho}{2}(\frac{1}{D}+\frac{1}{E})+U_y\frac{\rho}{2}(\frac{1}{D}-\frac{1}{E})]$$

$$or,\quad w_x\{E(z+1)+D(z-1)\}+w_y\{E(z+1)-D(z-1)\} = -\rho^2 f^{-2}[U_x(E+D)+U_y(E-D)]$$

$$or,\quad [w_x\{(x-y)(xy+1)+(x+y)(xy-1)\}+w_y\{(x-y)(xy+1)-(x+y)(xy-1)\}] = -\rho^2 f^{-2}[2xU_x-2yU_y]$$

$$or,\quad w_x(2x^2y-2y)+w_y(2x-2xy^2) = -f^{-2}(x^2-1)(1-y^2)(2xU_x-2yU_y)$$

$$or,\quad w_x\,2y(x^2-1)+w_y\,2x(1-y^2) = -f^{-2}(x^2-1)(1-y^2)(2xU_x-2yU_y)$$

$$or,\quad \frac{2y\,w_x}{(1-y^2)}+\frac{2xw_y}{(x^2-1)} = -f^{-2}(2xU_x-2yU_y)$$

Comparing the co-efficient of $2x$ and $2y$ from both sides of the above equation, we obtain

$$\frac{w_x}{(1-y^2)} = f^{-2}U_y \quad ; \quad \frac{w_y}{(x^2-1)} = -f^{-2}U_x$$

$$or,\ w_x = (1-y^2)f^{-2}U_y \quad ; \quad or,\ w_y = (1-x^2)f^{-2}U_x \qquad 6.45$$

Let us express (6.37) and (6.38) in terms of x *and* y and so compute the followings

$$f_\rho = f_x\frac{\partial x}{\partial\rho}+f_y\frac{\partial y}{\partial\rho}$$

$$= f_x\frac{\rho}{2}(\frac{1}{D}+\frac{1}{E})+f_y\frac{\rho}{2}(\frac{1}{D}-\frac{1}{E})$$

$$= \frac{\rho}{2ED}[\ f_x(E+D)+f_y(E-D)\ ]$$

$$= \frac{\rho}{2ED}\left[2xf_x+2y(-f_y)\right]$$

$$\therefore\ f_\rho = \frac{\rho}{ED}(xf_x-yf_y)$$

And $f_\rho^2 = \dfrac{\rho^2}{E^2D^2}(x^2f_x^2-2xyf_xf_y+y^2f_y^2)$

Now,

$$f_z = f_x\frac{\partial x}{\partial z}+f_y\frac{\partial y}{\partial z}$$

$$= \frac{f_x}{2}(\frac{z+1}{D}+\frac{z-1}{E})+\frac{f_y}{2}(\frac{z+1}{D}-\frac{z-1}{E})$$

$$= \frac{1}{2ED}[\ f_x\{E(z+1)+D(z-1)\}+f_y\{E(z+1)-D(z-1)\}]$$

$$= \frac{1}{2ED}\left[f_x\{(x-y)(xy+1)+(x+y)(xy-1)\}+f_y\{(x-y)(xy+1)-(x+y)(xy-1)\}\right]$$

$$or, \quad f_z = \frac{1}{2ED} [\ f_x (2x^2 y - 2y) + f_y (2x - 2xy^2)\ ]$$

$$\therefore \ f_z = \frac{1}{ED} [\ f_x (x^2 y - y) + f_y (x - xy^2)]$$

$$And \quad f_z^2 = \frac{1}{E^2 D^2} [\ f_x^2 (x^4 y^2 - 2x^2 y^2 + y^2) + f_y^2 (x^2 - 2x^2 y^2 + x^2 y^4) + 2 f_x f_y (x^3 y - x^3 y^3 - xy + xy^3)]$$

Again

$$U_\rho = U_x \frac{\partial x}{\partial \rho} + U_y \frac{\partial y}{\partial \rho}$$

$$= \frac{\rho}{2ED} [\ U_x (E + D) + U_y (E - D)\ ]$$

$$\therefore \ U_\rho = \frac{\rho}{ED} (xU_x - yU_y)$$

$$And \quad U_\rho^2 = \frac{\rho^2}{E^2 D^2} (x^2 U_x^2 + y^2 U_y^2 - 2xyU_x U\ )$$

Similarly we get

$$U_z = \frac{1}{ED} \{\ U_x (x^2 y - y) + U_y (x - xy^2)\}$$

$$And \quad U_z^2 = \frac{1}{E^2 D^2} [\ U_x^2 (x^4 y^2 + y^2 - 2x^2 y^2) + U_y^2 (x^2 + x^2 y^4 - 2x^2 y^2) + 2U_x U_y (x^3 y - x^3 y^3 - xy + xy^3)\ ]$$

Therefore

$$U_\rho^2 - U_z^2 = \frac{1}{E^2 D^2} [\ U_x^2 (\rho^2 x^2 - x^4 y^2 - y^2 + 2x^2 y^2) + U_y^2 (\rho^2 y^2 - x^2 - x^2 y^4 + 2x^2 y^2) - 2U_x U_y (xy\rho^2 + x^3 y - x^3 y^3 - xy + xy^3)\ ]$$

And

$$f_\rho^2 - f_z^2 = \frac{1}{E^2 D^2} [\ f_x^2 (\rho^2 x^2 - x^4 y^2 - y^2 + 2x^2 y^2) + f_y^2 (\rho^2 y^2 - x^2 - x^2 y^4 + 2x^2 y^2) - 2 f_x f_y (xy\rho^2 + x^3 y - x^3 y^3 - xy + xy^3)\ ]$$

From equation (6.37) we get

$$\mu'_x \frac{\partial x}{\partial \rho} + \mu'_y \frac{\partial y}{\partial \rho} = \frac{1}{2} \rho f^{-2} \{(f_\rho^2 - f_z^2) + (U_\rho^2 - U_z^2)\}$$

$$or, \quad \frac{\rho}{2DE} [\ \mu'_x (E + D) + \mu'_y (E - D)] = \frac{1}{2} \rho f^{-2} \{(f_\rho^2 - f_z^2) + (U_\rho^2 - U_z^2)\}$$

$$or,\ \frac{1}{DE}[\,2x\mu'_x-2y\mu'_y\,]=\frac{f^{-2}}{D^2E^2}\ [\,\{\,f_x^2\,(\rho^2x^2-x^4y^2-y^2+2x^2y^2)+f_y^2(\rho^2y^2-x^2-x^2y^4+2x^2y^2)$$
$$-2f_xf_y\,(xy\rho^2+x^3y-x^3y^3-xy+xy^3)\,\}+\{U_x^2\,(\rho^2x^2-x^4y^2-y^2+2x^2y^2)$$
$$+U_y^2(\rho^2y^2-x^2-x^2y^4+2x^2y^2)-2U_xU_y\,(xy\rho^2+x^3y-x^3y^3-xy+xy^3)\}]$$

$$or,\ 2x\mu'_x-2y\mu'_y\ =\ \frac{f^{-2}}{ED}[\,\{\,x^2(x^2-x^2y^2-1+y^2)-y^2(x^4-2x^2+1)\}\,(f_x^2+U_x^2)$$
$$+\{\,y^2(x^2-x^2y^2-1+y^2)-x^2(1+y^4-2y^2)\}(f_y^2+U_y^2)$$
$$-2\{\,2xy(x^2-1)(1-y^2)\}(f_xf_y+U_xU_y)]$$

$$or,\ 2x\mu'_x-2y\mu'_y\ =\ \frac{f^{-2}}{ED}[\,x\{\,x(x^2-1)(1-y^2)\}M+x\{x(1-y^2)(y^2-1)\}N$$
$$-2x\{y(x^2-1)(1-y^2)\}O-y\,\{\,y(x^2-1)^2\}M$$
$$+\,y\{y(x^2-1)(1-y^2)\}N-2y\{x(x^2-1)(1-y^2)\}O]$$

where $M=(f_x^2+U_x^2)\,,N=(f_y^2+U_y^2)\ \ and\ \ O=(f_xf_y+U_xU_y)$

$$or,\ 2x\mu'_x-2y\mu'_y\ =\ \frac{f^{-2}}{ED}[\,x\{\,x(x^2-1)(1-y^2)M+x(1-y^2)(y^2-1)N$$
$$-2y(x^2-1)(1-y^2)O\}+y\,\{-y(x^2-1)^2M$$
$$+\,y(x^2-1)(1-y^2)N-2x(x^2-1)(1-y^2)O\}]$$

Equating the co-efficient of $x\ and\ y$ from both sides of the above equation

$$2\mu'_x=\frac{f^{-2}}{DE}[\,x(x^2-1)(1-y^2)M+x(1-y^2)(y^2-1)N-2y(x^2-1)(1-y^2)O$$
$$\therefore\mu'_x=\frac{f^{-2}(1-y^2)}{2(x^2-y^2)}[x(x^2-1)(f_x^2+U_x^2)+x(y^2-1)(f_y^2+U_y^2)-2y(x^2-1)(f_xf_y+U_xU_y)\qquad 6.46$$

and

$$-2\,\mu'_y=\frac{f^{-2}}{DE}\ [\,-y(x^2-1)^2M+y(x^2-1)(1-y^2)N-2x(x^2-1)(1-y^2)O\,]$$
$$\therefore\ \mu'_y=\frac{f^{-2}(x^2-1)}{2(x^2-y^2)}\ [\,y(x^2-1)(f_x^2+U_x^2)-y(1-y^2)(f_y^2+U_y^2)+2x(1-y^2)(f_xf_y+U_xU_y)\qquad 6.47$$

Effective feature of Ernst equation (5.44) is that the Kerr solution is given by the following simple solution of it,

$$\xi=px-i\,qy\qquad 6.48$$

where $p\ and\ q$ are constant with $p^2+q^2=1$.Now putting the value of (6.48) in (6.41) we get

$$E=\frac{px-iqy-1}{px+iqy+1}$$

$$=\frac{(p^2x^2-1)+iqy(px-1)-iqy(px+1)+q^2y^2}{\{(px+1)-iqy)\}\{(px+1)+iqy\}}$$

$$=\frac{p^2x^2+q^2y^2-1}{(px+1)^2+q^2y^2}-\frac{2i\,qy}{(px+1)^2+q^2y^2} \qquad 6.49$$

Comparing the real and imaginary part of (6.39) and (6.49)

$$f=\frac{p^2x^2+q^2y^2-1}{(px+1)^2+q^2y^2} \quad ; \quad U=\frac{-2qy}{(px+1)^2+q^2y^2} \qquad 6.50$$

From (6.50) we get

$$U_y=\frac{-2q\{(px+1)^2+q^2y^2\}+4q^3y^2}{\{(px+1)^2+q^2y^2\}^2}$$

Let us find the value of w from (6.45).We get

$$w_x=(1-y^2)\,f^{-2}\,U_y$$

$$or,\quad w_x=(1-y^2)\,\frac{\{(px+1)^2+q^2y^2\}^2}{(p^2x^2+q^2y^2-1)^2}\;\frac{-2q\{(px+1)^2+q^2y^2\}+4q^3y^2}{\{(px+1)^2+q^2y^2\}^2}$$

$$or,\quad w_x=2q(1-y^2)\,\frac{(q^2y^2-p^2x^2-1-2px)}{(p^2x^2+q^2y^2-1)^2}$$

$$or,\quad w_x=2q(1-y^2)\frac{\{(p^2x^2+q^2y^2-1)-2px(px+1)\}}{(p^2x^2+q^2y^2-1)^2}$$

$$or,\quad w_x=2p^{-1}q(1-y^2)\frac{\{p(p^2x^2+q^2y^2-1)-2p^2x(px+1)\}}{(p^2x^2+q^2y^2-1)^2}$$

$$or,\quad w_x=2p^{-1}q(1-y^2)\frac{d}{dx}\,\{\frac{px+1}{p^2x^2+q^2y^2-1}\}$$

Integrating on both sides with respect to x we get

$$w=\frac{2p^{-1}q(1-y^2)(px+1)}{(p^2x^2+q^2y^2-1)}+w_0 \qquad 6.51$$

where w_0 is an integrating constant. Again to obtain the value of μ compute the followings:

$$f = \frac{p^2x^2 + q^2y^2 - 1}{(px+1)^2 + q^2y^2}$$

$$or,\ f_x = \frac{2p^2x\{(px+1)^2 + q^2y^2\} - 2p(px+1)(p^2x^2 + q^2y^2 - 1)}{\{(px+1)^2 + q^2y^2\}^2}$$

$$= \frac{2p^3x^2 + 4p^2x - 2pq^2y^2 + 2p}{\{(px+1)^2 + q^2y^2\}^2}$$

$$= \frac{2p\{(px+1)^2 - q^2y^2\}}{\{(px+1)^2 + q^2y^2\}^2}$$

$$\therefore\ f_x^2 = \frac{4p^2\{(px+1)^2 - q^2y^2\}^2}{\{(px+1)^2 + q^2y^2\}^4}$$

Similarly,

$$f_y = \frac{4q^2y(px+1)}{\{(px+1)^2 + q^2y^2\}^2}$$

$$\therefore\ f_y^2 = \frac{16q^4y^2(px+1)^2}{\{(px+1)^2 + q^2y^2\}^4}$$

Again,

$$U = \frac{-2qy}{(px+1)^2 + q^2y^2}$$

$$or,\ U_x = \frac{4pqy(px+1)}{\{(px+1)^2 + q^2y^2\}^2}$$

$$\therefore\ U_x^2 = \frac{16p^2q^2y^2(px+1)^2}{\{(px+1)^2 + q^2y^2\}^4}$$

And

$$U_y = \frac{-2q(p^2x^2 + 2px + 1 + q^2y^2) + 4q^3y^2}{\{(px+1)^2 + q^2y^2\}^2}$$

$$or,\ U_y = \frac{-2q\{(px+1)^2 - q^2y^2\}}{\{(px+1)^2 + q^2y^2\}^2}$$

$$\therefore\ U_y^2 = \frac{4q^2\{(px+1)^2 - q^2y^2\}^2}{\{(px+1)^2 + q^2y^2\}^4}$$

Now compute

$$f_x^2 + U_x^2 = \frac{4p^2\{(px+1)^2 - q^2y^2\}^2}{\{(px+1)^2 + q^2y^2\}^4} + \frac{16p^2q^2y^2(px+1)^2}{\{(px+1)^2 + q^2y^2\}^4}$$

$$or,\ f_x^2 + U_x^2 = \frac{4p^2[\{(px+1)^2 - q^2y^2\}^2 + 4q^2y^2(px+1)^2]}{\{(px+1)^2 + q^2y^2\}^4} \qquad 6.52$$

And

$$f_y^2 + U_y^2 = \frac{4q^2\{(px+1)^2 - q^2y^2\}}{\{(px+1)^2 + q^2y^2\}^4} + \frac{16q^4y^2(px+1)^2}{\{(px+1)^2 + q^2y^2\}^4}$$

$$or,\ f_y^2 + U_y^2 = \frac{4q^2[\{(px+1)^2 - q^2y^2\} + 4q^2y^2(px+1)^2]}{\{(px+1)^2 + q^2y^2\}^4} \qquad 6.53$$

Again

$$f_x f_y + U_x U_y = \frac{8pq^2y(px+1)\{(px+1)^2 - q^2y^2\}}{\{(px+1)^2 + q^2y^2\}^4} - \frac{8pq^2y(px+1)\{(px+1)^2 - q^2y^2\}}{\{(px+1)^2 + q^2y^2\}^4}$$

$$= 0 \qquad 6.54$$

Putting the value of (6.52), (6.53) and (6.54) in equation (6.46)

$$\mu'_x = \frac{(1-y^2)f^{-2}}{2(x^2-y^2)}[\ 4p^2x(x^2-1)\ \frac{\{(px+1)^2 - q^2y^2\}^2 + 4q^2y^2(px+1)^2}{\{(px+1)^2 + q^2y^2\}^4}$$

$$+4q^2x(y^2-1)\frac{\{(px+1)^2 - q^2y^2\}^2 + 4q^2y^2(px+1)^2}{\{(px+1)^2 + q^2y^2\}^4}\ ]$$

$$or,\ \mu'_x = \frac{(1-y^2)}{2(x^2-y^2)}\ \frac{\{(px+1)^2 + q^2y^2\}^2}{(p^2x^2+q^2y^2-1)^2}[4p^2(x^3-x)\ \frac{\{(px+1)^2 + q^2y^2\}^2}{\{(px+1)^2 + q^2y^2\}^4}$$

$$+4q^2(xy^2-x)\frac{\{(px+1)^2 + q^2y^2\}^2}{\{(px+1)^2 + q^2y^2\}^4}\ ]$$

$$or,\ \mu'_x = \frac{(1-y^2)}{2(x^2-y^2)}\ \frac{1}{(p^2x^2+q^2y^2-1)^2}\ [\ 4p^2(x^3-x) + 4q^2(xy^2-x)\ ]$$

$$or,\ \mu'_x = \frac{2x(1-y^2)(p^2x^2+q^2y^2-1)}{(x^2-y^2)(p^2x^2+q^2y^2-1)^2} \qquad \sin ce\ p^2+q^2=1$$

$$or,\ \mu'_x = \frac{2x-2xy^2}{(x^2-y^2)(p^2x^2+q^2y^2-1)}$$

$$or,\ \mu'_x = \frac{2x-2xy^2(p^2+q^2)+2p^2x^3-2p^2x^3}{(x^2-y^2)(p^2x^2+q^2y^2-1)}$$

$$or,\ \mu'_x = \frac{2p^2x(x^2-y^2)-2x(p^2x^2+q^2y^2-1)}{(x^2-y^2)(p^2x^2+q^2y^2-1)}$$

$$or,\ \mu'_x = \frac{2p^2x}{(p^2x^2+q^2y^2-1)} - \frac{2x}{(x^2-y^2)}$$

Integrating on both sides with respect to x

$$\mu' = \log(p^2x^2 + q^2y^2 - 1) - \log(x^2 - y^2) + \log A$$

$$or, \quad \mu' = \log\frac{A(p^2x^2 + q^2y^2 - 1)}{(x^2 - y^2)}$$

$$or, \quad e^{\mu'} = \frac{A(p^2x^2 + q^2y^2 - 1)}{(x^2 - y^2)} \qquad 6.55$$

where A is an arbitrary constant.

But from definition of $\mu'(e^{\mu'} = fe^{\mu})$ we get

$$e^{\mu} = \frac{A\,[(px+1)^2 + q^2y^2]}{(x^2 - y^2)} \qquad 6.56$$

Let us introduce co-ordinate r and θ related to x and y by the relation given below

$$px + 1 = pr \quad and \quad z = Cos\,\theta \qquad 6.57$$

That implies $x = (r - \frac{1}{p})$.And also introduce constants m and a related to p and q as follows:

$$p^{-1} = m \;,\; p^{-1}q = a \quad and \quad m^2 - a^2 = 1 \qquad 6.58$$

The mass and angular momentum of the Kerr solution will turn out be m and ma respectively while these constant being evaluated here in units such that m and a are related as in (6.58). To transform the Kerr solution to its standard form i.e. Boyer - Lindquist (1967) form let us start with the form given by (6.14).

$$ds^2 = fdt^2 - 2kdtd\varphi - ld\varphi^2 - e^{\mu}(d\rho^2 + dz)$$

For the Kerr solution $f, k = (fw), e^{\mu}$ are given by the relations (6.50), (6.51) and (6.52) respectively which are functions of x and y.Also the function l is given by $f^{-1}(\rho^2 - k^2)$.

With the use of (6.43), (6.57) and (6.58) let express (ρ, z) in terms of (r, θ) as follows:

$$\begin{aligned}
\rho &= (x^2 - 1)^{1/2}\ (1 - y^2)^{1/2} \\
&= \{\,(r - 1/p)^2 - 1\}^{1/2}\ (1 - Cos^2\theta)^{1/2} \\
&= \left(\frac{r^2p^2 - 2pr + 1 - p^2}{p^2}\right)^{1/2} Sin\theta \\
&= (r^2 - 2\frac{r}{p} + \frac{1}{p^2} - 1)^{1/2}\ Sin\theta
\end{aligned}$$

$$or, \quad \rho = (r^2 - 2mr + m^2 - 1)^{1/2}\,Sin\theta$$

$$And \quad z = xy = (r - 1/p)\ Cos\theta$$

$$or, z = (r - m)Cos\theta$$

$$\therefore \quad \rho = (r^2 - 2mr + a^2)^{1/2}\,Sin\theta$$

$$\therefore \quad z = (r - m)Cos\theta \qquad 6.59$$

Let us evaluate e^{μ} in terms of (r,θ) by the help of equation (6.57) and (6.59)

$$e^{\mu} = \frac{A[(px+1)^2 + q^2y^2]}{(x^2 - y^2)}$$

$$= A \; \frac{\{p(r - 1/p) + 1\}^2 + q^2Cos^2\theta}{(r - 1/p)^2 - Cos^2\theta}$$

$$= A \frac{(r^2 + \frac{q^2}{p^2}Cos^2\theta)}{(\frac{r^2}{p^2} - \frac{2r}{p^3} + \frac{1}{p^4} - \frac{Cos^2\theta}{p^2})}$$

$$or,\; e^{\mu} = A \; \frac{(r^2 + a^2Cos^2\theta)}{(r^2m^2 - 2rm^3 + m^4 - m^2Cos^2\theta)}$$

$$\therefore \; e^{\mu} = \frac{A}{m^2} \; \frac{(r^2 + a^2Cos^2\theta)}{(r-m)^2 + (a^2 - m^2)Cos^2\theta} \qquad 6.60$$

From (6.59) we get

$$d\rho = \frac{1}{2}(r^2 - 2mr + a^2)^{-1/2} \; 2(r-m)\,dr \; Sin\theta + (r^2 - 2mr + a^2)^{1/2} Cos\theta\, d\theta$$

$$\therefore . d\rho^2 = (r^2 - 2mr + a^2)^{-1} \; (r-m)^2 \, dr^2 \; Sin^2\theta + (r^2 - 2mr + a^2) Cos^2\theta\, d\theta^2 + 2(r-m) Sin\theta Cos\theta\, dr\, d\theta$$

And

$$dz = -(r-m) Sin\theta\, d\theta + Cos\theta\, dr$$

$$\therefore \; dz^2 = (r-m)^2 Sin^2\theta\, d\theta^2 + Cos^2\theta\, dr^2 - 2(r-m) Sin\theta\, Cos\theta\, d\theta\, dr$$

Hence

$$d\rho^2 + dz^2 = (r^2 - 2mr + a^2)^{-1} \; (r-m)^2 \, dr^2 \; Sin^2\theta + (r^2 - 2mr + a^2) Cos^2\theta\, d\theta^2$$
$$+ 2(r-m) Sin\theta Cos\theta\, dr\, d\theta + (r-m)^2 Sin^2\theta\, d\theta^2 + Cos^2\theta\, dr^2 - 2(r-m) Sin\theta Cos\theta\, d\theta\, dr$$

$$or,\; d\rho^2 + dz^2 = (r^2 - 2mr + a^2)^{-1} \; (r-m)^2 \, dr^2 \; Sin^2\theta + (r^2 - 2mr + a^2) Cos^2\theta\, d\theta^2$$
$$+ (r-m)^2 Sin^2\theta\, d\theta^2 + Cos^2\theta\, dr^2$$

$$or,\; d\rho^2 + dz^2 = (r^2 - 2mr + a^2)^{-1} \; (r-m)^2 \, dr^2 - (r^2 - 2mr + a^2)^{-1} \; (r-m)^2 \, Cos^2\theta\, dr^2 + (r-m)^2 d\theta^2$$
$$- (r-m)^2 Cos^2\theta\, d\theta^2 + (r^2 - 2mr + a^2) Cos^2\theta\, d\theta^2 + Cos^2\theta\, dr^2$$

$$or,\; d\rho^2 + dz^2 = (r^2 - 2mr + a^2)^{-1} \; (r-m)^2 \, dr^2 + (r-m)^2 d\theta^2$$
$$+ Cos^2\theta\, d\theta^2 \,(r^2 - 2mr + a^2 - r^2 + 2mr + m^2) + Cos^2\theta\, dr^2 \left[\frac{-(r-m)^2}{r^2 - 2mr + 1} + 1 \right]$$

$$or,\; d\rho^2 + dz^2 = (r^2 - 2mr + a^2)^{-1} \; (r-m)^2 \, dr^2 + (a^2 - m^2) Cos^2\theta\, d\theta^2 + (r-m)^2 d\theta^2$$
$$+ (r^2 - 2mr + 1)^{-1} (a^2 - m^2) Cos^2\theta\, dr^2$$

$$\therefore \; d\rho^2 + dz^2 = [(r-m)^2 + (a^2 - m^2) Cos^2\theta]\,[(r^2 - 2mr + a^2)^{-1} dr^2 + d\theta^2] \qquad 6.61$$

Again,

$$k = fw$$

$$= \frac{(p^2x^2 + q^2y^2 - 1)}{(px+1)^2 + q^2y^2} \; \frac{2qp^{-1}(1-y^2)(px+1)}{(p^2x^2 + q^2y^2 - 1)}$$

$$= \frac{2qp^{-1}(1-y^2)(px+1)}{(p^2x^2 + q^2y^2 - 1)^2}$$

$$= \frac{2qp^{-1}\,(1-Cos^2\theta)\,\{p\,(r - \frac{1}{p}) + 1\}}{\{p\,(r - \frac{1}{p}) + 1\}^2 + q^2Cos^2\theta}$$

$$= \frac{2qrSin^2\theta}{r^2p^2 + q^2Cos^2\theta}$$

$$= \frac{2\,p^{-2}qrSin^2\theta}{(r^2 + \frac{q^2}{p^2}Cos^2\theta)}$$

$$or,\; k = \frac{2\,marSin^2\theta}{(r^2 + a^2Cos^2\theta)}$$

$$= 2\,marSin^2\theta\,(r^2 + a^2Cos^2\theta)^{-1} \qquad 6.62$$

And

$$f = \frac{(p^2x^2 + q^2y^2 - 1)}{(px+1)^2 + q^2y^2}$$

$$or,\; f = \frac{r^2 - 2mr + a^2Cos^2\theta}{r^2 + a^2Cos^2\theta}$$

$$= (1 - 2rm\textstyle\sum_1^{-1}) \qquad 6.63$$

$$where\; \textstyle\sum_1 = r^2 + a^2Cos^2\theta$$

Again

$$l = f^{-1}(\rho^2 - k^2)$$

$$or,\; l = \frac{(r^2 + a^2Cos^2\theta)}{(r^2 - 2mr + a^2Cos^2\theta)}\,[\,(r^2 - 2mr + a^2)Sin^2\theta - \frac{4a^2m^2r^2Sin^4\theta}{(r^2 + a^2Cos^2\theta)^2}\,]$$

$$or,\; l = \frac{(r^2 + a^2Cos^2\theta)Sin^2\theta}{(r^2 - 2mr + a^2Cos^2\theta)}\,[\frac{(r^2 - 2mr + a^2)(r^2 + a^2Cos^2\theta)^2 - 4a^2m^2r^2Sin^4\theta}{(r^2 + a^2Cos^2\theta)^2}\,]$$

$$or,\; l = \frac{Sin^2\theta}{(r^2 - 2mr + a^2Cos^2\theta)(r^2 + a^2Cos^2\theta)}\,[\,(r^2 - 2mr + a^2\,Cos^2\theta)$$

$$(r^4 + r^2a^2 + r^2a^2Cos^2\theta - 2mra^2\,Cos^2\theta + 2ma^2r + a^4Cos^2\theta)]$$

$$or,\; l = 2mra^2Sin^4\theta(r^2 + a^2Cos^2\theta)^{-1} + (r^2 + a^2)Sin^2\theta$$

$$or,\; l = 2mra^2Sin^4\theta\textstyle\sum_1^{-1} + (r^2 + a^2)Sin^2\theta \qquad 6.64$$

Putting the value of k, l, f, e^{μ} in equation (6.14) we get the required Kerr solution i.e. the Kerr metric (setting $A = m^2$ *and* $w_0 = 0$)

$$ds^2 = (1 - 2mr\sum_1^{-1})\,dt^2 - 4amr\,Sin^2\theta \sum_1^{-1} d\varphi\,dt$$
$$- (2ma^2 r\,Sin^2\theta \sum_1^{-1} + r^2 + a^2)Sin^2\varphi\,d\varphi^2 - \sum_1 \left(\sum_2^{-1} dr^2 + d\varphi^2 \right) \tag{6.65}$$

$where \sum_1 \equiv (r^2 + a^2 Cos^2\theta)$ *and* $\sum_2 \equiv (r^2 - 2mr + a^2)$.

By examining the Kerr metric in the asymptotic region $r \to \infty$ we get

$$ds^2 = (1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1} dr^2 - r^2 (d\theta^2 + Sin^2\theta\,d\varphi^2) - \frac{4amSin^2\theta}{r} d\varphi\,dt$$

Writing $(x^0, x^1, x^2, x^2) = (t, r, \theta, \varphi)$, let us write the inverse metric components of (6.65)

$$g^{00} = (\sum_1 \sum_2)^{-1}[(r^2 + a^2)^2 - \sum_2 a^2 Sin^2\theta]$$

$$g^{11} = -\sum_1^{-1} \sum_2 \quad ; \; g^{22} = -\sum_1^{-1} \quad ; \; g^{33} = -(\sum_1 \sum_2 Sin^2\theta)^{-1}(\sum_1 - 2mr)$$

$$and \;\; g^{03} = (\sum_1 \sum_2)^{-1}(2amr)$$

For the following discussion let us assume $m^2 \rangle a^2$.In the Schwarzschild metric the horizon is determined by the equation $g_{00} = 0$ and it is null surface but in the Kerr solution the equation correspond to the surface is

$$r^2 + a^2 Cos^2\vartheta - 2mr = 0 \tag{6.66}$$
$$or, r = m \pm m(m^2 - a^2 Cos^2\theta)^{1/2}$$

which is not null surface. So these can not be the horizons of the Kerr metric. Now consider instead (6.66) the surface $\sum_2 = 0$ i.e.

$$r^2 - 2mr + a^2 = 0$$
$$or,\; r = m \pm (m^2 - a^2)^{1/2} \tag{6.67}$$

Let us these surfaces by $\sum_+$ *and* $\sum_-$.These surfaces are null surfaces since they satisfy the form F=0 with $g^{\mu\nu} F_{,\mu} F_{,\nu} = 0$.No outgoing null or time like geodesics cross surface $\sum_+$, so $\sum_+$ is the horizon for the Kerr metric. Let us denote the surfaces of 6.66) S_+ *and* S_-.The meaning of these is as follows: The killing vector corresponding to the time independence of (6.65) is $\frac{\partial}{\partial t}$ i.e. the vector $\xi^{\mu} - (1,0,0,0)$. This vector is time like only outside of S_+ and inside of S_- , it is null on S_+ and space like in between S_+ *and* S_-. The surface S_+ is called stationary limit surface since it is only outside this surface that a material particle can remain at rest with respect to infinity.

The surface S_+ is time like except at two points on the axis where it coincides with $\sum_+$ and where it is null. The region between S_+ and $\sum_+$ is called ergosphere .Particles can escape to infinity from this region but not from inside $\sum_+$. Also in the ergosphere it is possible for a material particle or light wave to remain at rest with respect observer at infinity.

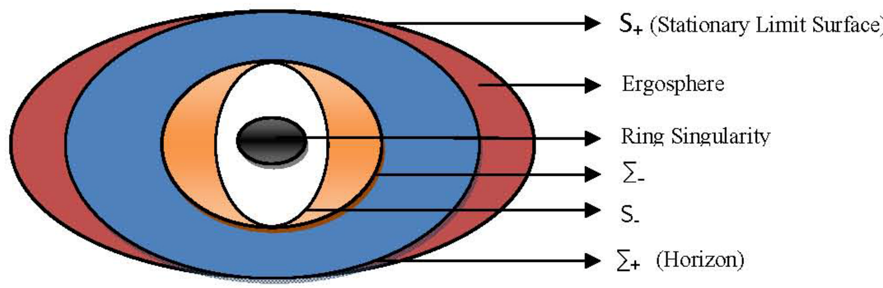

FIG: 01

The metric (6.65) has a ring singularity within the surface S_-.The surfaces $S_\pm$ *and* $\sum_\pm$ are non singular. Inside the surface $\sum_-$ one gets the closed time like curves, so one gets violation of causality and thus unphysical behavior. Such a violation of causality does not occur outside $\sum_-$. Thus the unphysical region is covered by the region between $\sum_-$ and $\sum_+$, from which material particles and signals can not emerge to the region outside $\sum_+$, to communicate with a distant observer. For this reason the unphysical nature of the geometry within $\sum_-$ is thought to be acceptable and the Kerr solution for $m^2 \rangle a^2$ is believed to represent the field of highly collapsed rotating star- a rotating black hole. For $a^2 \rangle m^2$ violations of causality occur in the regions accessible to distant observers and hence in this case the metric is unphysical.

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