The Open Monophonic Number of a Graph

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Abstract— For a connected graph G of order n, a subset S of vertices of G is a monophonic set of G if each vertex v in G lies on a x-y monophonic path for some elements x and y in S. The minimum cardinality of a monophonic set of G is defined as the monophonic number of G, denoted by $m(G)$. A monophonic set of cardinality $m(G)$ is called a m–set of G. A set S of vertices of a connected graph G is an open monophonic set of G if for each vertex v in G, either v is an extreme vertex of G and $v \in S$, or v is an internal vertex of a x-y monophonic path for some $x, y \in S$. An open monophonic set of minimum cardinality is a minimum open monophonic set and this cardinality is the open monophonic number, $om(G)$. The open monophonic number of certain standard graphs are determined. For positive integers r, d and $l \geq 2$ with $r \leq d \leq 2r$, there exists a connected graph of radius r, diameter d and open monophonic number l. It is proved that for a tree T of order n and diameter d, $om(T) = n - d + 1$ if and only if T is a caterpillar. Also for integers n, d and k with $2 \leq d < n$, $2 \leq k < n$ and $n - d - k + 1 \geq 0$, there exists a graph G of order n, diameter d and open monophonic number k. It is proved that $om(G) - 2 \leq om(G') \leq om(G) + 1$, where G' is the graph obtained from G by adding a pendant edge to G. Further, it is proved that if $om(G') = om(G) + 1$, then v does not belong to any minimum open monophonic set of G, where G' is a graph obtained from G by adding a pendant edge uv with v a vertex of G and u not a vertex of G.

Keywords— *Distance, geodesic, geodetic number, open geodetic number, monophonic number, open monophonic number*.

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1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m, respectively. For basic graph theoretic terminology we refer to Harary [4]. The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An u-v path of length $d(u,v)$ is called an *u-v geodesic*. It is known that this distance is a metric on the vertex set $V(G)$. For any vertex v of G, the *eccentricity* $e(v)$ of v is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, *rad G* and the maximum eccentricity is its *diameter*, *diam G* of G. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices which are adjacent with v. The vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete. For a cutvertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ is called a *branch* of G at v. A *geodetic* set of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The *geodetic number* $g(G)$ of G is the cardinality of a minimum geodetic set. A vertex x is said to *lie* on a *u-v geodesic* P if x is a vertex of P and x is called an *internal vertex* of P if $x \neq u, v$. A set S of vertices of a connected graph G is an *open geodetic set* of G if for each vertex v in G, either v is an extreme vertex of G and $v \in S$, or v is an internal vertex of a x-y geodesic for some $x, y \in S$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the *open geodetic number* $og(G)$. It is clear that every open geodetic set is a geodetic set so that $g(G) \leq og(G)$. The geodetic number of a graph was introduced and studied in [**1, 2**]. The open geodetic number of a graph was introduced and studied in [3**, 5, 7**] in the name open geodomination in graphs. A chord of a path $u_1, u_2, \dots, u_n$ in G is an edge u_iu_j with $j \geq i + 2$. For two vertices u and v in a connected graph G, a u-v path is called a *monophonic path* if it contains no chords. A *monophonic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a monophonic path joining some pair of vertices in S. The *monophonic number* $m(G)$ of G is the cardinality of a minimum monophonic set.

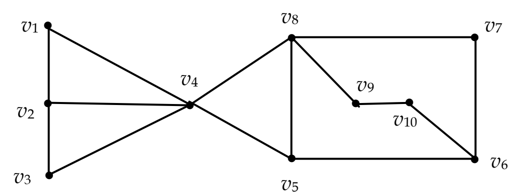

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Fig. 1.1: A graph with monophonic number 3.

For the graph G given in Fig. 1.1, the set $S = \{v_1, v_3, v_6\}$ is a minimum monophonic set so that $m(G) = 3$.

Since every extreme vertex v is either an initial vertex or a terminal vertex of a path containing v, it follows that every monophonic set S of graph G contains all its extreme vertices. Hence we have the following theorem.

Theorem 1.1 *Every extreme vertex of a connected graph G belongs to each monophonic set of G. In particular, if the set S of all extreme vertices of G is a monophonic set of G, then S is the unique minimum monophonic set of G.*

2 Open Monophonic Number of a Graph

2.1 Definition

A set S of vertices in a connected graph G is an *open monophonic set* if for each vertex v in G, either v is an extreme vertex of G and $v \in S$, or v is an internal vertex of an *x-y monophonic path* for some $x, y \in S$. An open monophonic set of minimum cardinality is a *minimum open monophonic set* and this cardinality is the *open monophonic number* $om(G)$ of G. An open monophonic set of cardinality $om(G)$ is called an *om-set* of G.

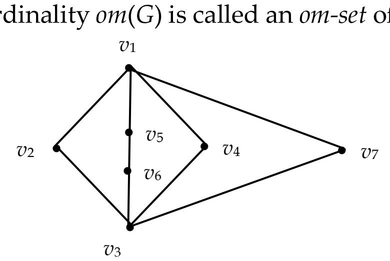

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Fig. 1: A graph with open monophonic number 4.

For the graph G is Fig. 1, the set $S = \{v_1, v_3\}$ is a monophonic set of G so that $m(G) = 2$. It is easily checked that neither a 2-element subset nor a 3-element subset of vertices is an open monophonic set of G. Since $S = \{v_1, v_2, v_3, v_4\}$ is an open monophonic set of G, it follows that S is a minimum open monophonic set and so $om(G) = 4$. Also $S_1 = \{v_1, v_2, v_3, v_7\}$, $S_2 = \{v_1, v_3, v_4, v_5\}$, $S_3 = \{v_1, v_3, v_4, v_6\}$ are minimum open monophonic sets. Thus, there can be more than one minimum open monophonic set for a connected graph. This example also shows that the monophonic number and open monophonic number of a graph are different.

It clear that an open monophonic set needs at least two vertices and so $om(G) \geq 2$ Also the set of all vertices of G is an open monophonic set of G so that $om(G) \leq n$. Hence we have the following theorem.

Theorem 2.2 *For any connected graph G of order n, $2 \leq om(G) \leq n$.*

Remark 2.3 We observe that the bounds in Theorem 2.2 are sharp. For the complete graph $K_n (n \geq 2)$, $om(K_n) = n$. The set of two end vertices of a path $P_n(n \geq 2)$ is its unique minimum open monophonic set so that $om(P_n) = 2$. Thus the complete graph K_n has largest possible open monophonic number n and that non-trivial paths have the smallest open monophonic number 2.

We observe that every open monophonic set of a graph G is a monophonic set so that $m(G) \leq om(G)$. This combined with Theorem 2.2 gives the following result.

Theorem 2.4 *For a connected graph G, $2 \leq m(G) \leq om(G) \leq n$.*

Since every open monophonic set of a graph G is also a monophonic set of a graph G, the next theorem follows from Theorem 1.1.

Theorem 2.5 *Every open monophonic set of a graph G contains its extreme vertices. Also, if the set S of all extreme vertices of G is an open monophonic set, then S is the unique minimum open monophonic set of G.*

Corollary 2.6 For the complete graph $K_n (n \geq 2)$, $om(K_n) = n$.

Remark 2.7 If $om(G) = n$ for a connected graph G of order n, then it need not be true that G is complete. It is clear that for the cycle $G = C_4$, $om(G) = 4$.

Now, Corollary 2.6 leads us to ask the question whether $m(G) = n$ for a connected graph G of order n implies $G = K_n$. If G is not a complete graph, then there exist two vertices x and y such that x and y are not adjacent. Hence there is a x-y geodesic P of length at least 2 so that P is also a x - y monophonic path of length at least 2. Let v be an internal vertex of the x-y monophonic path P. Then it is clear that $S = V - \{v\}$ is a monophonic set of G so that $m(G) \leq n - 1$, which is a contradiction. Thus we have the following theorem.

Theorem 2.8 *For a connected graph G of order n, $m(G) = n$ if and only if $G = K_n$.*

The same result is not true for open monophonic number of a graph. It is to be noted that for $G = C_4$, $om(G) = 4$.

Theorem 2.9 *If G is a non-trivial connected graph with no extreme vertices, then $om(G) \geq 3$.*

Proof. First, we observe that if G is a non-trivial connected graph having no extreme vertices, then the order of G is at least 4. Let S be an open monophonic set of G. If $u \in S$, then there exist vertices v and w such that u is an internal vertex of a v-w monophonic path. Hence it follows that $|S| \geq 3$, and so $om(G) \geq 3$.

Theorem 2.10 *For any cycle $G = C_n$ $(n \geq 4)$,*

$$om(G) = \begin{cases} 3 & \text{if } n \geq 6 \\ 4 & \text{if } n = 4,5. \end{cases}$$

Proof. Let the cycle $G = C_n (n \geq 6)$ be $C_n : v_1, v_2, \ldots, v_n, v_1$. Since G has no extreme vertices, it follows from Theorem 2.9 that $om(G) \geq 3$. It is clear that $S = \{v_1, v_3, v_5\}$ is a minimum open monophonic set of G so that $om(G) = 3$. For $G = C_4$, it is clear that no 3-element subset of vertices is an open monophonic set of G. Hence it follows that $om(G) = 4$. For $G = C_5$, it is easily seen that no 3-element subset of vertices is an open monophonic set of G. Since $S = \{v_1, v_2, v_3, v_4\}$ is an open monophonic set of G, it follows that $om(G) = 4$. Thus the proof of the theorem is complete.

Remark 2.11 Theorem 2.10 shows that the bound in Theorem 2.9 is sharp.

Theorem 2.12 *For the complete bipartite graph $G = K_{r,s} (2 \leq r \leq s)$, $om(G) = 4$.*

Proof. Let $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the partite sets of G. Since G contains no extreme vertices, by Theorem 2.9 $om(G) \geq 3$. It is easily verified that no 3-element subset of vertices of G is an open monophonic set of G so that $om(G) \geq 4$. Let S be any set of four vertices formed by taking two vertices from each of U and W. Then it is clear that S is an open monophonic set of G so that $om(G) = 4$.

Theorem 2.13 *If G is a connected graph having $k \geq 2$ extreme vertices, and if $m(G) = k$, then $om(G) = k$.*

Proof. Let S be the set of all extreme vertices of G. Since $m(G) = k$, by Theorem 1.1, S is the unique minimum monophonic set of G. We prove that S is also an open monophonic set of G. If $v \notin S$, then, since S is a monophonic set, v is an internal vertex of an x-y monophonic path for some $x, y \in S$. Therefore, S is an open monophonic set of G and so by Theorem 2.4 $om(G) = k$.

Theorem 2.14 *For any wheel $W_n = K_1 + C_{n-1}$ $(n \geq 5)$,*

$$om(W_n) = \begin{cases} 3 & \text{if } n \geq 7 \\ 4 & \text{if } n = 5,6. \end{cases}$$

Proof. Let $W_n = K_1 + C_{n-1}$ $(n \geq 5)$. Let $n \geq 7$. Since W_n has no extreme vertices, by Theorem 2.9, $om(G) \geq 3$. Since the set $S = \{v_1, v_3, v_5\}$ is an open monophonic set of W_n, it follows that $om(W_n) = 3$. Let $W_n = K_1 + C_{n-1}$ $(n = 5, 6)$. Since W_n has no extreme vertices, by Theorem 2.9, $om(W_n) \geq 3$. It is easily verified that no 3-element subset of vertices of W_n is an open monophonic set. Since $S = \{v_1, v_2, v_3, v_4\}$ is an open monophonic set of W_n, it follows that $om(W_n) = 4$. Thus the proof is complete.

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Theorem 2.15 *If G is a connected graph with a cutvertex v, then every open monophonic set of G contains at least one vertex from each component of* $G - v$.

Proof. Let v be a cut vertex of G. Let $G_1, G_2, \ldots, G_k$ $(k \geq 2)$ be the components of $G - v$. Let S be an open monophonic set of G. Suppose that S contains no vertex from a component say G_i $(1 \leq i \leq k)$. Let u be a vertex of G_i. Then by Theorem 2.5 u is not an extreme vertex of G. Since S is an open monophonic set of G, there exist vertices $x, y \in S$ such that u lies on a $x - y$ monophonic path $P : x = u_0, u_1, u_2, \ldots, u, \ldots, u_l = y$ with $u \neq x, y$. Then the $x - u$ subpath of P and the $u - y$ subpath of P both contain v. Hence it follows that P is not a path, which is a contradiction. Thus every open monophonic set of G contains at least one vertex from the component of $G - v$.

Corollary 2.16 Let G be a connected graph with cutvertices and let S be an open monophonic set of G. Then every branch of G contains an element of S.

Theorem 2.17 *Let G be a connected graph with cutvertices and S a minimum open monophonic set of G. Then no cut vertex of G belongs to S.*

Proof. Let S be any minimum open monophonic set of G. Let $v \in S$. We prove that v is not a cutvertex of G. Suppose that v is a cutvertex of G. Let $G_1, G_2, \ldots, G_k (k \geq 2)$ be the components of $G - v$. Then v is adjacent to at least one vertex of each G_i for $1 \leq i \leq k$. Let $S' = S - \{v\}$. We show that S' is an open monophonic set of G. Let x be a vertex of G. If x is an extreme vertex of G, then $x \neq v$ and so by Theorem 2.5, $x \in S'$. Suppose that x is not an extreme vertex of G. Since S is an open monophonic set of G, x lies as an internal vertex of a u - w monophonic path P for some $u, w \in S$. If $v \neq u, w$ then obviously $u, w \in S'$ and S' is an open monophonic set of G. If $v = u$, then $v \neq$ w. Assume without loss of generality that $w \in G_i$. By Theorem 2.15, S' contains a vertex w' from $G_i (2 \leq i \leq k)$. Then $w' \neq v$. Let P' be a v - w' monophonic path. Then, since v is a cutvertex of G, it follows that the path P followed by the path P' is a w - w' monophonic path of G. Hence x is an internal vertex of a w - w' monophonic path with $w, w' \in S'$. Thus S' is an open monophonic set of G with $|S'| < |S|$. This is a contradiction to S a minimum open monophonic set. Thus no cutvertex of G belongs to S.

Remark 2.18 If $om(G) = n$ for a connected graph G of order n, it follows from Theorem 2.17 that G is a block.

We leave the following problem as an open question.

Problem 2.19 Characterize the class of graphs G of order n for which $om(G) = n$.

Corollary 2.20 For any tree T, the open monophonic number $om(T)$ equals the number of endvertices of T. In fact, the set of all endvertices of T is the unique minimum open monophonic set of T.

Proof. This follows from Theorems 2.5 and 2.17.

Theorem 2.21 *For every pair k, n of integers with* $2 \leq k \leq n$, *there exists a connected graph G of order n such that* $om(G) = k$.

Proof. For $k = n$, let $G = K_n$. Then the result follows from Corollary 2.6. For $2 \leq k < n$, let G be a tree of order n with k endvertices. Then the result follows from the Corollary 2.20.

Theorem 2.22 *For a connected graph G of order* $n \geq 2$, $om(G) = 2$ *if and only if there exist exactly two extreme vertices u and v such that every vertex of G is on a monophonic* $u - v$ *path.*

Proof. Let $om(G) = 2$. Let $S = \{u, v\}$ be a minimum open monophonic set of G. Then, necessarily both u and v are extreme vertices of G. Hence every vertex of G lies as an internal vertex of a $u - v$ monophonic path. The converse is obvious.

Theorem 2.23. *Let G be a non-complete connected graph of order n. If G contains a vertex of degree* $n - 1$, *then* $om(G) \leq n - 1$.

Proof. Let x be a vertex of degree $n - 1$. Since G is not complete, x is not an extreme vertex of G. Let $S = V(G) - \{x\}$. We show that S is an open monophonic set of G. Since x is not an extreme vertex of G, there exist non-adjacent neighbors y and z of x. Hence it follows that x lies as an internal vertex of a $y - z$ monophonic path for some $y, z \in S$. Now, let $u \in S$. If u is an extreme vertex of G, then there is nothing to prove. Suppose that u is not an extreme vertex of G. If $\langle N(u) \rangle$ is complete in $\langle S \rangle$, then $\langle N(u) \cup \{x\} \rangle$ is complete in G. Hence u is an extreme vertex of G, which is a contradiction. Therefore, $\langle N(u) \rangle$ is not complete in $\langle S \rangle$. This means that there exist non-adjacent neighbors v, w of u such that $v, w \in S$. Hence it follows that u lies as an internal vertex of a $v - w$ monophonic path so that S is an open monophonic set of G. Thus $om(G) \leq |S| = n - 1$.

For the wheel $W_5 = K_1 + C_4$, $om(W_5) = 4$ so that the bound in Theorem 2.23 is sharp. For the graph G in Fig. 2, $S = \{v_1, v_3\}$ is a minimum open monophonic set of G, $om(G) = 2 < 4$, so that the bound in Theorem 2.23 can be strict.

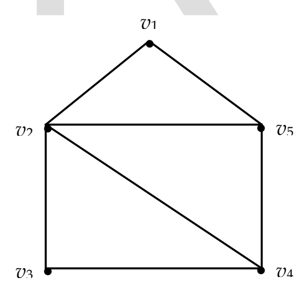

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Fig. 2: A noncomplete graph *G* with a vertex of deg 4 and *om*(*G*) < 4

Theorem 2.24 *For any tree T of order* $n \geq 3$, $om(T) = n - 1$ *if and only if T is the star* $K_{1,n-1}$.

Proof. This follows from Corollary 2.20, and also from the fact that a tree with exactly one cutvertex is a star.

In the following theorem, we construct a class of graphs G of order n for which $om(G) = n - 1$.

Theorem 2.25 *If* G_i $(1 \leq i \leq k)$ *are vertex disjoint connected graphs of order* $n_i \geq 2$, $k \geq 2$ *and* $om(G_i) = n_i$, *then* $om(K_1 + \cup G_i) = \sum n_i$.

Proof. Let $G = K_1 + \cup G_i$. Let $K_1 = \{v\}$. By Theorem 2.23, $om(G) \leq \sum n_i$. Suppose that $om(G) < \sum n_i$. Let S be a minimum open monophonic set of G. Then $|S| \leq \sum n_i - 1$. Since v is a cutvertex of G, by Theorem 2.17 $v \notin S$. Let $S_i = S \cap V(G_i)$ $(1 \leq i \leq k)$. $S_i \neq \phi$, by Theorem 2.15. Also $S = S_1 \cup S_2 \ldots \cup S_k$, $S_i \cap S_j$

$= \phi$, $i \neq j$. Since $|S| \leq \sum n_i - 1$, it follows that $|S_i| \leq n_i - 1$ for some i ($1 \leq i \leq k$). Hence S_i is a proper subset of vertices of G_i. We show that S_i is an open monophonic set of G_i. Let x be an extreme vertex of G_i. Then it is clear that x is also an extreme vertex of G so that by Theorem 2.5, $x \in S$. Hence $x \in S_i$. If x is not an extreme vertex of G_i, then since S is an open monophonic set of G, x lies as an internal vertex of a $y - z$ monophonic path P with $y, z \in S$. Now, since P is $y - z$ monophonic path and since v is a cutvertex of G, it follows that both $y, z \in S_i$. Thus S_i is an open monophonic set of G_i so that $om(G_i) \leq |S_i| \leq n_i - 1$, which is a contradiction to $om(G_i) = n_i$.

Now, we leave the following problem as an open question.

Problem 2.26 Characterize the class of graphs G of order n for which $om(G) = n - 1$.

For every connected graph G, $rad\ G \leq diam\ G \leq 2\ rad\ G$. Ostrand [6] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter, respectively, of some connected graph. Now, Ostrand's theorem can be extended so that the open monophonic number can also be prescribed, when $a < b \leq 2a$.

Theorem 2.27 *For positive integers r, d and $l \geq 2$ with $r < d \leq 2r$, there exists a connected graph G with $rad\ G = r$, $diam\ G = d$ and $om(G) = l$.*

Proof. When $r = 1$, let $G = k_{1,l}$. Then $d = 2$ and by Corollary 2.20 $om(G) = l$. For $r \geq 2$, we construct a graph G with the desired properties as follows:

Let $C_{2r} : v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order $2r$ and let P_{d-r+1}: $u_0, u_1, u_2, \ldots, u_{d-r}$ be a path of order $d - r + 1$. Let H be a graph obtained from C_{2r} and P_{d-r+1} by identifying v_1 in C_{2r} and u_0 in P_{d-r+1}. Let G be the graph obtained from H by adding $l - 2$ new vertices $w_1, w_2, \ldots, w_{l-2}$ to H and joining each vertex w_i ($1 \leq i \leq l - 2$) with the vertex u_{d-r-1} and also joining the edge $v_r v_{r+2}$. The graph G is show in Fig. 3. Then $rad\ G = r$ and $diam\ G = d$.

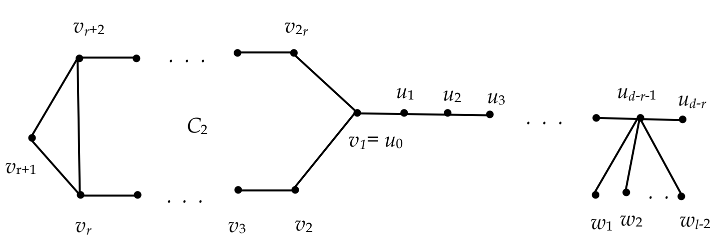

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Fig. 3: A graph *G* with radius *r*, diameter *d* and *om(G)* = *l*.

The graph G has $l - 1$ endvertices. Let $S = \{w_1, w_2, \ldots, w_{l-2}, u_{d-r}, v_{r+1}\}$. Then S is the set of all extreme vertices of G and it is clear that S is an open monophonic set of G so that by Theorem 2.5, $om(G) = l$.

3. The open monophonic number and diameter of a graph

For a graph G of order n and diameter d, it is proved that $g(G) \leq n - d + 1$. Since $m(G) \leq g(G)$, it follows that $m(G) \leq n - d + 1$. However, in the case of $om(G)$, it happens that $om(G) < n - d + 1$, $om(G) = n - d + 1$ and $om(G) > n - d + 1$. For the graph G given in Fig. 4 it is clear that $\{v_3, v_6\}$ is a minimum open monophonic set of G and so $om(G) = 2$. Since $n = 6$ and $d = 4$, we have $n - d + 1 = 3$ and so $om(G) < n - d + 1$. For the Wheel $W_5 = K_1 + C_4$, by Theorem 2.14, so $om(W_5) = 4$. Since $n = 5$ and $d = 2$, we have $n - d + 1 = 4$ and so $om(W_5) = n - d + 1$. Also for the graph G given in Fig. 5, it is clear that $\{v_1, v_2, v_3, v_6, v_7, v_8\}$ is a minimum open monophonic set of G and so $om(G) = 6$. Since $n = 8$ and $d = 4$ we have $n - d + 1 = 5$ and so $om(G) > n - d + 1$.

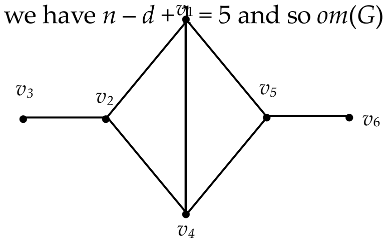

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Fig. 4: A graph with *om*(*G*) < *n*-*d*+1.

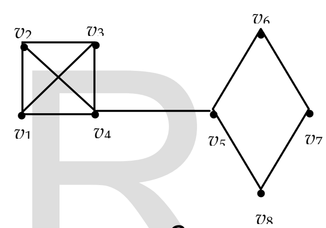

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Fig. 5: A graph with *om*(*G*) > *n*-*d*+1.

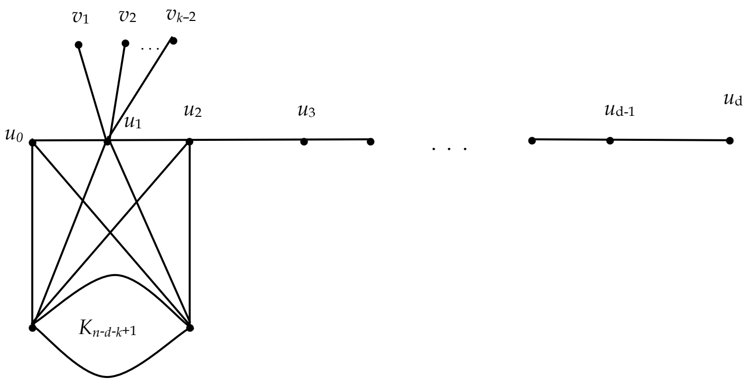
Theorem 3.1 For *every non-trivial tree T of order n, $om(T) = n - d + 1$ if and only if T is a caterpillar.*

Proof. Let T be a non-trivial tree. Let $d(u, v) = d$ and $P : u = v_0, v_1, v_2, \ldots, v_{d-1}, v_d = v$ be a diametral path. Let k be the number of endvertices of T and l the number of internal vertices of T other than $v_1, v_2, \ldots, v_{d-1}$. Then $n = d - 1 + k + l$. By Theorem 2.5, $om(T) = k$ and so $om(T) = n - d + 1$ if and only if $l = 0$, if and only if all the internal vertices of T lie on the diametral path P, if and only if T is a caterpillar.

Now, we prove the following realization result.

Theorem 3.2 *If n, d and k are integers such that $2 \leq d < n$, $2 \leq k < n$ and $n - d - k + 1 \geq 0$, then there exists a graph G of order n, diameter d and $om(G) = k$.*

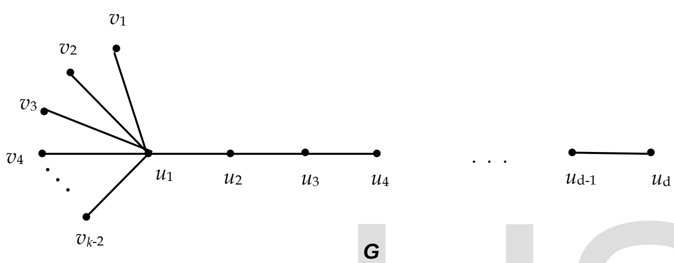
Proof. Let $P_d : u_0, u_1, u_2, \ldots, u_d$ be a path of length d. First, let $n - d - k + 1 \geq 1$. Let $K_{n-d-k+1}$ be the complete graph with vertex set $\{w_1, w_2, \ldots, w_{n-d-k+1}\}$. Let H be the graph obtained from P_d and $K_{n-d-k+1}$ by joining each vertex of $K_{n-d-k+1}$ to u_i for $i = 0, 1, 2$. Let G be the graph obtained from H by adding $k - 2$ new vertices $v_1, v_2, \ldots, v_{k-2}$ to H and by joining each vertex v_i ($1 \leq i \leq k - 2$) with the vertex u_1 of P_d. The graph G is shown in Fig. 6 and G has order n and diameter d. Let $S = \{u_0, u_d, v_1, v_2, \ldots, v_{k-2}\}$ be the set of extreme vertices of G. Then it is clear that S is an open monophonic set of G and so by Theorem 2.5 $om(G) = k$.



G

Fig. 6: A graph *G* with order *n*, diameter *d* and *om*(G) = *k*.

For $n - d - k + 1 = 0$, let G be the tree given in Fig. 7. Then it is clear that G has diameter d, order $d + k - 1 = n$ and $om(G) = k$.



G

Fig. 7: A graph with order *n* = *d* + *k* - 1, diameter *d* and *om*(G) = *k*.

4. Addition of a pendant edge and open monophonic number

A fundamental question in graph theory concerns how the value of a parameter is affected by making a small change in the graph. In this section, we study how the open monophonic number of a graph is affected by the addition of a pendant edge.

Theorem 4.1 *If G' is a graph obtained by adding a pendant edge to a connected graph* G, then $om(G) - 2 \le om(G') \le om(G) + 1$.

Proof. Let G' be the graph obtained from G by adding a pendant edge uv, where u is not a vertex of G and v is a vertex of G. Let S' be a minimum open monophonic set of G'. Then $om(G') = |S'|$. Since u is an endvertex of G', by Theorem 2.5, $u \in S'$. Also since v is a cutvertex of G', by Theorem 2.17, $v \notin S'$. We consider two cases.

Case 1. v is an extreme vertex of G.

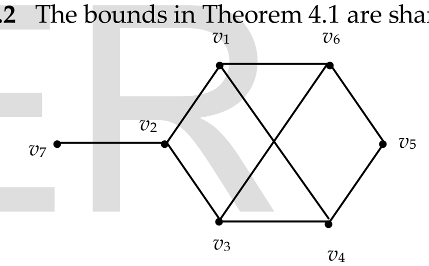
Let $S = (S' - \{u\}) \cup \{v\}$. Then it is clear that $|S| = |S'| = om(G')$. We show that S is an open monophonic set of G. Let x be a vertex of G. Suppose that x is an extreme vertex of G. If $x = v$, then $x \in S$. If $x \neq v$, then x is also an extreme vertex of G' and so $x \in S'$. Since $x \neq u$, v we have $x \in S$. Now, if x is not an extreme vertex of G, then $x \neq v$. Since S' is an open monophonic set of G', x lies as an internal vertex of a $y - z$ monophonic path with $y, z \in S'$. If $u \neq y, z$, then it is clear that x is an internal vertex of a $y - z$ monophonic path with $y, z \in S$. If $u = y$ or $u = z$, say $y = u$, then since $x \neq v$ it is easily verified that x is an internal vertex of a $v - z$ monophonic path with $v, z \in S$. Thus S is an open monophonic set of G so that $om(G) \le |S| = |S'| = om(G')$.

Case 2. v is not an extreme vertex of G.

Since v is not an extreme vertex of G, there exists vertices v', v'' such that v' and v'' are not adjacent in G, and v is adjacent to both v' and v''. Hence v lies in the v'- v'' geodesic of length 2 so that v lies on a v' - v'' monophonic path in G. Let $S = (S' - \{u\}) \cup \{v, v', v''\}$. Then $|S| \le |S'| + 2$. We show that S is an open monophonic set of G. Let x be a vertex of G such that $x \neq v$. If x is an extreme vertex of G, then it clear that x is also an extreme vertex of G'. Hence $x \in S'$. Also, since $x \neq u$, it follows that $x \in S$. Now, assume that x is not an extreme vertex of G. Since $x \neq u$, it is clear that x is also not an extreme vertex of G' and so x lies as internal vertex of a $y - z$ monophonic path. Then, proceeding as in Case 1, we see that S is an open monophonic set of G. Hence $om(G) \le |S| \le |S'| + 2 = om(G') + 2$. Combining both cases, we see that $om(G) - 2 \le om(G')$.

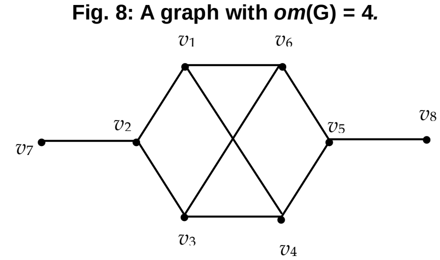
Now, we look for the upper bound of $om(G')$. Let S be a minimum open monophonic set of G. Since u is an extreme vertex of G', it is clear that $S \cup \{u\}$ is an is an open monophonic set of G' and so $om(G') \le |S \cup \{u\}| = om(G) + 1$. Thus $om(G) - 2 \le om(G') \le om(G) + 1$.

Remark 4.2 The bounds in Theorem 4.1 are sharp.



G

Fig. 8: A graph with *om*(G) = 4.



G'

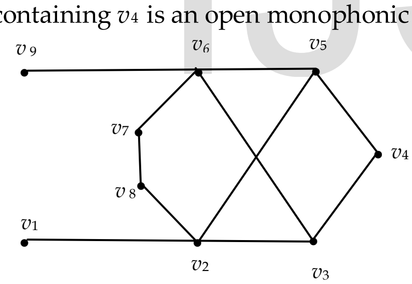
Fig. 9: A graph with *om*(G′) = *om*(G)+1

For the graph G given in Fig. 8, it is easily seen that no 3-element subset of vertices of G is an open monophonic set of G. Now, the set $S = \{v_4, v_5, v_6, v_7\}$ is an open monophonic set of G so that $om(G) = 4$. Let G' be the graph in Fig. 9 obtained from G by adding the pendant edge v_5v_8. Then $S' = \{v_7, v_8\}$ is a minimum open monophonic set of G' so that $om(G') = 2$. Thus $om(G) - 2 = om(G')$. For any path G of length at least 2, we have $om(G) = 2$. Let G' be the tree obtained from G by adding the pendant edge at a cutvertex of G. The $om(G') = 3$. Thus $om(G') = om(G) + 1$.

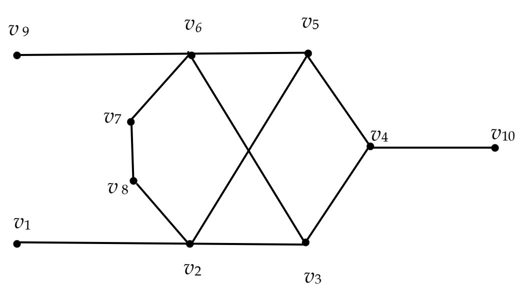
Theorem 4.3 *If G' is a graph obtained from a connected graph G by* *adding a pendant edge uv, where u is not a vertex of G and v is a vertex of G and if om(G′) = om(G) + 1, then v does not belong to any minimum open monophonic set of G.*

Proof. Assume that v belongs to some minimum open monophonic set S of G. Let $S' = (S - \{v\}) \cup \{u\}$. Then $|S| = |S'|$. We show that S' is an open monophonic set of G'. Let x be a vertex in G'. If x is an extreme vertex of G', then $x \neq v$. If $x = u$, then by definition of S′, $x \in S'$. If $x \neq u$, then x is an extreme vertex of G and so $x \in S$. Hence it follows that $x \in S'$. Suppose that x is not an extreme vertex of G'. Then $x \neq u$. It is clear that x is a vertex of G. If $x = v$, then x lies as an internal vertex of a $y - u$ monophonic path for any $y \in S$, with $y \neq x$. If $x \neq v$, then since S is an open monophonic set of G, x is an internal vertex of a $y - z$ monophonic path with $y, z \in S$. If $v \neq y, z$, then $y, z \in S'$. If $v = y$ or $v = z$, say $y = v$, then x lies as an internal vertex of a $v - z$ monophonic path with $v, z \in$ S. Since v is a cut vertex of G', it is clear that x is an internal vertex of a $u - z$ monophonic path with $u, z \in$ S′. Hence S' is an open monophonic set of G' so that $om(G') \leq |S'| = |S| = om(G)$, which is a contradiction.

Remark 4.4 The converse of Theorem 4.3 need not be true. For the graph G given in Fig. 10, it is easily seen that $S = \{v_1, v_3, v_5, v_9\}$ is a minimum open monophonic set so that $om(G) = 4$. Let G' be the graph given in Fig. 11, obtained from G by adding the pendant edge v_4v_{10}. Then $S' = \{v_1, v_9, v_{10}\}$ is the unique minimum open monophonic set of G' so that $om(G') = 3$. Thus $om(G') \neq om(G) + 1$. It is easily seen that no 4-element subset of vertices of G containing v_4 is an open monophonic set of G.



G
Fig. 10: A graph with *om*(G) = 4.



G′
Fig. 11: A graph with *om*(G′) ≠ *om*(G) + 1

We leave the following problem as an open question.

Problem 4.3 Characteristize the class of graphs G for which $om(G') = om(G) + 1$, where G' is the graph obtained from G by adding a pendant edge of G.

Conclusion

This paper introduces a new parameter known as open monophonic number of a graph. The open problems given in this paper are challenging. Further, this concept can be extended to conditional parameters.

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