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Review of Generalized Fermat Number & Few Properties of Prime Number with respect to N-equation

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ABSTRACT

It cannot be denial of the fact that with the help of N-equation theory we have received the answers of so many unsolved problems in number theory. N-equation is nothing but the systematic arrangement of all Pythagorean triplets, details of which was first published in this journal IJSER vide Aug-edition 2013 and by which first we have been able to penetrate the mystery of Beal equation. Then with further development of N-equation we unveiled the mystery of Generalized Fermat Number published in same journal Nov-edition, 2015 and so many related conjectures in between. Now it is felt necessary to review the proof of Generalized Fermat Number once again just to have a clear picture and to highlight few properties of prime numbers generated out of this. The most important highlight of which is how twin primes form with respect to N-equation.

Key words

Prime and Pure prime, Maximum Wing (W_{max}), Minimum Wing (W_{min}), Intermediate Wing (W_{int}), Functional consecutive prime etc

INTRODUCTION:

With usual notation of N-equation i.e. $a^2 + b^2 = c^2$ & its comparable equation $(\alpha^2 - \beta^2)^2 + (2\alpha\beta)^2 = (\alpha^2 + \beta^2)^2$ few assumptions and some important points are given below.

1. a, b or α, β are combination of odd even integers where a is odd and b is even element and (a, b, c) or (α, β) are prime to each other.
2. The natural constant i.e. c – max(a, b) of a N-equation is denoted by k.
3. a_p or c_p denotes prime whereas a_q or c_q denotes product of two primes of a N-equation.
4. e & o generally denotes even & odd integers respectively.
5. u_x is an integer with unit digit x.
6. Expressions of the form $(\alpha^2 \pm \beta^2)$ are usually said to be positive or negative wing.
7. P & P^2 both can be assumed as prime as because both cannot produce more than one positive or negative prime wing. But P^3 is to be considered as product of two primes as it can produce two positive or two negative wings, of course, one is prime wing & other is composite wing. P can be said as pure prime.
8. The primes which are capable of producing a positive wing are known as 2nd kind prime and rest of the primes which are not capable of producing positive wing are of 1st kind.
9. Any prime 2p + 1, of 1st kind or 2nd kind have a single expression of 'Negative prime wing' i.e. $(p + 1)^2 - p^2$ and all 2nd kind primes have a single expression of positive wing $\alpha^2 + \beta^2$.
10. All primes 1st kind or 2nd kind only satisfy 'a' of a N-equation under k = 1 but 2nd kind primes are distributed to 'c' of N-equation for all values of k.

1. Proof of a Generalized Fermat Number (GF_n) that if it is composite for n = m then it will be always composite for n ≥ m.

By N_d operation in a product of two numbers we have,
$(e_1^2 - o_1^2)(e_2^2 - o_2^2) = (e_1e_2 + o_1o_2)^2 - (e_1o_2 + e_2o_1)^2 = (e_1e_2 - o_1o_2)^2 - (e_1o_2 - e_2o_1)^2$
Here, odd & even elements of maximum values & minimum values lie on opposite side of equality.
Sum of maximum elements = $(e_1 + o_1)(e_2 + o_2)$ which is composite & Sum of minimum elements = $(e_1 \sim o_1)(e_2 \sim o_2)$ which may be composite or may not be
Let us consider a number which is a product of n number of primes i.e. $N = x_1x_2x_3 \ldots\ldots x_n$ in ascending order of primes. It has 2^{n-1} nos. of negative prime wings out of which one wing (W_{max}) contains maximum sum of elements and one wing (W_{min}) contains minimum sum of elements. All other can be said as intermediate wings (W_{int}). For W_{max} elements are obviously consecutive. As both the elements increase their values the wing is approaching towards W_{max}.
Now for $n \geq 3$ & $x_n < (x_1x_2x_3 \ldots\ldots x_{n-1})$ we have $N = \alpha_1^2 - \beta_1^2 = \alpha_2^2 - \beta_2^2 = \alpha_3^2 - \beta_3^2 = \ldots\ldots = \alpha_t^2 - \beta_t^2$ and in all cases $(\alpha_i + \beta_i)$ is composite. Because W_{min} can be equated with any other wing and this pair of wings have definitely come by the product of two group of factors i.e. x_n & $(x_1x_2x_3 \ldots\ldots x_{n-1})$. x_n being the minimum W_{min} is also composite. But for $x_n > (x_1x_2x_3 \ldots\ldots x_{n-1})$ it is composite in all cases except $(\alpha + \beta)_{min} = x_n$.
Let us take an example N = 3.5.7 where 7 < 3.5.
Here $N = 53^2 - 52^2 = 19^2 - 16^2 = 13^2 - 8^2 = 11^2 - 4^2$ & in all cases sum of elements is composite.
But for N = 3.5.17 where 17 > 3.5, $N = 128^2 - 127^2 = 44^2 - 41^2 = 28^2 - 23^2 = 16^2 - 1^2$
Here sum of elements for all cases is composite except or W_{min} where it is 17 i.e. prime. Reasons for these two kinds are easily understood.
Now let us consider the case of Generalized Fermat Number (GF_n).
$\alpha^4 - \beta^4$ i.e.$(\alpha^2)^2 - (\beta^2)^2 = (\alpha^2 + \beta^2)(\alpha + \beta)(\alpha - \beta)$. If α, β are consecutive $(\alpha - \beta)$ factor is to be ignored.
Here, $(\alpha^2 + \beta^2) > (\alpha + \beta)(\alpha - \beta)$. Hence $(\alpha^2)^2 - (\beta^2)^2 = w_{min}$. So $.\alpha^2 + \beta^2$ i.e. sum of the elements may be composite or may be prime. If it is found to be composite say pq then $(\alpha^2)^2 - (\beta^2)^2 = pq(\alpha + \beta)(\alpha - \beta)$ where the factor of maximum value has been reduced with respect to which $(\alpha^2)^2 - (\beta^2)^2$ is an intermediate wing.
Now for $(\alpha^4)^2 - (\beta^4)^2 = (\alpha^4 + \beta^4)[pq(\alpha + \beta)(\alpha - \beta)]$ where elements of LHS have increased their values i.e. wing is approaching towards W_{max} and it is bound to be an intermediate wing. Hence,$(\alpha^4 + \beta^4)$ is composite. Similarly, $(\alpha^8 + \beta^8)$ or higher exponents in Fermat order i.e. all GF_n are composite.
The fact can be shown by an example. In a product of 3.5.7 if we consider it as a product of two integers i.e. 15 & 7 definitely it will produce two wings, one is W_{max} and other is $W_{min=}$ by the product wings $(8^2 - 7^2)(4^2 - 3^2)$. Now if we split up 15 into 3.5 we will get total 4 wings where W_{min} will act as a W_{int}.
Hence, from $(\alpha^{2^\wedge n})^2 - (\beta^{2^\wedge n})^2 = (GF_n)(GF_{n-1})\ldots\ldots(GF_1)(GF_0)(\alpha - \beta)$ we can say,
LH side will act as a W_{min} so long all the GF factors of RH side are primes. If any GF factor of RH side is found to be composite then LH side will act as a W_{int}.
Note: In a N-equation $a^2 + b^2 = c^2$ for k = 1, c is in the form of $(\alpha^2 + \beta^2)$ where $(\alpha - \beta) = 1$and hence $a = (\alpha + \beta)$. As a & c both cannot be in power form we can say that if $(\alpha^2 + \beta^2)$ is prime, $(\alpha + \beta)$ is a prime (P) or square of a prime (P^2). So far leading factor GF_0 is concerned if GF_0 i.e. $(\alpha + \beta) = P^2$ then it should be considered as prime because P & P^2 both cannot produce more than one wing.
For non consecutive elements if $(\alpha^2 + \beta^2)$ is prime then $(\alpha + \beta)$ must be a pure prime but reverse is not true.
This implies, for a N-equation where $k \neq 1$ if $c = (\alpha^2 + \beta^2)$ is a prime corresponding $a = (\alpha^2 - \beta^2)$ contains greatest prime factor which is a pure prime. Similarly, for non consecutive elements if $\alpha + \beta$ is composite $(\alpha^2 + \beta^2)$ must be composite but reverse is not true.
In view of the above we can establish the following theorems.

2.1 In a N-equation $a^2 + b^2 = c^2$ for k = 1, if c is prime a is prime or square of a prime but reverse is not true. All c must be of the nature where unit digit is one and tenth digits is even excepting the cases $3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$, $25^2 + 312^2 = 313^2$.

2.2 For consecutive elements $c = (u_3)^2 + (u_2)^2$ or $c = (u_8)^2 + (u_7)^2$ cannot satisfy N-equation for k = 1, where a is a prime or square of a prime except the cases $5^2 + 12^2 = 13^2$ and $25^2 + 312^2 = 313^2$ and hence c is always composite except 13 & 313 [u_x represents an integer of unit digit x].

Obviously, c is of the form 4x + 1. Hence if unit digit of c is 1, 10^{th} digit must be even.
Now, c must be prime and sum of squares of two consecutive integers. Hence its possibilities are:
$c = (u_1)^2 + (u_0)^2$ or $= (u_3)^2 + (u_2)^2$ or $= (u_5)^2 + (u_4)^2$ or $= (u_6)^2 + (u_5)^2$ or $= (u_8)^2 + (u_7)^2$ or $= (u_0)^2 + (u_1)^2$
For $c = (u_3)^2 + (u_2)^2$ or $c = (u_8)^2 + (u_7)^2$ corresponding $a = u_5$ where accepted values of a can be 5 or 5^2 only. In all other cases where a is a prime or square of a prime, $c = u_1$ nature.
Hence, for consecutive elements $c = (u_3)^2 + (u_2)^2$ or $c = (u_8)^2 + (u_7)^2$ always produces composite except 13 & 313.

3.1 For non-consecutive two elements $c = (2x)^2 + (u_1 \text{ or } u_9)^2$ where x = 1, 2, 3, ……. cannot be prime both for x = n & x = n + 1.

3.2 For non-consecutive two elements $c = (2x + 1)^2 + (u_4 \text{ or } u_6)^2$ where x = 1, 2 3, ……. cannot be prime both for x = n & x = n + 1.

3.3 For 2^{nd} kind N-equation where $k = 2(u_1 \text{ or } u_9 \text{ or } u_4 \text{ or } u_6)^2$ we cannot receive two primes consecutively for c = f(x) and c = f(x + 1) except 5, 7 for $k = 2.1^2$ produced by $4^2 + 1$ & $6^2 + 1$.

3.4 Here, when sum of the elements represents twin primes for x = n & x = n + 1, both c = f(n) & f(n + 1) cannot be prime.

3.5 For all other values of k for a 2^{nd} kind N-equation or the expression c of 3.1 & 3.2, when c = f(n) and c = f(n + 1) both are found to be prime then there must exist twin primes by their sum of elements

3.6 If there exists infinitely many functional consecutive primes then the existence of twin primes is also infinite or vice-versa.

For $c_p = (2x)^2 + (u_1 \text{ or } u_9)^2$ obviously $x \neq u_1$ or u_4 or u_6 or $u_9 \Rightarrow$ accepted values of consecutive $x = u_2$, u_3 or u_7, u_8. But for both u_2 & u_3 or u_7 & u_8, $(2x + u_1 \text{ or } u_9)$ cannot be prime.
Similarly, for $c_p = (2x + 1)^2 + (u_4 \text{ or } u_6)^2$ obviously $x \neq u_1$ or u_3 or u_6 or $u_8 \Rightarrow$ accepted values of consecutive $x = u_4$, u_5 or u_9, u_0. But for both u_4 & u_5 or u_9 & u_0, $(2x + 1 + u_4 \text{ or } u_6)$ cannot be prime.
Nature of c for a 2^{nd} kind N-equation where $k = 2(u_1 \text{ or } u_9 \text{ or } u_4 \text{ or } u_6)^2$ follows the same sequence.
Regarding 3.4, 3.5 & 3.6 one thing is to be remembered that if $\alpha^2 + \beta^2$ is prime $\alpha + \beta$ must be prime for non-consecutive elements α & β. But it does not mean that if $\alpha^2 + \beta^2$ is composite $\alpha + \beta$ cannot be prime.

4. There exist infinitely many composites that satisfy c of a N-equation for any value of k.

It is quite obvious that c is always in the form of $\alpha^2 + \beta^2$ where if $\alpha + \beta$ is composite $\alpha^2 + \beta^2$ must be composite.
In a 1^{st} kind N-equation, nature of c is: $c = (x + \lambda)^2 + x^2$ where λ is constant & $k = \lambda^2$, x varies as x = 1, 2, 3,……..
So, we can choose infinitely many x so that sum of the elements i.e $(2x + \lambda)$ is composite.
Hence, c is extended up to infinity as a composite.
Similarly, for 2^{nd} kind $c = (2x)^2 + \lambda^2$ or $c = (2x + 1)^2 + \lambda^2$ where $k = 2\lambda^2$ & x = 1, 2, 3, ……..
Here also we can choose infinitely many x so that $(2x + \lambda)$ or $(2x + 1 + \lambda)$ is composite. Hence, c is extended up to infinity as a composite.

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This infinite existence of composite numbers also includes the product of two pure primes because infinitely many composites can be brought under the sum of elements among which we can choose those which can be divided into two parts α & β so that $\alpha^2 + \beta^2$ is a product of two 2nd kind primes.

5. Condition for $\gcd(c_i, c_j) = c_i$ for a 2nd kind N-equation $a^2 + b^2 = c^2$.

Say, $c_i = \alpha^2 + \beta^2$ for $k = 2\beta^2$ and $c_j = (x^2 + y^2)(\alpha^2 + \beta^2) = (\alpha x \pm \beta y)^2 + (\beta x -/+ \alpha y)^2$
If c_j satisfy the same equation $2\beta^2 = 2(\beta x - \alpha y)^2$ [one case]
$\Rightarrow y = \beta(x - 1)/\alpha$. As $\gcd(\alpha, \beta) = 1$ for integer of y say $x - 1 = \omega\alpha$ i.e. $x = \omega\alpha + 1$ & $y = \omega\beta$ or $x = \omega\alpha - 1$ & $y = \omega\beta$ and $y = \omega\alpha + 1$ & $x = \omega\beta$ or $y = \omega\alpha - 1$ & $x = \omega\beta$, obviously ω is even.
Hence, there exist infinitely many cases of $\gcd(c_i, c_j) = c_i$. and it confirms that $\gcd(c_i, c_{i+1}) = 1$

6.1 For a 2nd kind N-equation $a^2 + b^2 = c^2$ where $k = 2\beta^2$, $c_i.c_{i+1}$ always satisfies the 2nd kind N-equation of $k = 2(2\beta)^2$.

Say, $c_i = (2x)^2 + \beta^2$ where $k = 2\beta^2 \Rightarrow c_{i+1} = (2x + 2)^2 + \beta^2$ and $c_i.c_{i+1} = (4x^2 + 4x + \beta^2) + (2\beta)^2$ where $k = 2(2\beta)^2$
Again if we consider $c_i = (2x + 1)^2 + \beta^2$ where $k = 2\beta^2 \Rightarrow c_{i+1} = (2x + 3)^2 + \beta^2$ and $c_i.c_{i+1} = (4x^2 + 8x + 3 + \beta^2) + (2\beta)^2$ where $k = 2(2\beta)^2$. Hence proved.
Note: for $k = 2\beta^2$ where β is odd $c \neq c_i.c_{i+1}$ of any value of k. If β is even, $c = c_i.c_{i+1}$ of $k = 2(\beta/2)^2$.

6.2 For a 1st kind N-eq. $a^2 + b^2 = c^2$ where $k = \lambda^2$, $c_i.c_{i+1}$ always satisfies the 2nd kind N-equation of $k = 2\lambda^2$

Say, $c_i = (x + \lambda)^2 + x^2$ where $k = \lambda^2 \Rightarrow c_{i+1} = (x + 1 + \lambda)^2 + (x + 1)^2$ and $c_i.c_{i+1} = (2x^2 + 2x\lambda + \lambda^2 + 2x + \lambda) + \lambda^2$ where $k = 2\lambda^2$. $\Rightarrow$ In all 2nd kind N-equation c contains the product of consecutive two c i.e. $c_i.c_{i+1}$ of other k at an increasing intervals. This interval is filled up by new generated primes and some unidentified composites and then multiple of all c as per Point-5.

In general for a 1st kind N-equation of $k = \lambda^2$ the product of two c at a gap of g i.e. $f(x)f(x + g)$ satisfies c of N-eq. for $k = 2(g\lambda)^2$. Similarly for 2nd kind it will satisfy where $k = 2(2g\mu)^2$ against $k = 2\mu^2$ provided 2nd kind N-eq. is to be ensured by the ratio of the elements $> \sqrt{2} + 1$.

7. If c_1, c_2 both are of 2nd kind N-equation then one of the wings of c_1c_2 must satisfy a 2nd kind N-equation.

Say, $c_1 = \alpha_1^2 + \beta_1^2$ & $c_2 = \alpha_2^2 + \beta_2^2$ where $\alpha_1/\beta_1, \alpha_2/\beta_2 > \sqrt{2} + 1$ i.e. both are of 2nd kind.
Say $\alpha_1/\beta_1 = p + \omega_1$ & $\alpha_2/\beta_2 = p + \omega_2$ where $p = \sqrt{2} + 1$. Now one of the wings of $c_1c_2 = (\alpha_1\alpha_2 + \beta_1\beta_2)^2 + (\alpha_1\beta_2 - \alpha_2\beta_1)^2$ where ratio of the elements is $(\alpha_1\alpha_2 + \beta_1\beta_2)/(\alpha_1\beta_2 - \alpha_2\beta_1) = \{(p + \omega_1)(p + \omega_2) + 1\}/(\omega_1 - \omega_2) =$
$\{p^2 + p(\omega_1 + \omega_2) + \omega_1\omega_2\}/(\omega_1 - \omega_2) > p$. Hence this wing of c_1c_2 is of 2nd kind N-equation.

8. How twin primes form with respect to N-equation.

According to previous theorems we can say the N-equation of $k = 2(\lambda)^2$ has infinitely many composites of c which are the product of two primes say, P_1 & P_2 where P_1 & P_2 are the functional consecutive primes of $k = 2(\lambda/2)^2$ or λ^2 if λ is odd and where corresponding sum of the elements of P_1 & P_2 are twin primes.

In reverse way suppose $c_p = \alpha^2 + \beta^2$ satisfies a 2nd kind N-equation. $\Rightarrow \alpha + \beta$ is also a prime. Divide $\alpha + \beta + 2$ into two parts α_1 & β_1 so that $\alpha_1^2 + \beta_1^2$ is another prime of 2nd kind N-equation. Then $c = (\alpha^2 + \beta^2)(\alpha_1^2 + \beta_1^2)$ has two wings where one wing say $\alpha_2^2 + \beta_2^2$ must belong to 2nd kind N-equation of $k = 2(\beta_2)^2$.
Hence existence of twin primes is fully dependent on existence of c-consecutive primes of 2nd kind N-equation, product of which satisfy another 2nd kind N-equation.

CONCLUSION

I believe that with the help of this theory that for two non-consecutive elements if $\alpha^2 + \beta^2$ is prime $\alpha + \beta$ must be prime or if $\alpha + \beta$ is composite $\alpha^2 + \beta^2$ must be composite where reverse is not true for both the cases, we will be able to prove so many conjectures particularly related to prime numbers. In my earlier papers published during last six months several attempts were made to penetrate the mystery of Fermat number. But all those logics were not so sound as it reveals now. Whether $\alpha^2 + \beta^2$ will be prime or composite if $\alpha + \beta$ is prime is still unknown and needs further investigation but it does not depend upon the number of factors $(\alpha - \beta)$ have. $(\alpha - \beta)$ may have many factors. In my papers published in last Nov-edition it was written that to be $\alpha^2 + \beta^2$ composite $(\alpha + \beta)(\alpha - \beta)$ should have at least four factors. This was not the correct projection.
Regarding infinite existence of twin primes the proof seems to be not so sound. However, in the same line of thinking we do hope that in near future we will be able to have the answers of this along with so many conjectures of primes.

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