RMSE Based Performance Analysis of Marginalized Particle Filter and Rao Blackwellised Particle filter for Linear/Nonlinear State Space Models

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Abstract: - Particle filters and Rao Blackwellised particle filter have been widely used in solving nonlinear filtering problems. The particle filter is fairly easy to implement and tune, its main drawback is that it is quite computer intensive, with the computational complexity increasing quickly with the state dimension. One solution to this problem is to marginalize out the states appearing linearly in the dynamics. The result is that one Kalman filter is associated with each particle. The main contribution in this paper is to analyse the performance of the marginalized particle filter and Rao Blackwellised Particle filter for a general nonlinear state-space model. In an extensive Monte Carlo simulation different computational aspects are studied and compared with the derived theoretical results.

Index terms: - State estimation, particle filter, Kalman filter, marginalization, Nonlinear estimation, marginalization, complexity analysis, Rao Blackwellised particle filter.

I. INTRODUCTION

The nonlinear non Gaussian filtering problem considered here consists of recursively computing the posterior probability density function of the state vector in a general discrete time state- space model, given the observed measurements. Such a general model can be formulated as considered here consists of recursively computing the posterior probability density function of the state vector in a general discrete time state-space model, given the observed measurements. Such a general model can be formulated as:

$$x_{t+1} = f(x_t, w_t) \quad (1a)$$

$$y_t = h(x_t, e_t) \quad (1b)$$

Here, $y_t$is the measurement at time t, x_t is the state variable. $w_t$is the process noise, $e_t$is the measurement noise and f, h are two arbitrary nonlinear functions. The two noise densities $P_{w_t}$and p_{e_t} are independent and are assumed to be known.

The posterior density p(x_t $|Y_t$)where Y_t= {y_i}, i=0 to t is given by the following general measurement recursion:

$$p(x_t|Y_t) = \frac{p(y_t|x_t)p(x_t|Y_{t-1})}{p(y_t|Y_{t-1})} \quad (2a)$$

$$p(y_t \; |Y_{t-1}) = \int p(y_t|x_t) * p(x_t \; |Y_{t-1}) \, dx_t \quad (2b)$$

and the following time recursion

$$p(x_{t+1} \; |Y_t) = \int p(x_{t+1}|x_t) * p(x_t|Y_t) dx_t \quad (2c)$$

is initiated by $p(x_0 \; |Y_{-1}) = p(x_0)$. For linear Gaussian models the integrals can be solved analytically with a finite dimensional representation. This leads to the Kalman filter recursions, where the mean and the covariance matrix of the state are propagated [1]. More generally, no finite dimensional representation of the posterior density exists. Thus, several numerical approximations of the integrals (2) have been proposed. A recent important contribution is to use simulation based methods from mathematical statistics, sequential Monte Carlo methods, commonly referred to as particle filters" [2], [3], [5]".

The state vector can be represented as:

$$x_t = \begin{pmatrix} x_t{}^l \\ x_t{}^n \end{pmatrix} \quad (3)$$

Where $x_t{}^l$ denotes the state variable with conditionally linear dynamics and $x_t{}^n$ denotes the nonlinear state variable [6],[10].Using Bayes' theorem we can then marginalize out the linear state variables from (1) and estimate them using the Kalman filter [7], which is the optimal filter for this case. The nonlinear state variables are estimated using the particle filter. This technique is sometimes referred to as Rao-Blackwellisation [6]. The marginalized particle filter has been successfully used in several applications, for instance in aircraft navigation [10], underwater navigation [9], communications [4], [11], nonlinear system identification [10].

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In order to make presentation easy to follow, here we have considered one model. To illustrate the use of the marginalized particle filter a synthetic example is given in Section V the simulation result is given in section VI and the conclusion are stated in section VII.

II.PARTICLE FILTER

Particle filtering is a general Monte Carlo (sampling) method for performing inference in State-space models where the state of a system evolves in time and information about the state is obtained via noisy measurements made at each time step. In a general discrete-time state-space model, the state of a system evolves according to:

$$x_k = f_k(x_{k-1}, v_{k-1}) \tag{4a}$$

Where x_k is a vector representing the state of the system at time k, v_{k-1} is the state noise vector, $f_k$is a possibly non-linear and time-dependent function describing the evolution of the state vector. The state vector x_k is assumed to be latent or unobservable. Information about x_k is obtained only through noisy measurements of it, z_k, which are governed by the equation:

$$z_k = h_k(x_k, n_k) \tag{4b}$$

Where h_k a possibly non-linear and time-dependent is is function describing the measurement process and n_k is the measurement noise vector.

Particle filtering essentially combines the particles at a particular position into a single particle, giving that particle a weight to reflect the number of particles that were combined to form it. This eliminates the need to perform redundant computations without skewing the probability distribution. Particle filtering accomplishes this by sampling the system to create N particles, then comparing the samples with each other to generate an importance weight. After normalizing the weights, it resamples N particles from the system using these weights. This process greatly reduces the number of particles that must be sampled, making the system much less computationally intensive.

Particle filtering technique is used for filtering nonlinear dynamical systems driven by non-Gaussian noise processes. The purpose of particle filter is to estimate the states $\{S_1 \ldots\ldots\ldots . S_t\}$ recursively using the sampling technique.

To estimate the states, the particle filter approximates the posterior distribution $p(S_t|Z_1{:}t)$ with a set of samples $\{S_1 \ldots\ldots\ldots S_t\}$ and a noisy observation $\{Z_1 \ldots\ldots\ldots . Z_t\}$. In particle filtering, the probability density distribution of the target state is represented by a set of particles. The posterior density of the target state for a given input image is calculated and represented as a set of particles. In other words, a particle is a hypothesis of the target state, and each hypothesis is evaluated by assessing how well the hypothesis fits the current input data. Depending on the scores of the hypotheses, the set of hypotheses is updated and regenerated in the next time step. The particle filter consists of two components, state transition model and observation model. They can be written as:

Translation Model: St = Ft (St−1 , Nt),

Observation Model: Zt = Ht (St , Wt). (4c)

The transition function F_t approximates the dynamics of the object being tracked using previous state S_{t-1} and the system noise N_t . The measurement Ht models a relationship among the noisy observation Z_t the hidden state S_t, the observation noiseW_t. We can characterize transition probability $P(S_t|S_{t-1})$with the state transition model, and likelihood $P(Z_t|S_t)$ with the observation model.

III. RAO-BLACKWELLISED PARTICLE FILTER

An additional way to improve our computational efficiency is to reduce the complexity of each sample. If we sample 2 variables instead of 3, then we will reduce the number of dimensions in the system. RBPF reduces the number of variables that must be sampled by identifying variables that do not need to be sampled to be computed. The advantage of the Rao-Blackwellised particle filter is that it allows the state variables to be splitted into two sets, being of them analytically calculated from the posterior probability of the remaining ones. It has been applied to SLAM, non-linear regression, multi-target tracking, and appearance and position estimation. In the particle filter framework, if the dimension of the state space becomes higher, it would be inefficient sampling in high-dimensional spaces [12]. However, the state can be separated into tractable subspaces in some cases. If some of these subspaces can be analytically calculated, the size of the space over which particle filter samples will be drastically reduced. This kind of concept was first proposed in [12].

If we denote a state space model as s_t and observation model as z_t and observations are assumed to be conditionally independent given the process s_t of marginal distribution $p(S_t|Z_1{:}t)$. The aim is to estimate the joint posterior distribution $p(S_{0:t}|Z_{0:t})$. The pdf can be written in the recursive way:

$$\mathrm{p}(s_{0:t} \mid z_{1:t}) = \frac{p(z_t|s_t)\,p(s_t|s_{t-1})\,p(s_{0:t-1} \mid z_{1:t-1})}{p(z_t \mid z_{1:t-1})} \tag{5}$$

Where p $(z_t \mid z_{1:t-1})$ is a proportionality constant

IV. MARGINALIZATION

The variance of the estimates obtained from the standard particle filter can be decreased by exploiting linear sub-structures in the model. The corresponding variables are marginalized out and estimated using an optimal linear filter. This is the main idea behind the marginalized particle filter.

ALGORITHM 1: The marginalized particle filter

1) Initialization: For I =1, … , N, initialize the

Particles, $x_{0|-1}^{n,(i)} \sim p_{x_0^n}(x_0^n)$ and set $\{x_{0|-1}^{l,(i)} P_{0|-1}^{(i)}\} = \{\overline{x}_0^l, \overline{P}_0\}$.

2) For i = 1, …. , N, evaluates the importance weights

$q_t^{(i)} = p((y_t|X_t^{n,(i)}), Y_{t-1})$ and normalizes

$$\tilde{q}_t^i = \frac{q_t^l}{\sum_{j=1}^N q_t^{(j)}}$$

3) Particle filter measurement update

(resampling):Resample N particles with replacement,

$$\Pr(x_{t|t}^{n,(i)} = x_{t|t-1}^{n,(j)} = \tilde{q}_t^{(j)})$$

4) Particle filter time update and Kalman filter:

a) Kalman filter measurement update:

Model 1: (10)

Model 2: (10)

Model3: (22)

b) Particle filter time update (prediction): For i =

1, … , N, predict new particles.

$$x_{t+1|t}^{n,(i)} \sim p\left(\left(x_{t+1|t}^n \middle| X_{t+1|t}^n\right), Y_t\right)$$

c) Kalman filter time update:

Model1: (11)
Model2: (17)
Model 3: (23)

5) Set t: = t + 1 and iterate from step 2.

The particle filter is used to get an approximation of the posterior density $p(x_t|y_t)$ in the general model (1). In the following the particle filter, as it was introduced in [5], will be referred to as the standard particle filter. The marginalized and the standard particle filter are closely related. The marginalized particle filter is given in Algorithm 1 and neglecting steps 4a and 4c results in the standard particle filter algorithm.

It is interesting to consider which states to put in the nonlinear and the linear partition, respectively. Two relevant aspects with respect to this partitioning are how it will affect the computational complexity and the estimation performance.
This will be discussed using the following model.

$$x_{t+1}{}^n = f_t{}^n(x_t{}^n) + A_t{}^n(x_t{}^n)x_t{}^l + G_t{}^n(x_t{}^n)w_t{}^n \quad (6a)$$

$$x_{t+1}{}^l = f_t{}^l(x_t{}^n) + A_t{}^l(x_t{}^n)x_t{}^l + G_t{}^l(x_t{}^n)w_t{}^l \quad (6b)$$

$$y_t = h_t(x_t{}^n) + c_t(x_t{}^n)x_t{}^l + e_t \quad (6c)$$

Where the noise is assumed white and Gaussian distributed with

$$w_t = \begin{bmatrix} w_t^l \\ w_t^n \end{bmatrix} \sim \mathcal{N}(0, Q_t), Q_t = \begin{bmatrix} Q_t^l & Q_t^{ln} \\ (Q_t^{ln})^T & Q_t^n \end{bmatrix} \quad (7a)$$

The measurement noise is assumed white and Gaussian distributed according to

$$e_t \sim \mathcal{N}(0, R_t) \quad (7b)$$

Furthermore,

$$x_0^l \sim \mathcal{N}(\overline{x}_0, \overline{P}_0) \quad (7c)$$

The density of x_0^n can be arbitrary, but it is assumed known. In [8] the marginalized particle filter was applied to underwater navigation using a model corresponding to (6), save the fact that

$G_t^n = I,\ G_t^l = I,\ f_t^l = 0, A_n = 0.$

In [13] a model corresponding to linear state equations and a nonlinear measurement equation is applied to various problems, such as car positioning, terrain navigation.

V. AN ILLUSTRATING EXAMPLE

In order to illustrate the estimation of particle filter and marginalized (Rao-Blackwellised) particle filter we have taken the following examples where the two states X1 and X2 has to be estimated as described in equation (8a) and (8b).

$$\mathrm{x_{t+1}}^{\mathrm{n}} = \arctan \mathrm{x_t}^{\mathrm{n}} + (1\ 0\ 0)\mathrm{x_t}^{\mathrm{l}} + \mathrm{w_t}^{\mathrm{n}} \quad (8a)$$

$$\mathrm{x_{t+1}}^{\mathrm{l}} = \begin{pmatrix} 1 & 0.3 & 0 \\ 0 & 0.92 & -0.3 \\ 0 & 0.3 & 0.92 \end{pmatrix} \mathrm{x_t}^{\mathrm{l}} + \mathrm{w_t}^{\mathrm{l}} \quad (8b)$$

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$$y_t = \begin{pmatrix} 0.1(x_t^n)^2 & sgn(x_t^n) \\ & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} x_t^1 + e_t \quad (8c)$$

$$w_t = \begin{pmatrix} w_t^n \\ w_t^n \end{pmatrix} \sim \mathcal{N}(0, 0.01 I_{4*4}) \quad (8d)$$

$$e_t \sim \mathcal{N}(0, 0.01 e_{2*2}) \quad (8e)$$

$$x_0^n = \mathcal{N}(0,1) \quad (8f)$$

$$x_0^1 = \mathcal{N}(0_{3*3}, 0_{3*3}) \quad (8g)$$

Looking at the notation used in the above Model that is the model specified in (6) and (7)

$$f_t^n(x_t^n) = arctan\, x_t^n \quad (9a)$$

$$A_t^n(x_t^n) = (1 \quad 0 \quad 0) \quad (9b)$$

$$G_t^n(x_t^n) = 1 \quad (9c)$$

$$f_t^n(x_t^n) = (1 \quad 0 \quad 0)^T \quad (9d)$$

$$A_t^1(x_t^n) = \begin{pmatrix} 1 & 0.3 & 0 \\ 0 & 0.92 & -0.3 \\ 0 & 0.3 & 0.92 \end{pmatrix} \quad (9e)$$

$$G_t^1(x_t^n) = I_{3*3} \quad (9f)$$

$$h_t(x_t^n) = \begin{pmatrix} 0.1(x_t^n)^2 & sgn(x_t^n) \\ & 0 \end{pmatrix} \quad (9g)$$

$$C_t(x_t^n) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad (9h)$$

$$Q_t = 0.01 I_{4*4} \quad (9i)$$

$$R_t = 0.1 I_{2*2} \quad (9j)$$

$$\bar{x}_0 = (0 \quad 0 \quad 0)^T \quad (9k)$$

$$\bar{P}_0 = 0_{3*3} \quad (9l)$$

VI.SIMULATION RESULTS

In this section, we present the simulation result of the Rao Blackwellised particle filter and give a performance comparison between the Rao Blackwellised Particle filter and Particle filter based on RMSE. Here we have taken two states X1 and X2(Corresponding to equation 8(a) and 8(b).

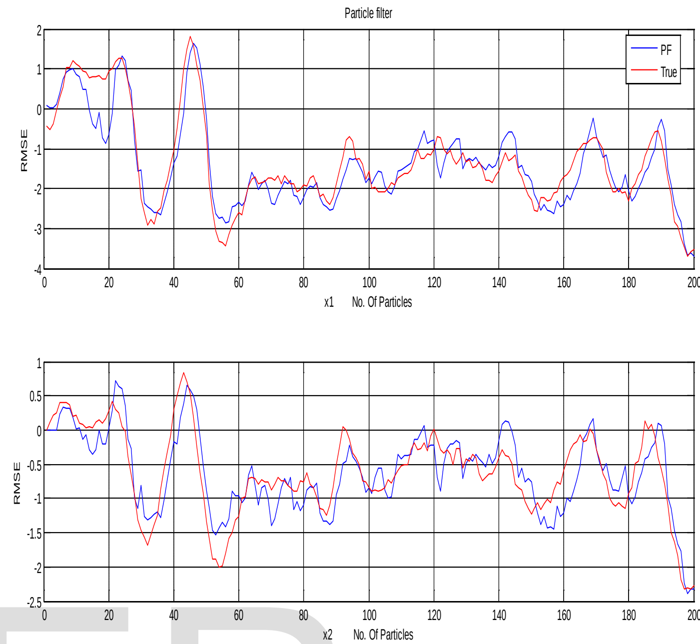

FIG-1: RMSE Estimation of Particle Filter using 10 Monte Carlo simulation with N = 200 particles

In fig.1 the red line indicates the true state estimation and blue line indicates the state estimation using particle filters. In fig.2 the red line indicates true state estimation and blue line indicates the state estimation using Rao Blackwellised particle filters. .In Fig. 1, the result is shown for the particle Filter when using N = 200 particles and 10 Monte Carlo Simulation. The performance is pretty bad, and it quickly deteriorates even more when the number of particles is decreased. The result from applying the Rao Blackwellised Particle filter using only N = 200 particles and 10 Monte Carlo Simulation is also shown in Fig. 2 and the performance enhancement is significant. It is shown that the estimates of the particle filter and the RBPF deviate from the true states very small at some time steps.

Table 1 gives RMSE evaluation of particle filter and Rao Blackwellised Particle Filter. From the Root mean square error (RMSE), we can clearly see that the Rao Blackwellised Particle Filter gives the better performance than particle filter.

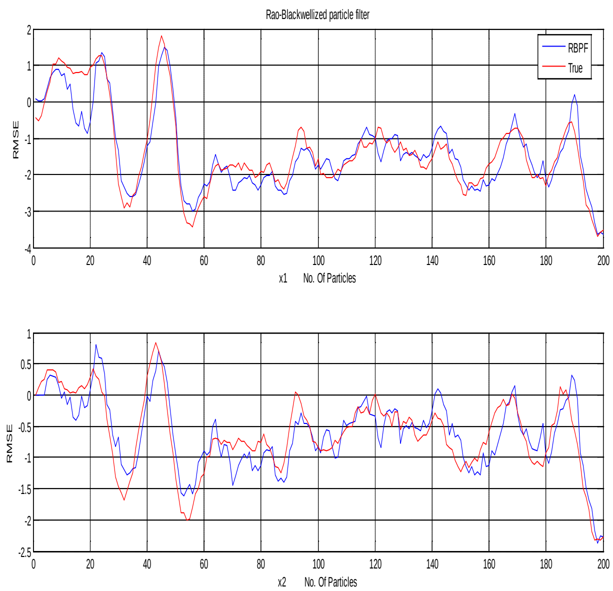

FIG-2: RMSE Estimation of Rao-Blackwellised Particle Filter using 10 Monte Carlo simulation with N = 200 particles.

Algorithm	**Maximum RMSE**	**Minimum RMSE**
Partcle Filter	0.4520	0.1713
Rao Blackwellised particle filter	0.4271	0.1690

Table1. RMSE of Particle Filter and Rao-Blackwellised Particle Filter over 10 Monte Carlo Iteration with N=200particles

VII.CONCLUSION

From Table no.1 shown above we conclude that The RMSE of RBPF is significantly lower in comparison to PF's. The main contribution in this paper is to analyse the performance of the marginalized particle filter and Rao Blackwellised Particle filter for a general nonlinear state-space model. The method is general and can be applied to a large class of problems. In an extensive Monte Carlo simulation different performance aspects are shown, and the theoretical results are illustrated and validated. Rao Blackwellised particle filter is used for Eigen Tracking, Multiple target tracking, INS/GPS Integration.

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