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REPRESENTATION OF SAMPLING OPERATORS ON BANACH SPACES

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Abstract— In the present paper we prove two theorems on the representation of the space of sampling operators and the spaces of their adjoint operators respectively. Our theorems include the corresponding results of Zimmermann [Z 94,theorems 6.2.1 and 6.2.5] for the particular value $\omega = 1$ of the moderate weight function.

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1 INTRODUCTION

Georg Zimmermann [Z 94, Chapter VI], using the concept of multiplier operators, has initiated the study of the representation of generalized sampling operators from a space of functions or distributions on the real line R to a space of sequences on Z, the set of all integers. These operators commuted with translations by integers and therefore, belong to the class of multiplier operators. He has demonstrated that if $X(R)$ is a translation invariant Banach space of functions or distributions on R, then the space

$$L_{\tau,z}(X(R), l^{\infty}(Z))$$

of sampling operators is isometric to the dual space $X^{'}(R)$, where $l^{\infty}(Z)$ is the Banach space of sequences $a = (a_n) \in Z$ with respect to the norm

$$\| a \|_{l^{\infty}(Z)} = \sup_{n \in Z} | a_n | < \infty.$$

In the present paper , using the concept of moderate weight functions, we define weighed sequence spaces $l_{\omega}^{p}(Z), 1 \le p \le \infty.$ We study the representation of sampling operators T from a translation-invariant Banach space $X(R)$ to a weighted sequence space $l_{\omega}^{\infty}(Z)$. Also, on the lines of Zimmermann (loc.cit) , We define the adjoint operator T^{*} of T and study its representation from the dual space of $l_{\omega}^{p}(Z)$ to $X^{'}(R).$ We prove two theorems on the representation of the space of sampling operators and the space of their adjoint operators respectively.Our theorems include the corresponding result of Zimmermann (loc.cit,theorems 6.2.1 and 6.2.5) for the particular $\omega = 1$ of the moderate weight function.

2 Notations and Basic Definitions.

We denote by R the real line. if f is a function on R, we define the translation operator t_x by

$$t_x f(y) = f(y - x), \forall x, y \in R.$$

Let X and Y be any two translation-invariant topological vector spaces of functions or distributions on R.We denote by T a continuous liner transformation

$$T : X \to Y$$

such that

$$Tt_x = t_x T, \forall x \in R.$$

The operator T is known as a multiplier operator on X .We denote by L_{loc}^{1} the space of locally integrable functions on the real line R consisting of all complex-valued functions

$$f : R \to C$$

such that

$$f \cdot \chi_K \in L^1$$

for every compact subset K of R , where χ_K is the characteristic function of K.

A positive and locally integrable function

$$m : R \to (0, \infty)$$

is called weight function. It is called submultiplicative provided

$$m(0) = 1$$

and

$$m(x + y) \le m(x) m(y); \forall x, y \in R.$$

It is well known that a submultiplicative weight function is locally bounded.

A weight function ω on R is called moderate with respect to a submultiplicative function m, if

$$\omega(x + y) \le \omega(x) m(y); \forall x, y \in R.$$

In the sequel treating the associated submultiplicative weight function m as implicit, we denote by ω a moderate weight function on R.

Let $L_{\omega}^{p}(R), 1 \le p < \infty,$ be the weighted Banach space of equivalence classes complex-valued measurable functions on R

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with respect to the norm

$$\| f \|_{L^p_\omega} = \left(\int_R | f(x) |^p \ w^p(x)\, dx \right)^{1/p} < \infty.....................(2.1)$$

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In case $p = \infty$, we define the banach space $L^\infty_w(R)$ under the norm

$$\| f \|_{L^p_\omega} = ess \sup_{x \in R} | f(x) | w(x) < \infty.......................(2.2) |$$

On account of Riesz representation Theorem, the topological dual space of $L^p_\omega(R)$ is the space $L^{p}_{\omega^{-1}}$ where p' is the dual exponent of p such that

$$1/p + 1/p' = 1.$$

It is known that $L^p_\omega(R), 1 \le p \le \infty,$ is closed under translations. for the proof see [Heil 90,pp.27-29].

3 Representation of Sampling Operators.

We denote by Z the set of all integers. Let $X(R)$ be a translation-invariant topological vector space of function or distribution on R and Y a shift-invariant topological vector space of sequence on Z .As in [Zimm 94,p.47],we denote by

$$L_{\tau,Z}((X(R), Y(Z))$$

the space of all continuous liner transformations $T : X(R) \to Y(Z)$, which commute with translations by integers such that

$$Tt_n = t_n T \ , \forall n \in Z.$$

It is evident that these operators belong to the class of multiplier operators.

In general,we denote by $a = (a_n)_{n \in Z}$ the sequence on $Z.$ Let $\omega(x_n) = \omega(n)$ be the value of the weight function at a point $x_n \in R.$ We define by $l^p_\omega(Z), 1 \le p < \infty,$ the Banach space of sequences with respect to the norm

$$\| a \|_{l^p_\omega(Z)} = (\sum_{n \in Z} | a_n |^p \ \omega^p(n))^{1/p} \ .\forall \ p \in [1, \infty)................(3.1)$$

In case $p = \infty,$ we define the Banach space $l^\infty_\omega(Z)$ under the norm

$$\| a \|_{l^\infty_\omega(Z)} = \sup_{n \in Z} | a_n | \omega(n)...................................(3.2)$$

It can be easily seen that the inclusion relation.

$$l^{p_1}_\omega(Z) \subset l^{p_2}_\omega(Z) \ , \forall \ 1 \le p_1 < p_2 \le \infty \ ,.$$

hold true.

We prove the following:

Theorem 3.1. If $X(R)$ and $l^\infty_\omega(Z)$ are translation-invariant Banach spaces , then the space of sampling operators $L_{t,z}(X(R), l^\infty_\omega(Z))$ is isometrically isomorphic to $X'(R)$, where $X'(R)$ is the dual of $X(R).$

We shall use the following lemmas in the proof of our theorem:

Lemma 3.2 If $X(R)$ an $Y(Z)$ are translations-invriant Banach spaces such that $\| \cdot \|_{Y(Z)} \ge \| \cdot \|_{L^\infty_\omega(Z)}$,then $L_{\tau Z}(X(R), Y(R))$ is continuously embedded in $X'(R)$.

It may be mentioned here that in case $\omega \equiv 1,$ then this lemma reduces to theorem 6.2.1 of Zimmermann (loc.cit.).

Proof of the Lemma: Let $T \in L_{\tau Z}(X(R), l^\infty_\omega(Z))$ Then we have

$$| (TF)(0) | \le \| TF \|_{l^\infty_\omega(Z)}$$

Since T is a bounded linear operators, the map $f \to (TF)(0)$ is a bounded linear functional on $X(R)$. This implies that

$$(TF)(0) = \langle f, \phi \rangle \ \ forsome \ \ \phi \in X'(R).$$

Hence we see that

$$\begin{aligned}(TF)^*(n) &= (\tau_{-n} TF)(0) \\ &= (T \tau_{-n} f)(0) \\ &= \langle \tau_{-n} f, \phi \rangle \\ &= \langle f, \tau_n \phi \rangle\end{aligned}$$

Also, we have

$$\begin{aligned}\| T \| &= \sup_{\| f \|_X = 1} \| TF \|_{l^\infty_\omega} (Z) \\ &\ge \sup_{\| f \|_X = 1} | (TF)(0) | \\ &= \| \phi \|_{X'} \ . \quad ....(3.3)\end{aligned}$$

$\Rightarrow L_{\tau Z}(X(R), l^\infty_\omega(Z))$ is continuously embedded in $X'(R)$.

Lemma 3.3 If $X(R)$ is a translation invariant Banach space, then $L_{\tau Z}(X(R), l^\infty_\omega(Z))$ is isometric to $X'(R)$.

Proof: We assume that $\phi \in X'$. Then, $\forall f \in X(R),$ we have

$$\| \{ \langle f, \tau_n \phi \}_{n \in Z} \|_{l^\infty_\omega(Z)} = \sup_{n \in Z} | \langle f, \tau_n \phi \rangle \omega(n) |$$

$$\le \| \phi \|_{X'(R)} \| f \omega \|_{X(R)}$$

$$\Rightarrow X'(R) \subseteq L_{\tau Z}(X(R), l^\infty_\omega(Z))$$

and

$$\| T \| \le \| \phi \|_{X'(R)} \ . \quad ....(3.4)$$

combining (3.3) and (3.4), the proof of the lemma holds true.

Proof of the theorem 3.1: Since $L_{\tau Z}(X(R), l^\infty_\omega(Z))$ is the space of continuous linear transformations, which belong to the class of multiplier operators, the Lemmas 3.2 and 3.3 ensure that the proof of the theorem is complete.

4 Representation of Adjoint Operators

Let $f \in X(R)$ and $X'(R)$. Then the convolution of f and $\tilde{\phi}$ (conjugate of ϕ) is defined by

$$(f * \tilde{\phi}) = \langle f, \tau_x \phi \rangle.$$

We now write the operator T in the form

$$T : f \to \{ f * \tilde{\phi}(n) \}_{n \in Z}$$

for some $\phi \in L^p_\omega(R)$.

We prove the following

Theorem 4.1: If $T \in L_{\tau Z}(X(R), l^p_\omega(Z))$ is defined by $f : f \to \{f * \tilde{\phi}(n)\}_{n\in Z}$, then its adjoint operator

$$T^* : L_{\tau Z}(l^q_{\omega^{-1}}(Z), X'(R))$$

is given by

$$a = (a_n)_{n\in Z} \to \sum_{n\in Z} a_n \tau_n \phi,$$

where $\frac{1}{p} + \frac{1}{q} = 1,\ 1 \le p < q < \infty$ and the series on the right-hand side of (4.1) is unconditionally convergent in the weak*-topology on $l^\infty_{\omega^{-1}}(Z)$. In case $q < \infty$, the series (4.1) is convergent in the norm topology of $l^q_{\omega^{-1}}(Z)$.

Proof of the theorem On he lines of Zimmermann (loc. cit.,p. 50), we assume that T is a sampling operator in $L_{\tau Z}(X(R), l^p_\omega(Z))$ suhc that

$$T : f \to \{f * \tilde{\phi}(n)\}_{n\in Z}$$

thus the adjoint operator T^* of T is given by

$$\langle f, T^* a\rangle = \langle Tf, a\rangle$$
$$= \sum_{n\in Z} \langle f, \tau_n \phi\rangle \overline{a_n}$$
$$= \langle f, \sum_{n\in Z} a_n\ \tau_n \phi\rangle$$

Since the sries on the right-hand side is unconditionally convergent, we may consider the limit over the set of finite if subsets F of Z, which are partially ordered by the inclustion relatioin. Hence we obtain

$$\langle f, T^* a\rangle = \lim_F \langle f, \sum_{n\in Z} a_n\ \tau_n \phi\rangle, \quad \forall \quad f \in X(R).$$

$\Rightarrow$ the series

$$\sum_{n\in Z} a_n\ \tau_n \phi$$

convergent to T^* in the weak*-topology.

In case $q < \infty$, we assume that $(a_n)_{n\in Z} \in l^q_{w^{-1}}(Z)$.

$\Rightarrow$ $\exists$ a finite subset F of Z such that

$$\left(\sum_{n\notin F} |a_n|^q\ (w^{-1})^q\right)^{1/q} < \frac{\varepsilon}{\|T^*\|}$$

Thus, for all finite sets $F_1, F_2 \supseteq F$, we have

$$\|\sum_{n\in F_1} a_n\ \tau_n\phi - \sum_{n\in F_2} a_n\ \tau_n \phi\|_{X'} = \|\sum_{n\in Z} b_n\ \tau_n \phi \|\ X'$$

$$= \|T^*\ b_n\|_{X;}$$

$$= \|T^*\| \|b_n\|_{l^q_{\omega^{-1}(Z)}}$$

where

$$b_n = \begin{cases} a_n, & if\ n \in F_1 \setminus F_2, \\ -a_n, & if\ n \in F2 \setminus F_1; \\ 0, & Otherwise. \end{cases}$$

$$\Rightarrow \|b-n\|_{l^q_{\omega^{-1}}}(Z) = \left(\sum_n |b_n|^q\ (\omega^{-1})^q\right)^{1/p}$$

$$\le \left(\sum_{n\notin F} |a_n|^q\ (\omega^{-1})^q\right)^{1/p}$$

$$< \frac{\varepsilon}{\|T^*\|}$$

This completets the proof of the theorem.

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