Quasi-Hadamard product of certain starlike and convex functions with respect to symmetric point

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Abstract— In this paper, we introduce establish certain results concerning the quasi-Hadmard product for two classes related to starlike and convex univalent functions with respect to symmetric points..

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1 INTRODUCTION

Throughout this paper, let S denote of the functions of the form :

$$f(z)=a_1z-\sum_{k=2}^{\infty}a_kz^k \qquad (a_1>0,a_k\geq 0),$$

$$f_r(z)=a_{1,r}z-\sum_{k=2}^{\infty}a_{k,r}z^k \qquad (r\in \mathrm{N},a_{1,r}>0,a_{k,r}\geq 0),$$

$$\breve{g}(z)=b_1z-\sum_{k=2}^{\infty}b_kz^k \qquad (b_1>0,b_k\geq 0)$$

and

$$\breve{g}_j(z)=b_{1,j}z-\sum_{k=2}^{\infty}b_{k,j}z^k \qquad (j\in \mathrm{N},b_j>0,b_{k,j}\geq 0)$$

which are analytic in the unit disc $U=\{z : |z|<1\}$. For a function $f(z)$(with $a_1=0$), we defined by (ref: 1.1),

$$D^0f(z)=f(z),$$

$$D^1f(z)=Df(z)=zf'(z)$$

and

$$D^nf(z)=D\left[D^{n-1}f(z)\right] \qquad (n\in \mathrm{N}=[1,2,...]),$$

where

$$D^nf(z)=z-\sum_{k=2}^{\infty}k^na_kz^k.$$

The differential operator D^n was introduced by Salagean [11]. With the help of Salagean D^n define the classes $S^*_{s,n}(\alpha,\beta)$ and $S^*_{c,n}(\alpha,\beta)$ as follows:

Denote by $S^*_{s,n}(\alpha,\beta)$, the class of functions $f(z)$ which satisfy the condition

$$\left|\frac{\frac{D^{n+1}f(z)}{D^nf(z)-D^nf(-z)}-1}{\alpha\frac{D^{n+1}f(z)}{D^nf(z)-D^nf(-z)}+1}\right|<\beta$$

where

$$n\in \mathrm{N}_0=\mathrm{N}\cup\{0\},0\leq\alpha\leq 1,0<\beta\leq 1,0\leq\frac{2(1-\beta)}{1+\alpha\beta}<1.$$

Let $S^*_{c,n}(\alpha,\beta)$ be the class of function $f(z)$ for which $zf'(z)\in S^*_{s,n}(\alpha,\beta)$.

We note that:

(i) $S^*_{s,0}(\alpha,\beta)=S^*_s(\alpha,\beta)$ and $S^*_{c,o}(\alpha,\beta)=S^*_c(\alpha,\beta)$ introduced by Sudharsan [15] (see also [14]).

(ii) $S^*_{s,0}(\alpha,0)=S^*_s(\alpha)$, these functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [12] (see also Robertson [10] , Stankiewics [13] Wu [16] and Owa et al. [6]).

(iii) $S^*_{c,0}(\alpha,0)=S^*_c(\alpha)$ introduced by El-Ashwah and Thomas [2].

Aouf et al. [1] , stales that $f(z)\in S^*_{s,n}(\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty}k^n\left[(1+\alpha\beta)k+(\beta-1)\left[1-(-1)^k\right]a_{k,r}\right]\leq\left[\beta(2+\alpha)-1\right]a_{1,r}$$

where $0\leq\alpha\leq 1$, $0<\beta\leq 1$, $0\leq\frac{2(1-\beta)}{1+\alpha\beta}<1$ and $z\in U$.

And $f(z)\in S^*_{c,n}(\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty}k^{n+1}\left[(1+\alpha\beta)k+(\beta-1)\left[1-(-1)^k\right]a_{k,r}\right]\leq\left[\beta(2+\alpha)-1\right]a_{1,r},$$

where $0\leq\alpha\leq 1$, $0<\beta\leq 1$, $0\leq\frac{2(1-\beta)}{1+\alpha\beta}<1$ and

$z \in U.$

We now introduce the following class of analytic function which plays an important role in the discussion that follows:

Afunction $f(z) \in S^*_{h,n}(\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty} k^h\left[(1+\alpha\beta)k + (\beta-1)\left[1-(-1)^k\right]a_{k,r}\right] \le [\beta(2+\alpha)-1]a_{1,r},$$

where $0 \le \alpha \le 1,\quad 0 < \beta \le 1,\quad 0 \le \frac{2(1-\beta)}{1+\alpha\beta} < 1$ and $z \in U$. Where h is an nonnegative real number. It is evident that $S^*_{s,k}(\alpha,\beta) \equiv S^*_{s,n}(\alpha,\beta)$ and,

$S^*_{s,k+1}(\alpha,\beta) = S^*_{c,n}(\alpha,\beta)$. Further,

$S^*_{s,h}(\alpha,\beta) \subset S^*_k(\alpha,\beta)$ if $h > k \ge 0$ the containment being proper. Hence, for any positive integer $h > k+1$, we have the inclusion relation

$$S^*_{s,h}(\alpha,\beta) \subset S^*_{s,h-1}(\alpha,\beta) \subset \quad \cdots$$
$$\subset S^*_{s.k+2}(\alpha,\beta) \subset S^*_{c,n}(\alpha,\beta) \subset S^*_{s,n}(\alpha,\beta).$$

We note that for every nonnegative real number h, the class $S^?_{s,n}(\alpha,\beta)$ is nonempty as the functions of the from

$$f(z) = a_1 z - \sum_{k=2}^{\infty} \frac{\beta(2+\alpha)-1}{k^h\left[(1+\alpha\beta)k + (\beta-1)\left[1-(-1)^k\right]\right]} a_1\lambda_k z^k,$$

where $a_1 > 0, \lambda_k \ge 0,$ and $\sum_{k=1}^{\infty}\lambda_k \le 0,$ satisfy the inequality (ref: 1.14).

The quasi-Hadamard product of two or more functions has recently been defined and used by Owa [7,8,9], Kumar [3,4,5] and others. Accordingly, the quasi-Hadamard product of two functions $f(z)$ and $\breve{g}(z)$ is given by

$$f * \breve{g}(z) = a_1 b_1 z - \sum_{k=2}^{\infty} a_k b_k z^k.$$

2 THE MAIN THEOREM

Theorem *A functions* $f_r(z)$ *defined by* (ref: 1.2) *in the class* $S^*_{c,n}(\alpha,\beta)$ *for each* $r = 1,2,\ldots,u;$ *and the functions* $\breve{g}_j(z)$ *in the class* $S^*_{s,n}(\alpha,\beta)$ *for each* $j = 1,2,\ldots,q.$ *Then we get the quasi-Hadamard product* $f_1 * f_2 * \ldots * f_u * \breve{g}_1 * \breve{g}_2 * \ldots * \breve{g}_q(z)$ ⍰ $S^*_{u(n+2)+q(n+1)-1}(\alpha,\beta)$.

Proof We denote the quasi-Hadamard product $f_1 * f_2 * \ldots * f_u * \breve{g}_1 * \breve{g}_2 * \ldots * \breve{g}_q(z)$ by the function $h(z)$, for the sake of the convenience.

Clearly,

$$h(z) = \left[\Pi_{r=1}^{u} a_{1,r}.\Pi_{j=1}^{q} b_{1,j}\right]z - \sum_{k=2}^{\infty}\left[\Pi_{r=1}^{u} a_{k,r}.\Pi_{j=1}^{q} b_{k,j}\right]z^k.$$

To prove the theorem, we need to show that

$$\sum_{k=2}^{\infty}\left\{(k)^{u(n+2)+q(n+1)-1}\left[(1+\alpha\beta)k + (\beta-1)\left(1-(-1)^k\right)\right]\left[\Pi_{r=1}^{u} a_{k,r}.\Pi_{j=1}^{q} b_{k,j}\right]\right\}$$

$$\le [\beta(2+\alpha)-1]\left(\Pi_{r=1}^{u} a_{1,r}.\Pi_{j=1}^{q} b_{1,j}\right).$$

Since $f_r(z) \in S^*_{c,n}(\alpha,\beta),$ we have

$$\sum_{k=2}^{\infty} k^{n+1}\left[(1+\alpha\beta)k + (\beta-1)\left[1-(-1)^k\right]\right] \le [\beta(2+\alpha)-1]a_{1,r},$$

for each $r = 1,2,\ldots,u.$ Therefore

$$k^{n+1}\left[(1+\alpha\beta)k + (\beta-1)\left[1-(-1)^k\right]a_{k,r}\right] \le [\beta(2+\alpha)-1]a_{1,r}$$

or

$$a_{k,r} \le \left\{\frac{\beta(2+\alpha)-1}{k^{n+1}\left[(1+\alpha\beta)k + (\beta-1)\left[1-(-1)^k\right]\right]}\right\}a_{1,r}$$

for each $r = 1,2,\ldots,u.$ The right-hand expression of this last inequality is not greater then $k^{-(n+1)}a_{0,r.}$. Hence

$$a_{n,r} \le k^{-(n+2)}a_{1,r.}$$

for each $r = 1,2,\ldots,u.$ Similarly, for $\breve{g}_j(z) \in S^*_{s,n}(\alpha,\beta),$ we have

$$\sum_{k=2}^{\infty} k^n\left[(1+\alpha\beta)k + (\beta-1)\left[1-(-1)^k\right]b_{k,j}\right] \le [\beta(2+\alpha)-1]b_{1,j}$$

for each $j = 1,2,\ldots,q.$ Hence we get

$$b_{k,j} \le k^{-(n+1)}b_{1,j}.$$

By (ref: 2.3) for $r = 1,2,\ldots,u,$ (ref: 2.5) for $j = 1,2,\ldots,q-1,$ and (ref: 2.4) for $j = q,$ we get

$$\sum_{k=2}^{\infty}\left\{(k)^{u(n+2)+q(n+1)-1}\left[(1+\alpha\beta)k+(\beta-1)\right]\left[1-(-1)^{k}\right]\left[\Pi_{r=1}^{u}a_{k,r}.\Pi_{j=1}^{q}b_{k,j}\right]\right\}$$

$$\leq\sum_{k=2}^{\infty}\left\{(k)^{u(n+2)+q(n+1)-1}\left[(1+\alpha\beta)k+(\beta-1)\left(1-(-1)^{k}\right)\right]\right\}$$
$$\times k^{-u(n+2)}.k^{-(n+1)(q-1)}\left(\Pi_{r=1}^{u}a_{1,r}.\Pi_{j=1}^{q-1}b_{1,j}\right)b_{q,k}$$

$$\sum_{k=2}^{\infty}\left\{k^{n}(1+\alpha\beta)k+(\beta-1)\left[1-(-1)^{k}\right]\right\}\left(\Pi_{r=1}^{u}a_{1,r}.\Pi_{j=1}^{q}b_{1,j}\right)b_{q,k}$$
$$\leq\left[\beta(2+\alpha)-1\right]\left(\Pi_{r=1}^{u}a_{1,r}.\Pi_{j=1}^{q}b_{1,j}\right).$$

Hence $h(z)\in S^{*}_{u(n+2)+q(n+1)-1}(\alpha,\beta)$. This completes the proof of Theorem th2.

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