KEY PROPERTIES OF HESITANT FUZZY SOFT TOPOLOGICAL SPACES

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Abstract— In this paper, we define hesitant fuzzy soft (‘HF soft’ in short) topological spaces and its fundamental properties such as HF soft interior, HF soft closure, HF soft exterior, HF soft boundary of a HF soft set. Moreover, we investigate some roperties on HF soft compact space, especially finite intersection property on HF soft topological spaces.

Key words — Hesitant fuzzy soft topological space, HF soft sub space, HF soft interior, HF soft exterior, HF soft boundary, HF soft closure, HF Soft open cover, HF soft compact space.

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1 INTRODUCTION

Ambiguity imperfection and uncertainty and are main factors in the real life. The significance and uses of ambiguity is identified by researchers for the last 5 decades. It is pointed out as a complicated one to be dealt within some specifications. To excel in dealing uncertainty, different theories such as vague, fuzzy and rough theories are initiated and developed drastically. In 1965, a new mathematical tool to cope up with uncertainty, [2] Zadeh has introduced fuzzy sets and this is used in various fields. In that process, [16] Molodtsov's has given one more useful approach, i.e. Soft set theory in the year 1999. It is an advanced frame work to handle ambiguity. Parameterization is the main concept in it. Without any specific condition on parameters, the soft theory has established recently in presenting uncertain concepts in a systematic manner. Torra [8] has generalized the fuzzy sets and proposed hesitant fuzzy sets which permit the membership degree to avail a set of possible values and the membership is contained in the closed interval [0, 1]. This new concept has grabbed the attention of number of authors. In 2013 K. V. Babithaa and Sunil Jacob John [6] have combined the hesitant fuzzy sets and soft sets. i.e. They have initiated the notion of hesitant fuzzy soft sets. A vigorous study on topological structures on soft sets is continued by number of researchers. S. Atmaca and ˙I. Zorlutuna have explored the topological structures of fuzzy parameterized soft sets and mappings on fuzzy parameterized soft sets. They have given the notion of quasi-coincidence for fuzzy parameterized soft sets and have examined some basic concepts in fuzzy parameterized soft topological spaces. In 2014, [17] S. Broumi, F. Smarandach have worked on new operations over interval valued intuitionistic hesitant fuzzy sets. Hesitant fuzzy topological space and some related properties are introduced by Serkan Karata [13]. Later [9] Verma and Sharma have given the fundamental operations on hesitant fuzzy sets. Few more operations are established by the author J. Wang et al. on Hesitant fuzzy soft sets. Their purpose is to use the hesitant fuzzy soft sets in multi criteria group decision making problem.

In this paper, we present hesitant fuzzy soft sets (Shortly HF Soft) and HF soft topological spaces and prove related properties. We also define HF Soft point, HF Soft neighborhood system and verify corresponding results. Later we define HF Soft interior point, HF Soft closure, HF Soft boundary point, HF Soft exterior points and examine few of their properties. We also define HF Soft compactness, for this we introduce HF Soft open cover and HF Soft finite sub cover. Next we prove some theorems related to them.

Let us suppose that X is any universal set and E is a set of parameters corresponding to the elements of X throughout the paper.

2 PRELIMINARIES

Definition 2.1. [16] A soft set is defined as a pair (*s*, A), denoted by sX^A over X, where s: A → ρ(X), A ⊆ E, ϱ(X) is the set of all subsets of X. The class of all soft sets defined on X with respect to the parameters in A is denoted by $s\tilde{X}^A$.

Definition 2.2 If sX^A and qX^B over a common universe X are soft sets then sX^A OR qX^B denoted by $sX^A \vee qX^B$ is defined as $sX^A \vee qX^B = (f, A \times B)$ where $f(a, b) = s(a) \tilde{\cup} q(b)$.

Definition 2.3. [16] A fuzzy soft set is defined as a pair ($\widetilde{fs}$, A) denoted by $\widetilde{fs}X^A$ over X, where $\widetilde{fs}$: A → ϱ(X), A ⊆ E, ϱ(X) is a set of all fuzzy sets in X. The class of all fuzzy soft sets over X with respect to the parameters in A is denoted by $\widetilde{fs}(\tilde{X}^A)$.

Definition 2.4. [8] A hesitant fuzzy set is defined as a function $\widetilde{hf}$ that when applied on X returns a subset of [0, 1]. i.e.$\widetilde{hf}$: X → [0, 1]. A hesitant fuzzy set over X is denoted by $\widetilde{hf}^X$ and the class of all hesitant fuzzy sets defined over X is denoted by HF$\widetilde{XE}$.

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Definition 2.5. [8] The lower and upper bound of a hesitant fuzzy set $\widetilde{hf}^{X}$ over X are defined as below.

Lower bound of $\widetilde{hf}^{X}$ is defined and denoted by $\widetilde{hf}^{X-}(x) = \min\{\widetilde{hf}(x)$: for all $x \in \tilde{X}\}$.

Upper bound of $\widetilde{hf}^{X}$ is defined and denoted by $\widetilde{hf}^{X+}(x) = \max\{\widetilde{hf}(x)$: for all $x \in \tilde{X}\}$.

Example 2.6. Let X = {h1, h2, h3} and $\widetilde{hf}^{X1}, \widetilde{hf}^{X2} \in \widetilde{hf}(\tilde{X})$. Such that

$\widetilde{hf}^{X1}$ = {(h1,[0.2, 0.45]) , (h2, [0.6, 0.8]) , (h3,[0.1, 0.7])},

$\widetilde{hf}^{X2}$ = {(h1,[0.05, 0.5]) , (h2, [0.7, 0.9]) , (h3,[0.2, 0.7])},

Then, we have that $\widetilde{hf}^{X1-}(x) = \min\{\widetilde{hf}(x)$: for all $h \in \tilde{X}\}$= {0.1},

$\widetilde{hf}^{X1+}(x) = \max\{\widetilde{hf}(x)$: for all $x \in \tilde{X}\}$. ={0.8},

$\widetilde{hf}^{X2-}(x) = \min\{\widetilde{hf}(x)$: for all $h \in \tilde{X}\}$= {0.05},

$\widetilde{hf}^{X2+}(x) = \max\{\widetilde{hf}(x)$: for all $x \in \tilde{X}\}$= {0.9}.

Definition 2.7. [8] The complement of any hesitant fuzzy set $\widetilde{hf}^{X}(x)$ is defined as

$\widetilde{hf}^{Xc}(x) = \cup_{\gamma \in \widetilde{hf}x}\{1 - \gamma\}$.

Example 2.8. $\widetilde{hf}^{X1'}(x) = \cup_{\gamma \in \widetilde{hf}x}\{1 - \gamma\}$ = {(h1,[0.8, 0.55]) , (h2, [0.4, 0.2]) , (h3,[0.9, 0.3])}, Complement of $\widetilde{hf}^{X1}$.

Definition 2.9. [8] Let $\widetilde{hf}^{X1}, \widetilde{hf}^{X2} \in \widetilde{hf}(\tilde{X})$. Then $\widetilde{hf}^{X1}$ is said to be a subset of $\widetilde{hf}^{X2}$ if $\widetilde{hf}^{X1}(x) \,\tilde{\subseteq}\, \widetilde{hf}^{X2}(x)$ for all x ∈X and is denoted by $\widetilde{hf}^{X1} \,\tilde{\subseteq}\, \widetilde{hf}^{X2}$.

Definition 2.10. [8] Two hesitant fuzzy sets $\widetilde{hf}^{X1}, \widetilde{hf}^{X2}$ are said to be equal if $\widetilde{hf}^{X1} = \widetilde{hf}^{X2}$ if and only if $\widetilde{hf}^{X1} \,\tilde{\subseteq}\, \widetilde{hf}^{X2}$ and $\widetilde{hf}^{X2} \,\tilde{\subseteq}\, \widetilde{hf}^{X1}$.

Example 2.11 It can be seen from example 2.6 clearly that $\widetilde{hf}^{X1} \,\tilde{\subseteq}\, \widetilde{hf}^{X2}$.

Definition 2.12. [8] The intersection of two hesitant fuzzy sets $\widetilde{hf}^{X1}$ and $\widetilde{hf}^{X2}$ is a hesitant fuzzy set $\widetilde{hf}^{X1} \,\tilde{\cap}\, \widetilde{hf}^{X2}$ such that $(\widetilde{hf}^{X1} \,\tilde{\cap}\, \widetilde{hf}^{X2})(x) = \{h \in (\frac{\widetilde{hf}X1(x) \,\tilde{\cup}\, \widetilde{hf}X2(x)}{h}) / h \leq \min(\widetilde{hf}^{X1+}, \widetilde{hf}^{X2+})\}$.

Definition 2.13. [8] The union of two hesitant fuzzy sets $\widetilde{hf}^{X1}$ and $\widetilde{hf}^{X2}$ is a hesitant fuzzy set $\widetilde{hf}^{X1} \,\tilde{\cup}\, \widetilde{hf}^{X2}$ such that $(\widetilde{hf}^{X1} \,\tilde{\cup}\, \widetilde{hf}^{X2})(x) = \{h \in (\widetilde{hf}^{X1}(x) \,\tilde{\cup}\, \widetilde{hf}^{X2}(x)) / h \geq \max(\widetilde{hf}^{X1-}, \widetilde{hf}^{X2-})\}$.

Example 2.14 Let X = {h1, h2, h3} and $\widetilde{hf}^{X1}, \widetilde{hf}^{X2} \in \widetilde{hf}(\tilde{X})$ such that

$\widetilde{hf}^{X1}$ = {(h1,[0.4, 0. 5]) , (h2, [0.6, 0.9]) , (h3,[0.3, 0.85])},

$\widetilde{hf}^{X2}$ = {(h1,[0. 5, 0.2]) , (h2, [0.8, 0.7]) , (h3,[0.1, 0.2])},

Then, we have that $\widetilde{hf}^{X1-}(x) = \min\{\widetilde{hf}(x)$: for all $x \in \tilde{X}\}$= {0.3},

$\widetilde{hf}^{X1+}(x) = \max\{\widetilde{hf}(x)$: for all $x \in \tilde{X}\}$. ={0.9},

$\widetilde{hf}^{X2-}(x) = \min\{\widetilde{hf}(x)$: for all $x \in \tilde{X}\}$= {0.1},

$\widetilde{hf}^{X2+}(x) = \max\{\widetilde{hf}(x)$: for all $x \in \tilde{X}\}$= {0.8}.

$\max(\widetilde{hf}^{X1-}, \widetilde{hf}^{X2-})$ = {0.3} and $h \geq \max(\widetilde{hf}^{X1-}, \widetilde{hf}^{X2-}) \Rightarrow h \geq 0.3$ also $(\widetilde{hf}^{X1}(x) \,\tilde{\cup}\, \widetilde{hf}^{X2}(x))$ = {(h1,[0.2, 0.4, 0. 5]) , (h2, [0.6, 0.7, 0.8, 0.9]) , (h3,[0.1, 0.2, 0.3, 0.85])}

$\Rightarrow (\widetilde{hf}^{X1} \,\tilde{\cup}\, \widetilde{hf}^{X2})(x) = \{h \in (\widetilde{hf}^{X1}(x) \,\tilde{\cup}\, \widetilde{hf}^{X2}(x)) / h \geq \max(\widetilde{hf}^{X1-}, \widetilde{hf}^{X2-})\}$= {(h1,[0.4, 0. 5]) , (h2, [0.6, 0.7, 0.8, 0.9]) , (h3,[0.3, 0.85])}.

Definition 2.15. [6] We define the hesitant fuzzy null set as a HF set such that $\widetilde{hf}(x)$ = {0} for all x ∈X. It is denoted by $\widetilde{hf}^{\varphi}$.

Definition 2.16. [6] We define the hesitant fuzzy full set as a HF set such that $\widetilde{hf}(x)$ = {1} for all x ∈X. It is denoted by $\widetilde{hf}^{X}$.

Definition 2.17. [6] A pair ($\widetilde{hfs}$, A) is called a hesitant fuzzy soft set (HF soft set in short cut) if $\widetilde{hfs}$ is a mapping such that $\widetilde{hfs}: A \to \widetilde{hf}(\tilde{X})$. i.e. $\widetilde{hfs}(e) \in \widetilde{hf}(\tilde{X})$, for all e ∈ A where A ⊆ E. The set of all hesitant fuzzy soft sets is denoted by HFS$\widetilde{XE}$.

Example 2.18. Let X be set of team members played in a game. X ={h1, h2, h3, h4}. Let A = {cooperation, energetic, communication, commitment}, a set of parameters. And $\widetilde{hfs}: B \to \widetilde{hf}(\tilde{X})$. Then hesitant fuzzy soft set $\widetilde{hfs}X^{A}$ given below is the evaluation of the performance of 4 players p1, p2, p3, p4 by four judges.

$\widetilde{hfs}$(cooperation) = {h1 = (0.75,1,0.8, 0.8), h2 = (0.6,0.65,0.7,0.75), h3 = (0.8,0.75,1,0.9), h4 = (0.6,0.75,0.9,0.85)}; $\widetilde{hfs}$(energetic) = {h1 = (0.75,0.7,0.8,0.6), h2 = (0.85,0.7,0.8,0.8), h3 = (0.7,0.7,0.9,0.8), h4 = (1,0.9,0.99,0.9)}; $\widetilde{hfs}$(communication) = {h1 = (0.75,0.9,0.8,0.9), h2 = (0.6,0.85,0.8,0.9), h3 = (0. 8,0.9,0.75,0.7), h4 = (0.9,0.65,0.85,0.7)}; $\widetilde{hfs}$(commitment) = {h1 = (0.8,0.9,0.9,1), h2 = (0.7,0.6,0.7,0.8), h3 = (0.8,1,0.9,0.9), h4 = (0.9,0.7,0.75,0.8)}.

Definition 2.19. [6] For two hesitant fuzzy soft sets $\widetilde{hfs}X^{A}$ and $\widetilde{hfq}X^{B}$ over a common universe X we say that $\widetilde{hfs}X^{A}$ is hesitant fuzzy soft subset of $\widetilde{hfq}X^{B}$. It is denoted by

$\widetilde{hfs}X^{A} \,\tilde{\subseteq}\, \widetilde{hfq}X^{B}$. Here $\widetilde{hfq}: B \to \widetilde{hf}(\tilde{X})$.

if (i) $A \subseteq B$ (ii) $\widetilde{hfs}(a) \,\tilde{\subseteq}\, \widetilde{hfq}(a)$ for every a in A.

Definition 2.20. [6] Let $\widetilde{hfs}X^{A}$ be hesitant fuzzy soft set. Then the complement of $\widetilde{hfs}X^{A}$ is defined by $\widetilde{hfs}X^{A'} = (\widetilde{hf}', A)$ where $\widetilde{hf}'(a)$ is the complement of the hesitant fuzzy set $\widetilde{hf}(a)$.

Note. (i). Let $\widetilde{hfs}X^{A}$ and $\widetilde{hfq}X^{B}$ be two hesitant fuzzy soft sets over X. Then they satisfy Demorgan laws.

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(ii). $[\widetilde{hfs}X^{A}\ \widetilde{\cup}\ \widetilde{hfq}X^{B}]' = \widetilde{hfs}X^{A'}\ \widetilde{\cap}\ \widetilde{hfq}X^{B'}$.

(iii). $[\widetilde{hfs}X_{A}\ \widetilde{\cap}\ \widetilde{hfq}X^{B}]' = (\widetilde{hfs}X_{A})'\ \widetilde{\cup}\ (\widetilde{hfq}X^{B})'$.

Definition 2.21. The hesitant fuzzy soft null set is a HF soft set $\widetilde{hfs}X^{A}$ over X is said to be HF soft null set denoted by $\widetilde{hfs}X^{\varphi}$ if $\forall e \in A$, $\widetilde{hfs}$ (e) is the hesitant fuzzy null set $\widetilde{hf}^{\varphi}$ of X where $\widetilde{hf}$ (x) = {0}, $\forall$ x $\in$ X.

Definition 2.22. A HF soft set $\widetilde{hfs}X^{A}$ over X is said to be HF soft full set denoted by $\widetilde{hfs}X^{X}$ if $\forall e \in A$, $\widetilde{hfs}$ (e) is the hesitant fuzzy full set $\widetilde{hf}^{X}$ of X where $\widetilde{hf}$ (x) = {1}, $\forall$ x $\in$ X.

In this section, we will introduce hesitant fuzzy soft topological spaces and related properties.

3 Hesitant fuzzy soft topological space

Definition 3.1.[13] A hesitant fuzzy topology HFτ is a class of hesitant fuzzy sets over X which satisfies the following conditions:

(i) $\widetilde{hf}^{\varphi}$, $\widetilde{hf}^{X} \in$ HFτ,

(ii) If $\widetilde{hf}^{A}$, $\widetilde{hf}^{B} \in$ HFτ , then $\widetilde{hf}^{A}\ \widetilde{\cap}\ \widetilde{hf}^{B} \in$ HFτ,

(iii) If $\{\widetilde{hf}^{A_i}\}_{i\in I}\ \widetilde{\subseteq}$ HFτ , then $\widetilde{\cup}_{i\in I}\ \widetilde{hf}^{A_i} \in$ HFτ .

The pair (HF$\widetilde{XE}$, HFτ) is called a hesitant fuzzy topological space. Every element of HFτ is said to be hesitant fuzzy open set. The complement of a hesitant fuzzy open set is said to be a hesitant fuzzy closed set.

Definition 3.2. A hesitant fuzzy soft topology HFSτ is a class of hesitant fuzzy soft sets over X which satisfies the following conditions:

(i) $\widetilde{hfs}X^{\varphi}$, $\widetilde{hfs}X^{X} \in$ HFSτ,

If $\widetilde{hfs}X^{A}$, $\widetilde{hfs}X^{B} \in$ HFSτ, then $\widetilde{hfs}X^{A}\ \widetilde{\cap}\ \widetilde{hfs}X^{B} \in$ HFSτ,

If $\{\widetilde{hfs}X^{H_i}\}_{i\in I}\ \widetilde{\subseteq}$ HFSτ, then $\widetilde{\cup}_{i\in I}\ \widetilde{hfs}X^{H_i} \in$ HFSτ .

The pair (HFS$\widetilde{XE}$, HFSτ) is called a hesitant fuzzy soft topological space. Every element of HFSτ is said to be hesitant fuzzy soft open set.

Definition 3.3. If the complement of a HF soft set $\widetilde{hfs}X^{A}$ is a hesitant fuzzy soft open set, then $\widetilde{hfs}X^{A}$ is called HF soft closed set.

Example 3.5. Consider a collection HFSτ of HF Soft sets over X as HFSτ = {$\widetilde{hfs}X^{\varphi}$, $\widetilde{hfs}X^{X}$ $\widetilde{hfs}X^{A}$, $\widetilde{hfs}X^{B}$ }.

(i). $\widetilde{hfs}X^{\varphi}$, $\widetilde{hfs}X^{X} \in$ HFSτ.

(ii). $\widetilde{hfs}X^{\varphi}\ \widetilde{\cap}\ \widetilde{hfs}X^{X} = \widetilde{hfs}X^{\varphi}$; $\widetilde{hfs}X^{\varphi}\ \widetilde{\cap}\ \widetilde{hfs}X^{A} = \widetilde{hfs}X^{\varphi}$; $\widetilde{hfs}X^{\varphi}\ \widetilde{\cap}\ \widetilde{hfs}X^{B} = \widetilde{hfs}X^{\varphi}$; $\widetilde{hfs}X^{X}\ \widetilde{\cap}\ \widetilde{hfs}X^{A} = \widetilde{hfs}X^{A}$; $\widetilde{hfs}X^{X}\ \widetilde{\cap}\ \widetilde{hfs}X^{B} = \widetilde{hfs}X^{B}$; $\widetilde{hfs}X^{A}\ \widetilde{\cap}\ \widetilde{hfq}X^{B} = \widetilde{hfs}X^{B}$.

(iii). $\widetilde{hfs}X^{\varphi}\ \widetilde{\cup}\ \widetilde{hfs}X^{X} = \widetilde{hfs}X^{X}$; $\widetilde{hfs}X^{\varphi}\ \widetilde{\cup}\ \widetilde{hfs}X^{A} = \widetilde{hfs}X^{A}$; $\widetilde{hfs}X^{\varphi}\ \widetilde{\cup}\ \widetilde{hfs}X^{B} = \widetilde{hfs}X^{B}$,

$\widetilde{hfs}X^{X}\ \widetilde{\cup}\ \widetilde{hfs}X^{A} = \widetilde{hfs}X^{X}$; $\widetilde{hfs}X^{X}\ \widetilde{\cup}\ \widetilde{hfs}X^{B} = \widetilde{hfs}X^{X}$; $\widetilde{hfs}X^{A}\ \widetilde{\cup}\ \widetilde{hfs}X^{B} = \widetilde{hfs}X^{A}$;

$\widetilde{hfs}X^{X}\ \widetilde{\cup}\ \widetilde{hfs}X^{A}\ \widetilde{\cup}\ \widetilde{hfs}X^{B} = \widetilde{hfs}X^{X}$ and also $\widetilde{hfs}X^{\varphi}\ \widetilde{\cup}\ \widetilde{hfs}X^{X}\ \widetilde{\cup}\ \widetilde{hfs}X^{A}\ \widetilde{\cup}\ \widetilde{hfs}X^{B} = \widetilde{hfs}X^{X}$.

Thus (HFS$\widetilde{XE}$, HFSτ) is a HFS Topological space.

The HFS open sets are $\widetilde{hfs}X^{\varphi}$, $\widetilde{hfs}X^{X}$, $\widetilde{hfs}X^{A}$, $\widetilde{hfs}X^{B}$ and their complements are HF Soft closed sets.

Theorem 3.6. Let (HFS$\widetilde{XE}$, HFSτ) be a HF soft topological space and HFSτ* denote the collection of all HF soft closed sets in (HFS$\widetilde{XE}$, HFSτ).Then

(i). $\widetilde{hfs}X^{\varphi}$, $\widetilde{hfs}X^{X} \in$ HFSτ*

(ii). If $\widetilde{hfs}X^{A}$, $\widetilde{hfs}X^{B} \in$ HFSτ* then $\widetilde{hfs}X^{A}\ \widetilde{\cup}\ \widetilde{hfs}X^{B} \in$ HFSτ* and

(iii). If $\widetilde{hfs}X^{A\alpha} \in$ HFSτ* for all $\alpha \in \Delta$, then $\widetilde{\cap}\ \widetilde{hfs}X^{A\alpha} \in$ HFSτ* for all $\alpha \in \Delta$.

Proof. (i). Since (HFS$\widetilde{XE}$, HFSτ) is a HF Soft topological space clearly $\widetilde{hfs}X^{\varphi}$, $\widetilde{hfs}X^{X} \in$ HFSτ.

$\Rightarrow$ $\widetilde{hfs}X^{\varphi}$, $\widetilde{hfs}X^{X}$ are HF Soft open sets in (HFS$\widetilde{XE}$, HFSτ).

$\Rightarrow$ $\widetilde{hfs}X^{\varphi\prime}$, $\widetilde{hfs}X^{X\prime}$ are HF Soft closed sets in (HFS$\widetilde{XE}$, HFSτ).

$\Rightarrow$ $\widetilde{hfs}X^{\varphi\prime}$, $\widetilde{hfs}X^{X\prime} \in$ HFSτ*.

$\Rightarrow$ $\widetilde{hfs}X^{\varphi}$, $\widetilde{hfs}X^{X} \in$ HFSτ* because $\widetilde{hfs}X^{\varphi\prime} = \widetilde{hfs}X^{X}$ and $\widetilde{hfs}X^{X\prime} = \widetilde{hfs}X^{\varphi}$.

(ii). Suppose $\widetilde{hfs}X^{A}$, $\widetilde{hfs}X^{B} \in$ HFSτ*

$\Rightarrow$ $\widetilde{hfs}X^{A}$, $\widetilde{hfs}X^{B}$ are HF Soft closed sets in (HFS$\widetilde{XE}$, HFSτ).

$\Rightarrow$ $\widetilde{hfs}X^{A\prime}$, $\widetilde{hfs}X^{B\prime}$ are HF Soft open sets in (HFS$\widetilde{XE}$, HFSτ).

$\Rightarrow$ $\widetilde{hfs}X^{A\prime}$, $\widetilde{hfs}X^{B\prime} \in$ HFSτ.

$\Rightarrow$ $\widetilde{hfs}X^{A\prime}\ \widetilde{\cap}\ \widetilde{hfs}X^{B\prime} \in$ HFSτ, By definition 5.3.2.

$\Rightarrow$ $\widetilde{hfs}X^{A\prime}\ \widetilde{\cap}\ \widetilde{hfs}X^{B\prime}$ is a HF Soft open set in (HFS$\widetilde{XE}$, HFSτ).

$\Rightarrow$ $(\widetilde{hfs}X^{A\prime}\ \widetilde{\cap}\ \widetilde{hfs}X^{B\prime})'$ is a HF Soft closed set in (HFS$\widetilde{XE}$, HFSτ).

i.e. $\widetilde{hfs}X^{A}\ \widetilde{\cup}\ \widetilde{hfs}X^{B}$ is a HF Soft open set, by Demorgan laws and definition of HF Soft closed set.

$\Rightarrow \widetilde{hfs}X^A \widetilde{\cup}\ \widetilde{hfs}X^B \in HFS\tau^*$ by definition of HFSτ*.

(iii). Suppose $\widetilde{hfs}X^{A\alpha} \in HFS\tau^*$ for all $\alpha \in \Delta$.

$\Rightarrow$ For each α, $\widetilde{hfs}X^{A\alpha}$ is a HF Soft closed set in (HFS$\widetilde{XE}$, HFSτ).

$\Rightarrow$ For each α, $(\widetilde{hfs}X^{A\alpha})'$ is a HF Soft open set in (HFS$\widetilde{XE}$, HFSτ).

$\Rightarrow \cup_{\alpha\in\Delta}\widetilde{hfs}X^{A\alpha'} \in HFS\tau$ By definition 5.3.2

$\Rightarrow \cup_{\alpha\in\Delta}\widetilde{hfs}X^{A\alpha'}$ is HF Soft open set in (HFS$\widetilde{XE}$, HFSτ).

$\Rightarrow (\cup_{\alpha\in\Delta}\widetilde{hfs}X^{A\alpha'})'$ is HF Soft closed set in (HFS$\widetilde{XE}$, HFSτ).

i.e. $\cap\, \alpha \in \Delta \widetilde{hfs}X^{A\alpha}$ (by Demorgan's law) is HF Soft closed set in (HFS$\widetilde{XE}$, HFSτ).

$\therefore \cap \widetilde{hfs}X^{A\alpha} \in HFS\tau^*$ for all $\alpha \in \Delta$.

Hence proved.

Definition 3.7. Let HFSτ and HSHτ* be two HF Soft topologies on (HFS$\widetilde{XE}$, HFSτ). HFSτ is said to be coarser than HFSτ* or that is HFSτ* finer than HFSτ iff HFSτ $\widetilde{\subseteq}$ HFSτ*. If neither HFSτ $\widetilde{\subseteq}$ HFSτ* nor HFSτ* $\widetilde{\subseteq}$ HFSτ then these topologies are not comparable.

Definition 3.8. Let (HFS$\widetilde{XE}$, HFSτ) be a HF Soft topological space and for Y ⊆ X, B ⊆ E, $\widetilde{hfq}$: B → ϱ(X). Then the HF Soft topology, HFSτ*= {$\widetilde{hfq}Y^B \widetilde{\cap}\ \widetilde{hfs}X^A$: $\widetilde{hfs}X^A \in$ HFSτ} equipped with the class of HF soft subsets HFQ$\widetilde{YB}$ is called HF Soft subspace of (HFS$\widetilde{XE}$, HFSτ) and is denoted by (HFQ$\widetilde{YB}$, HFSτ*).

Definition 3.9. A HF Soft set $\widetilde{hfs}X^A$ is said to be a HF Soft point in (HFS$\widetilde{XE}$, HFSτ), denoted by $\widetilde{hfs}X^Ae$, if for the element e∈A, $\widetilde{hfs}(e) \neq \varphi$ and $\widetilde{hfs}(e') = \varphi$, ∀e'∈A -{e}.

Definition 3.10. A HF Soft point $\widetilde{hfs}X^Ae$ is said to be in a HF Soft topological space (HFS$\widetilde{XE}$, HFSτ) denoted by $\widetilde{hfs}X^Ae \in \widetilde{hfs}X^B$, if for the element e∈ A, $\widetilde{hfs}X^A(e) \leq \widetilde{hfs}X^B(e)$.

Definition 3.11. Let (HFS$\widetilde{XE}$, HFSτ) be a HF Soft topological space over X;
$\widetilde{hfs}X^B$ be a HF soft subset of $\widetilde{hfs}X^A$ and $\widetilde{hfs}X^Ax$ be a HF soft point in (HFS$\widetilde{XE}$, HFSτ), then $\widetilde{hfs}X^B$ is called a HF Soft neighbourhood of $\widetilde{hfs}X^Ax$ if there exists a HF Soft open set $\widetilde{hfs}X^C$ such that $\widetilde{hfs}X^Ax \in \widetilde{hfs}X^C \widetilde{\subseteq} \widetilde{hfs}X^B$. The neighbourhood system of a HF Soft point $\widetilde{hfs}X^Ae$, denoted by $N_{HFS_\tau}(\widetilde{hfs}X^Ae)$, is the collection of all its neighbourhoods.

Definition 3.12. A HF Soft set $\widetilde{hfs}X^N$ in a HF Soft topological space (HFS$\widetilde{XE}$, HFSτ) is called a HF Soft neighbourhood (shortly nbd) of the HF Soft set $\widetilde{hfs}X^A$ if there exists a HF Soft open set $HFSX_B$ such that $\widetilde{hfs}X^A \widetilde{\subseteq} \widetilde{hfs}X^B \widetilde{\subseteq} \widetilde{hfs}X^N$.

Theorem 3.13. The neighbourhood system $N_{HFS_\tau}(\widetilde{hfs}X^Ae)$ at $\widetilde{hfs}X^Ae$ in a HF Soft topological space (HFS$\widetilde{XE}$, HFSτ) possesses the following properties:

(i) If $\widetilde{hfs}X^B \widetilde{\in} N_{HFS_\tau}(\widetilde{hfs}X^Ae)$, then $\widetilde{hfs}X^Ae \in \widetilde{hfs}X^B$,

(ii) If $\widetilde{hfs}X^B \widetilde{\in} N_{HFS_\tau}(\widetilde{hfs}X^Ae)$ and $\widetilde{hfs}X^B \widetilde{\subseteq} \widetilde{hfs}X^D$ then $\widetilde{hfs}X^D \widetilde{\in} N_{HFS_\tau}(\widetilde{hfs}X^Ae)$,

(iii) If $\widetilde{hfs}X^B$, $\widetilde{hfs}X^D \widetilde{\in} N_{HFS_\tau}(\widetilde{hfs}X^Ae)$, then $\widetilde{hfs}X^B \widetilde{\cap} \widetilde{hfs}X^D \widetilde{\in} N_{HFS_\tau}(\widetilde{hfs}X^Ae)$,

(iv) If $\widetilde{hfs}X^B \widetilde{\in} N_{HFS_\tau}(\widetilde{hfs}X^Ae)$, then there is a $\widetilde{hfs}X^M \widetilde{\in} N_{HFS_\tau}(\widetilde{hfs}X^Ae)$ such that $\widetilde{hfs}X^B \widetilde{\in} N_{HFS_\tau}(\widetilde{hfs}X^Ae')$ for each $\widetilde{hfs}X^Ae' \widetilde{\in} \widetilde{hfs}X^M$.

Proof. (i). If $\widetilde{hfs}X^B \widetilde{\in} N_{HFS_\tau}(\widetilde{hfs}X^Ae)$, then there is a $\widetilde{hfs}X^D \widetilde{\in}$ HFSτ
such that $\widetilde{hfs}X^Ae \widetilde{\in} \widetilde{hfs}X^D \widetilde{\subseteq} \widetilde{hfs}X^B$.

Therefore, we have $\widetilde{hfs}X^Ae \widetilde{\in} \widetilde{hfs}X^B$.

(ii). Let $\widetilde{hfs}X^B \widetilde{\in} N_{HFS_\tau}(\widetilde{hfs}X^Ae)$ and $\widetilde{hfs}X^B \widetilde{\subseteq} \widetilde{hfs}X^D$.

Since $\widetilde{hfs}X^B \widetilde{\in} N_{HFS_\tau}(\widetilde{hfs}X^Ae)$, then there is a $\widetilde{hfs}X^M \widetilde{\in}$ HFSτ such that $\widetilde{hfs}X^Ae \widetilde{\in} \widetilde{hfs}X^M \widetilde{\subseteq} \widetilde{hfs}X^B$.

Therefore, we have $\widetilde{hfs}X^Ae \widetilde{\in} \widetilde{hfs}X^M \widetilde{\subseteq} \widetilde{hfs}X^B \widetilde{\subseteq} \widetilde{hfs}X^D$ and so $\widetilde{hfs}X^D \widetilde{\in} N_{HFS_\tau}(\widetilde{hfs}X^Ae)$.

(iii). If $\widetilde{hfs}X^B$, $\widetilde{hfs}X^D \widetilde{\in} N_{HFS_\tau}(\widetilde{hfs}X^Ae)$, then there exist $\widetilde{hfs}X^M$, $\widetilde{hfs}X^K \widetilde{\in}$ HFSτ
such that $\widetilde{hfs}X^Ae \widetilde{\in} \widetilde{hfs}X^M \widetilde{\subseteq} \widetilde{hfs}X^B$
and $\widetilde{hfs}X^Ae \widetilde{\in} \widetilde{hfs}X^K \widetilde{\subseteq} \widetilde{hfs}X^D$.

Hence $\widetilde{hfs}X^Ae \widetilde{\in} [\widetilde{hfs}X^M \widetilde{\cap}\ \widetilde{hfs}X^K] \widetilde{\subseteq} [\widetilde{hfs}X^B \widetilde{\cap} \widetilde{hfs}X^D]$.

Since $[\widetilde{hfs}X^M \widetilde{\cap}\ \widetilde{hfs}X^K] \widetilde{\in}$ HFSτ, we have $[\widetilde{hfs}X^B \widetilde{\cap}\ \widetilde{hfs}X^D] \widetilde{\in} N_{HFS_\tau}(\widetilde{hfs}X^Ae)$.

(iv). If $\widetilde{hfs}X^B \widetilde{\in} N_{HFS_\tau}(\widetilde{hfs}X^Ae)$, then there is a $\widetilde{hfs}X^K \widetilde{\in}$ HFSτ such that $\widetilde{hfs}X^Ae \widetilde{\in} \widetilde{hfs}X^K \widetilde{\subseteq} \widetilde{hfs}X^B$.

Put $\widetilde{hfs}X^M = \widetilde{hfs}X^K$. Then for each $\widetilde{hfs}X^Ae' \widetilde{\in} \widetilde{hfs}X^M$,
$\widetilde{hfs}X^Ae' \widetilde{\in} \widetilde{hfs}X^M = \widetilde{hfs}X^K \widetilde{\subseteq} \widetilde{hfs}X^B$.

Hence $\widetilde{hfs}X^B \widetilde{\in} N_{HFS_\tau}(\widetilde{hfs}X^Ae')$.

Definition 3.14. The HF Soft interior of a HF Soft set $\widetilde{hfs}X^A$ denoted by $\mathbf{I}(\widetilde{hfs}X^A)$ is defined by the union of all HF Soft

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open sub sets of $\widetilde{hfs}X^A$.

i.e = $\bigcup_i \{\widetilde{hfs}X^{Bi}$: $\widetilde{hfs}X^{Bi}$ is HF Soft open sub set of $\widetilde{hfs}X^A\}$.

Theorem 3.15. By the above definition, the following statements can be proved easily.

(i). I($\widetilde{hfs}X^A$) is a HF Soft open set, because arbitrary union of HF Soft open sets is HF Soft open.

(ii). I($\widetilde{hfs}X^A$) $\subset \widetilde{hfs}X^A$, because union of subsets of a set is again subset of it, and

(iii). If $\widetilde{hfs}X^A$ is HF Soft open then $\widetilde{hfs}X^A$ = **I**($\widetilde{hfs}X^A$).

(iv). I($\widetilde{hfs}X^A$) is the largest HF Soft open set contained in $\widetilde{hfs}X^A$.

Definition 3.16. The HF Soft closure of a HF Soft set $\widetilde{hfs}X^A$ denoted by $\overline{\widetilde{hfs}X^A}$ is defined by the intersection of all HF Soft closed super sets of $\widetilde{hfs}X^A$.

i.e. $\overline{\widetilde{hfs}X^A} = \bigcap_i \{\widetilde{hfs}X^{Bi}$: $\widetilde{hfs}X^{Bi}$is HF Soft closed super set of $\widetilde{hfs}X^A\}$

Theorem 3.17. For any HF Soft set $\widetilde{hfs}X^A$, we have the following.

(i). $\overline{\widetilde{hfs}X^A}$ is a HF Soft closed set (Since $\overline{\widetilde{hfs}X^A}$ is the arbitrary intersection of HF Soft closed sets).

(ii). $\widetilde{hfs}X^A \widetilde{\subseteq} \overline{\widetilde{hfs}X^A}$. (Since $\overline{\widetilde{hfs}X^A}$ is the intersection of HF Soft super sets of $\widetilde{hfs}X^A$)

and (iii). If $\widetilde{hfs}X^A$ is HF Soft closed then $\widetilde{hfs}X^A = \overline{\widetilde{hfs}X^A}$.

Proof: Let (HFS$\widetilde{XE}$, HFSτ) be a HF Soft topological space over X

Let $\widetilde{hfs}X^A$ be a HF Soft set over X such that $\widetilde{hfs}X^A = \overline{\widetilde{hfs}X^A}$.

To prove that $\widetilde{hfs}X^A$ is HF Soft closed.

We have $\overline{\widetilde{hfs}X^A} = \bigcap_i \{\widetilde{hfs}X^{Bi}$: $\widetilde{hfs}X^{Bi}$ is HF Soft closed super set of $\widetilde{hfs}X^A\}$ ……..(1)

Being an arbitrary intersection of HF Soft closed sets, $\overline{\widetilde{hfs}X^A}$ is HF Soft closed set,

and since $\widetilde{hfs}X^A = \overline{\widetilde{hfs}X^A}$, we get $\widetilde{hfs}X^A$ is also HF Soft closed.

Conversely, suppose that $\widetilde{hfs}X^A$ is HF Soft closed in (HFS$\widetilde{XE}$, HFSτ).

To prove that $\widetilde{hfs}X^A = \overline{\widetilde{hfs}X^A}$.

It is clear from definition (1), being a superset, $\widetilde{hfs}X^A \widetilde{\subseteq} \overline{\widetilde{hfs}X^A}$ …….(2).

And for any HF Soft closed super set $\widetilde{hfq}X^B$ of $\widetilde{hfs}X^A$,

we have $\overline{\widetilde{hfs}X^A} \widetilde{\subseteq} \widetilde{hfq}X^B$ from (1), being intersection of all HF Soft closed super sets.

Similarly, as $\widetilde{hfs}X^A$ is HF Soft closed and also we know that $\widetilde{hfs}X^A \widetilde{\subseteq} \widetilde{hfs}X^A$, we get that $\overline{\widetilde{hfs}X^A} \widetilde{\subseteq} \widetilde{hfs}X^A$ ………..(3)

Hence from (2) and (3), $\widetilde{hfs}X^A = \overline{\widetilde{hfs}X^A}$.

Definition 3.18. A HF Soft point $\widetilde{hfs}X^A e$ is said to be a HF Soft boundary point of a HF Soft set $\widetilde{hfs}X^A$ if $\widetilde{hfs}X^A{}_e \in \overline{\widetilde{hfs}X^A} \widetilde{\cap} \overline{\widetilde{hfs}X^{A'}}$.

Definition 3.19. The set of all HF Soft boundary points over a HF Soft set $\widetilde{hfs}X^A$ is called HF Soft boundary of the set $\widetilde{hfs}X^A$ and is denoted by **B**($\widetilde{hfs}X^A$).

Definition 3.20. Let (HFS$\widetilde{XE}$, HFSτ) be a HF Soft topological space. Let $\widetilde{hfs}X^A$ be a

HF Soft set over X. Then the HF Soft exterior of $\widetilde{hfs}X^A$, denoted by $\widetilde{hfs}X^{Ao}$,

is defined as $\widetilde{hfs}X^{Ao}$ = **I**($\widetilde{hfs}X^{Ac}$).i.e. the exterior of a HF Soft set $\widetilde{hfs}X^A$ is interior of the complement of the HF Soft set $\widetilde{hfs}X^A$.

Theorem 3.21. Let (HFS$\widetilde{XE}$, HFSτ) be a HF Soft topological space. Let $\widetilde{hfs}X^A$ and

$\widetilde{hfq}X^B$ are fuzzy soft sets over X. Then

(1) $(\widetilde{hfs}X^A)^o = (\widetilde{hfs}X^{Ac})^o$.

(2) $((\widetilde{hfs}X^A) \widetilde{\cup} (\widetilde{hfq}X^B))^o = (\widetilde{hfs}X^A)^o \widetilde{\cap} (\widetilde{hfq}X^{B\,o})$.

(3) $(\widetilde{hfs}X^A)^o \widetilde{\cup} (\widetilde{hfq}X^B)^o \widetilde{\subseteq} ((\widetilde{hfs}X^A) \widetilde{\cap} (\widetilde{hfq}X^B))^o$.

Theorem 3.22. Let (HFS$\widetilde{XE}$, HFSτ) be a HF Soft topological space. Let $\widetilde{hfs}X^A$ be HF Soft set over X. Then

(1) (**B**($\widetilde{hfs}X^A$))′ = $(\widetilde{hfs}X^A)^o \widetilde{\cup} ((\widetilde{hfs}X^{A'})^o = (\widetilde{hfs}X^A)^o \widetilde{\cup}$ **I**($\widetilde{hfs}X^A$).

(2) $\overline{\widetilde{hfs}X^A} = (\widetilde{hfs}X^A)^o \widetilde{\cup}$ **B**($\widetilde{hfs}X^A$).

(3) $(\widetilde{hfs}X^A)^o = (\widetilde{hfs}X^A) \widetilde{\setminus}$ **B**($\widetilde{hfs}X^A$).

(4) **B**($\widetilde{hfs}X^A$) = $\overline{\widetilde{hfs}X^A} \widetilde{\cap} \overline{\widetilde{hfs}X^{A\prime}} = \overline{\widetilde{hfs}X^A} \widetilde{\setminus} (\widetilde{hfs}X^A)^o$.

Proof. (1) $(\widetilde{hfs}X^A)^o \widetilde{\cup} ((\widetilde{hfs}X^{A'})^o = (((\widetilde{hfs}X^A)^o)')' \widetilde{\cup} ([(\widetilde{hfs}X^{A'})^o]')'$.

$= [((\widetilde{hfs}X^A)^o)' \widetilde{\cap} (((\widetilde{hfs}X^{A'})^o)']' = [\overline{\widetilde{hfs}X^A} \widetilde{\cap} \overline{\widetilde{hfs}X^{A\prime}}] = ($**B**$(\widetilde{hfs}X^A))'$.

(2) $(\widetilde{hfs}X^A)^o \widetilde{\cup}$ **B**($\widetilde{hfs}X^A$) $= (\widetilde{hfs}X^A)^o \widetilde{\cup} \overline{\widetilde{hfs}X^A} \widetilde{\cap} \overline{\widetilde{hfs}X^{A'}}$

$= [(\widetilde{hfs}X^A)^o \widetilde{\cup} \overline{\widetilde{hfs}X^A}] \widetilde{\cap} [(\widetilde{hfs}X^A)^o \widetilde{\cup} \overline{\widetilde{hfs}X^{A\prime}}]$

$= \overline{\widetilde{hfs}X^A} \widetilde{\cap} [(\widetilde{hfs}X^A)^o \widetilde{\cup} \overline{\widetilde{hfs}X^{A\prime}}]$

$= \overline{\widetilde{hfs}X^A} \widetilde{\cap} \widetilde{hfs}X^X = (\widetilde{hfs}X^A)$.

(3) $(\widetilde{hfs}X^A) \widetilde{\setminus}$ **B**($\widetilde{hfs}X^A$) $= (\widetilde{hfs}X^A) \widetilde{\cap}$ (**B**($\widetilde{hfs}X^A$))′

$= (\widetilde{hfs}X^A) \widetilde{\cap} ((\widetilde{hfs}X^A)^o \widetilde{\cup} ((\widetilde{hfs}X^A)')^o)$ (by (a))

$= [(\widetilde{hfs}X^A) \tilde{\cap} ((\widetilde{hfs}X^A)^o] \tilde{\cup} [(\widetilde{hfs}X^A) \tilde{\cap} ((\widetilde{hfs}X^A)')^o]$

$= (\widetilde{hfs}X^A)^o \tilde{\cup} \widetilde{hfs}X^{\varphi} = (\widetilde{hfs}X^A)^o$.

(4) $\mathbf{B}(\widetilde{hfs}X^A) = \overline{\widetilde{hfs}X^A} \tilde{\setminus} (\widetilde{hfs}X^A)^o = \overline{\widetilde{hfs}X^A} \tilde{\cap} ((\widetilde{hfs}X^A)^o)' = \overline{\widetilde{hfs}X^A} \tilde{\cap} \overline{\widetilde{hfs}X^A}'$.

Theorem 3.23. Let (HFS$\widetilde{XE}$, HFSτ) be a HF Soft topological space. Let $\widetilde{hfs}X^A$ be fuzzy

soft set over X. Then

(1) $\mathbf{B}(\widetilde{hfs}X^A) \tilde{\cap} (\widetilde{hfs}X^A)^o = \widetilde{hfs}X^{\varphi}$.

(2) $\mathbf{B}(\widetilde{hfs}X^A) \tilde{\cap} (\widetilde{hfs}X^A)^o = \widetilde{hfs}X^{\varphi}$.

Proof. (1) $\mathbf{B}(\widetilde{hfs}X^A) \tilde{\cap} (\widetilde{hfs}X^A)^o$

$= (\widetilde{hfs}X^A)^o \tilde{\cap} \mathbf{B}(\widetilde{hfs}X^A)$

$= (\widetilde{hfs}X^A)^o \tilde{\cap} (\overline{\widetilde{hfs}X^A} \tilde{\cap} \overline{\widetilde{hfs}X^{A'}})$

$= (\widetilde{hfs}X^A)^o \tilde{\cap} \overline{\widetilde{hfs}X^A} \tilde{\cap} \overline{\widetilde{hfs}X^A}'$

$= \widetilde{hfs}X^{\varphi}$.

(2) $\mathbf{B}(\widetilde{hfs}X^A) \tilde{\cap} \mathbf{I}(\widetilde{hfs}X^A)$

$= ((\widetilde{hfs}X^{A'})^o \tilde{\cap} (\overline{\widetilde{hfs}X^A} \tilde{\cap} \overline{\widetilde{hfs}X^{A'}})$

$= ((\widetilde{hfs}X^{A'})^o \tilde{\cap} \overline{\widetilde{hfs}X^A} \tilde{\cap} \overline{\widetilde{hfs}X^{A'}}$

$= \overline{\widetilde{hfs}X^{A'}} \tilde{\cap} \overline{\widetilde{hfs}X^A} \tilde{\cap} \overline{\widetilde{hfs}X^{A\prime}}$.

$= \widetilde{hfs}X^{\varphi}$.

Theorem 3.24. The following properties can be easily established with the help of the above definition.

For any HF Soft topological space (HFS$\widetilde{XE}$, HFSτ) and $\widetilde{hfs}X^C \in$ HFSτ then

(i). $\mathbf{B}(\widetilde{hfs}X^C) \tilde{\subseteq} \overline{\widetilde{hfs}X^C}$.

i.e. The HF Soft boundary of a HF Soft set is subset of the HF Soft boundary of the set.

(ii). $\widetilde{hfs}X^A$ is HF soft open set if and only if $\widetilde{hfs}X^A \tilde{\cap} \mathbf{B}(\widetilde{hfs}X^A) = \widetilde{hfs}X^{\varphi}$.

(iii) $\widetilde{hfs}X^A$ is HF Soft closed if and only if and only if $\mathbf{B}(\widetilde{hfs}X^A) \tilde{\subseteq} \widetilde{hfs}X^A$.

(iv). $\mathbf{B}(\widetilde{hfs}X^A) = \overline{\widetilde{hfs}X^A} \tilde{\setminus} (\widetilde{hfs}X^A)^o$.

Proof. (i) By definition it is clear.

(ii) Let $\widetilde{hfs}X^A$ be a HF Soft open set over X.

Then $\widetilde{hfs}X^A = (\widetilde{hfs}X^A)^o$.

$\Rightarrow \widetilde{hfs}X^A \tilde{\cap} \mathbf{B}(\widetilde{hfs}X^A) = (\widetilde{hfs}X^A)^o \tilde{\cap} \mathbf{B}(\widetilde{hfs}X^A) = \widetilde{hfs}X^{\varphi}$. (by theorem 3.23.)

Conversely, let $\widetilde{hfs}X^A \tilde{\cap} \mathbf{B}(\widetilde{hfs}X^A) = \widetilde{hfs}X^{\varphi}$.

Then $\widetilde{hfs}X^A \tilde{\cap} \overline{\widetilde{hfs}X^A} \tilde{\cap} \overline{\widetilde{hfs}X^{A\prime}} = \widetilde{hfs}X^{\varphi}$.

That is, $\widetilde{hfs}X^A \tilde{\cap} \overline{\widetilde{hfs}X^{A\prime}} = \widetilde{hfs}X^{\varphi}$.

. So $\overline{\widetilde{hfs}X^{A\prime}} \tilde{\subseteq} \widetilde{hfs}X^{A\prime}$ It shows that $\widetilde{hfs}X^{A\prime}$ is a HF Soft closed set.

Hence $\widetilde{hfs}X^A$ is HF Soft open.

(iii) Let $\widetilde{hfs}X^A$ be HF Soft closed set over X. Then $\widetilde{hfs}X^A = \overline{\widetilde{hfs}X^A}$.

Now $\mathbf{B}(\widetilde{hfs}X^A) = \overline{\widetilde{hfs}X^A} \tilde{\cap} \overline{\widetilde{hfs}X^{A\prime}} \tilde{\subseteq} \overline{\widetilde{hfs}X^A} = \widetilde{hfs}X^A$.

That is, $\mathbf{B}(\widetilde{hfs}X^A) \tilde{\subseteq} \widetilde{hfs}X^A$.

Conversely, let $\mathbf{B}(\widetilde{hfs}X^A) \tilde{\subseteq} \widetilde{hfs}X^A$.

Then $\mathbf{B}(\widetilde{hfs}X^A) \tilde{\cap} \widetilde{hfs}X^{A\prime} = \widetilde{hfs}X^{\varphi}$.

Since $\mathbf{B}(\widetilde{hfs}X^A) = \mathbf{B}(\widetilde{hfs}X^{A\prime}) = \widetilde{hfs}X^{\varphi}$, we have $\mathbf{B}(\widetilde{hfs}X^{A\prime}) \tilde{\cap} \widetilde{hfs}X^{A\prime} = \widetilde{hfs}X^{\varphi}$.

Then by (1) $\widetilde{hfs}X^{A\prime}$ is HF Soft open and hence $\widetilde{hfs}X^A$ is a HF Soft closed set over X.

(iii). $\overline{\widetilde{hfs}X^A} \tilde{\setminus} (\widetilde{hfs}X^A)^o = \overline{\widetilde{hfs}X^A} \tilde{\cap} ((\widetilde{hfs}X^A)^o)^c = \overline{\widetilde{hfs}X^A} \tilde{\cap} [\tilde{\cup} (\widetilde{hfs}X^{A\alpha}]'$

(By definition 2.19 and 3.2., $\widetilde{hfs}X^{A\alpha} \tilde{\subseteq} (\widetilde{hfs}X^A)$, $\widetilde{hfs}X^{A\alpha} \in$HFSτ))

$= \overline{\widetilde{hfs}X^A} \tilde{\cap} (\tilde{\cap}[\overline{\widetilde{hfs}X^{A\alpha}}]') = \overline{\widetilde{hfs}X^A} \tilde{\cap} \overline{\widetilde{hfs}X^{A'}} = \mathbf{B}(\widetilde{hfs}X^A)$.

(Since $\widetilde{hfs}X^{A\alpha} \in$HFSτ, $\widetilde{hfs}X^{A\alpha}$ is HF Soft open set for each α, $\widetilde{hfs}X^{A\alpha'}$ is HF Soft closed for each α and arbitrary intersection of HF Soft closed sets is HF Soft closed set and we know that $\tilde{\cap}[\widetilde{hfs}X^{A\alpha}]' = \widetilde{hfs}X^{A\prime}$).

Theorem 3.25. Let (HFS$\widetilde{XE}$, HFSτ) be a HF Soft topological space. Let $\widetilde{hfs}X^A$ and

$\widetilde{hfq}X^B$ are HF Soft sets over X. Then

(1) $\mathbf{B}(\widetilde{hfs}X^A \tilde{\cup} \widetilde{hfq}X^B) \tilde{\subseteq} \mathbf{B}[\widetilde{hfs}X^A \tilde{\cap} \widetilde{hfq}X^{B\prime}] \tilde{\cup} [\mathbf{B}(\widetilde{hfq}X^B) \tilde{\cap} \overline{\widetilde{hfs}X^{A\prime}}] \tilde{\subseteq} [\mathbf{B}(\widetilde{hfs}X^A \tilde{\cup} \mathbf{B}(\widetilde{hfq}X^B)]$

(2) $\mathbf{B}(\widetilde{hfs}X^A \tilde{\cap} \widetilde{hfq}X^B) \tilde{\subseteq} [\mathbf{B}(\widetilde{hfs}X^A) \tilde{\cap} \overline{\widetilde{hfq}X^B}] \tilde{\cup} [\mathbf{B}(\widetilde{hfq}X^B) \tilde{\cap} \overline{\widetilde{hfs}X^A}] \tilde{\subseteq} [\mathbf{B}(\widetilde{hfs}X^A \tilde{\cup} \mathbf{B}(HFQX_B]$.

Proof. (1) $\mathbf{B}(\widetilde{hfs}X^A \tilde{\cup} \widetilde{hfq}X^B) = \overline{(\widetilde{hfs}X^A \tilde{\cup} \widetilde{hfq}X^B)} \tilde{\cap} \overline{(\widetilde{hfs}X^A \tilde{\cup} \widetilde{hfq}X^B)'}$.

$= [\overline{\widetilde{hfs}X^A} \tilde{\cup} \overline{\widetilde{hfq}X^B}] \tilde{\cap} [\overline{\widetilde{hfs}X^{A\prime} \tilde{\cap} \widetilde{hfq}X^{B\prime}}]$.

$\tilde{\subseteq} [\overline{\widetilde{hfs}X^A} \tilde{\cup} \overline{\widetilde{hfq}X^B}] \tilde{\cap} [\overline{\widetilde{hfs}X^{A\prime}} \tilde{\cap} \overline{\widetilde{hfq}X^{B\prime}}]$.

$= [\overline{\widetilde{hfs}X^A} \tilde{\cap} (\overline{\widetilde{hfs}X^{A\prime}} \tilde{\cap} \overline{\widetilde{hfq}X^{B\prime}})] \tilde{\cup} [\overline{\widetilde{hfq}X^B} \tilde{\cap} (\overline{\widetilde{hfs}X^{A\prime}} \tilde{\cap} \overline{\widetilde{hfq}X^{B\prime}})]$

$= [\overline{\widetilde{hfs}X^A} \tilde{\cap} (\overline{\widetilde{hfs}X^{A\prime}} \tilde{\cap} \overline{\widetilde{hfq}X^{B\prime}})] \tilde{\cup} [\overline{\widetilde{hfq}X^B} \tilde{\cap} \overline{\widetilde{hfq}X^{B\prime}} \tilde{\cap} \overline{\widetilde{hfs}X^{A\prime}}]$

$= [\mathbf{B}(\widetilde{hfs}X^A) \tilde{\cap} \overline{\widetilde{hfq}X^{B\prime}}] \tilde{\cup} [\mathbf{B}(\widetilde{hfq}X^B) \tilde{\cap} \overline{\widetilde{hfs}X^{A\prime}}]$

$\tilde{\sqsubseteq}$ [**B**($\widetilde{hfs}X^{A}$) $\tilde{\cup}$ **B**($\widetilde{hfq}X^{B}$)]

(2) **B** ($\widetilde{hfs}X^{A}$ $\tilde{\cap}$ $\widetilde{hfq}X^{B}$) = $[\overline{\widetilde{hfs}X^{A} \tilde{\cap} \widetilde{hfq}X^{B}}] \tilde{\cap} [\overline{\widetilde{hfs}X^{A} \tilde{\cap} \widetilde{hfq}X^{B}}]'$.

= $[\overline{\widetilde{hfs}X^{A}} \tilde{\cap} \overline{\widetilde{hfq}X^{B}}] \tilde{\cap} [\overline{\widetilde{hfs}X^{A'}} \tilde{\cup} \overline{\widetilde{hfq}X^{B'}}]$.

$\tilde{\sqsubseteq}$ $[\overline{\widetilde{hfs}X^{A}} \tilde{\cap} \overline{\widetilde{hfq}X^{B}}] \tilde{\cap} [\overline{\widetilde{hfs}X^{A}{}'} \tilde{\cup} \overline{\widetilde{hfq}X^{B}{}'}]$

= $[\overline{\widetilde{hfs}X^{A}} \tilde{\cap} \overline{\widetilde{hfq}X^{B}} \tilde{\cap} \overline{\widetilde{hfs}X^{A}{}'} \tilde{\cup} [\overline{\widetilde{hfs}X^{A}} \tilde{\cap} \overline{\widetilde{hfq}X^{B}} \tilde{\cap} \overline{\widetilde{hfq}X^{B}{}'}]$

= $[\overline{\widetilde{hfs}X^{A}} \tilde{\cap} \overline{\widetilde{hfs}X^{A}{}'} \tilde{\cap} \overline{\widetilde{hfq}X^{B}}] \tilde{\cup} [\overline{\widetilde{hfs}X^{A}} \tilde{\cap} \overline{\widetilde{hfq}X^{B}} \tilde{\cap} \overline{\widetilde{hfq}X^{B}{}'}$

= [**B**($\widetilde{hfs}X^{A}$) $\tilde{\cap}$ $\overline{\widetilde{hfq}X^{B}}$ $\tilde{\cup}$ [**B**($\widetilde{hfq}X^{B}$) $\tilde{\cap}$ $\overline{\widetilde{hfs}X^{A}}$].

$\tilde{\sqsubseteq}$ [**B**($\widetilde{hfs}X^{A}$) $\tilde{\cup}$ **B**($\widetilde{hfq}X^{B}$)].

4 Hesitant fuzzy soft compact spaces

In this section we define HF Soft open cover, HF Soft sub cover, HF Soft compact space, HF Soft compact subspace.

Definition 4.1. Let (HFS$\widetilde{XE}$, HFSτ) be a HF Soft topological space. A collection {$\widetilde{hfs}X^{Gi}$} of HF Soft subsets of HFS$\widetilde{XE}$ is said to be a HF Soft open cover of HFS$\widetilde{XE}$ if each HF Soft point in HFS$\widetilde{XE}$ is in at least one $\widetilde{hfs}X^{Gi}$. i.e. HFS$\widetilde{XE}$= $\cup_i$ $\widetilde{hfs}X^{Gi}$.

Definition 4.2. A sub class {$\widetilde{hfs}X^{G1}$, $\widetilde{hfs}X^{G2}$, $\widetilde{hfs}X^{G3}$ …. $\widetilde{hfs}X^{Gn}$} of a HF Soft open cover {$\widetilde{hfs}X^{Gi}$} which itself forms an open cover for HFS$\widetilde{XE}$ is called a HF Soft sub cover of HFS$\widetilde{XE}$.

i.e. When HFS$\widetilde{XE}$= $\widetilde{hfs}X^{G1}$ $\tilde{\cup}$ $\widetilde{hfs}X^{G2}$ $\tilde{\cup}$ $\widetilde{hfs}X^{G3}$……….. $\tilde{\cup}$ $\widetilde{hfs}X^{Gn}$.

Definition 4.3. A HF Soft compact space is a HF Soft topological space in which every HF Soft open cover has a finite HF Soft sub cover.

Definition 4.4. A HF Soft compact subspace of a HF Soft topological space is a HF Soft subspace which itself is a HF Soft compact as a HF Soft topological space in its own right.

Theorem 4.5. Any HF Soft closed subspace of a HF Soft compact space is HF Soft compact.

Proof. Let (HFS$\widetilde{XE}$, HFSτ) be a HF Soft compact space.

Let $\widetilde{hfs}X^{A}$ be a HF Soft closed subspace of HFS$\widetilde{XE}$.

Now we have to prove that $\widetilde{hfs}X^{A}$ is HF Soft compact space.

For this we show that every HF Soft open cover of $\widetilde{hfs}X^{A}$ contains a finite HF Soft sub cover.

Suppose {$\widetilde{hfs}X^{Gi}$} is a HF Soft open cover of $\widetilde{hfs}X^{A}$.

By definition 3.2, $\widetilde{hfs}X^{A}$ = $\tilde{\cup}$ $\widetilde{hfs}X^{Gi}$ (1)

Since each $\widetilde{hfs}X^{Gi}$ is a HF Soft open set in $\widetilde{hfs}X^{A}$,

$\widetilde{hfs}X^{Gi}$ = $\widetilde{hfs}X^{Hi}$ $\tilde{\cap}$ HFS$\widetilde{XE}$…………..(2),

where $\widetilde{hfs}X^{Hi}$ $\tilde{\sqsubseteq}$ HFS$\widetilde{XE}$ for each i by definition of relative HF Soft topology.

Since $\widetilde{hfs}X^{A}$ is HF Soft closed subspace of HFS$\widetilde{XE}$,

$\widetilde{hfs}X^{A}{}'$ is HF Soft open subspace of HFS$\widetilde{XE}$.

And it implies that HFS$\widetilde{XE}$ = $\widetilde{hfs}X^{A}$ $\tilde{\cup}$ $\widetilde{hfs}X^{A'}$…………….(3)

= $\tilde{\cup}$ $\widetilde{hfs}X^{Gi}$ $\tilde{\cup}$ $\widetilde{hfs}X^{B'}$ (Using (1).

= $\tilde{\cup}$ ($\widetilde{hfs}X^{Gi}$ $\tilde{\cup}$ $\widetilde{hfs}X^{A'}$)

= $\tilde{\cup}$ [($\widetilde{hfs}X^{Hi}$) $\tilde{\cup}$ $\widetilde{hfs}X^{A'}$], Using (2).

= $\tilde{\cup}$ [($\widetilde{hfs}X^{Hi}$ $\tilde{\cup}$ $\widetilde{hfs}X^{A'}$) $\tilde{\cap}$ ($\widetilde{hfs}X^{A}$ $\tilde{\cup}$ $\widetilde{hfs}X^{A'}$)] By distributive property.

= [($\widetilde{hfs}X^{Hi}$ $\tilde{\cup}$ $\widetilde{hfs}X^{A'}$) $\tilde{\cap}$ HFSX_E], Using (3).

= [$\widetilde{hfs}X^{Hi}$ $\tilde{\cup}$ $\widetilde{hfs}X^{A'}$], Since each $\widetilde{hfs}X^{Hi}$ and $\widetilde{hfs}X^{A}{}'$ are HF Soft subsets of HFS$\widetilde{XE}$.

∴{{$\widetilde{hfs}X^{Hi}$}, $\widetilde{hfs}X^{A}{}'$} is a HF Soft cover of HFS$\widetilde{XE}$.

Since HFS$\widetilde{XE}$ is HF Soft compact, this HF Soft open cover has a finite HF Soft sub cover.

Let it be {$\widetilde{hfs}X^{H1}$, $\widetilde{hfs}X^{H2}$, $\widetilde{hfs}X^{H3}$, ………..$\widetilde{hfs}X^{Hn}$, $\widetilde{hfs}X^{A}{}'$}.

⇒ HFS$\widetilde{XE}$ = $\widetilde{hfs}X^{H1}$ $\tilde{\cup}$ $\widetilde{hfs}X^{H2}$ $\tilde{\cup}$ $\widetilde{hfs}X^{H3}$……….. $\tilde{\cup}$ $\widetilde{hfs}X^{Hn}$ …………….(4)

Since $\widetilde{hfs}X^{A}$ $\tilde{\sqsubseteq}$ HFS$\widetilde{XE}$ we get $\widetilde{hfs}X^{A}$ = HFS$\widetilde{XE}$ $\tilde{\cap}$ $\widetilde{hfs}X^{A}$.

⇒ $\widetilde{hfs}X^{A}$ = [$\widetilde{hfs}X^{H1}$ $\tilde{\cup}$ $\widetilde{hfs}X^{H2}$ $\tilde{\cup}$ $\widetilde{hfs}X^{H3}$ $\tilde{\cup}$……..$\tilde{\cup}$ $\widetilde{hfs}X^{Hn}$ $\tilde{\cup}$ $\widetilde{hfs}X^{A'}$] $\tilde{\cap}$ $\widetilde{hfs}X^{A}$, Using (4).

$\widetilde{hfs}X^{A}$=[($\widetilde{hfs}X^{H1}$ $\tilde{\cap}$ $\widetilde{hfs}X^{A}$) $\tilde{\cup}$($\widetilde{hfs}X^{H2}$ $\tilde{\cap}$ $\widetilde{hfs}X^{A}$)…………($\widetilde{hfs}X^{Hn}$ $\tilde{\cap}$ $\widetilde{hfs}X^{A}$)] $\tilde{\cup}$ ($\widetilde{hfs}X^{A}{}'$ $\tilde{\cap}$ $\widetilde{hfs}X^{A}$).

$\widetilde{hfs}X^{A}$ = $\widetilde{hfs}X^{G1}$ $\tilde{\cup}$ $\widetilde{hfs}X^{G2}$ $\tilde{\cup}$……….$\tilde{\cup}$ $\widetilde{hfs}X^{Gn}$ $\tilde{\cup}$ $\widetilde{hfs}X^{\varphi}$.

⇒ $\widetilde{hfs}X^{A}$ = $\widetilde{hfs}X^{G1}$ $\tilde{\cup}$ $\widetilde{hfs}X^{G2}$ $\tilde{\cup}$ $\widetilde{hfs}X^{G3}$……….$\tilde{\cup}$ $\widetilde{hfs}X^{Gn}$.

∴ {$\widetilde{hfs}X^{G1}$, $\widetilde{hfs}X^{G2}$, $\widetilde{hfs}X^{G3}$, $\widetilde{hfs}X^{G4}$……….., $\widetilde{hfs}X^{Gn}$} is a finite HF Soft sub cover of $\widetilde{hfs}X^{A}$.

Hence every HF Soft open cover of $\widetilde{hfs}X^{A}$ has a finite HF Soft sub cover.

And hence $\widetilde{hfs}X^{A}$ is a HF Soft compact space.

Thus any HF Soft closed subspace of a HF Soft compact space is HF Soft compact.

Hence the theorem is proved.

5 CONCLUSION

In this chapter, we have presented mappings on hesitant fuzzy soft sets,HF soft topological spaces and proved related properties. We have also defined HF Soft point, HF Soft neighborhood system and verified corresponding results. Later we have defined HF Soft interior point, HF Soft closure, HF Soft boundary point, and HF Soft exterior points and their properties are verified. We have also defined HF Soft compactness, for this we have introduced HF Soft open cover and HF Soft finite sub cover. Next we have proved some theorems related

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to them.

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