# Interpretation of Dirichlet, Bartlett, Hanning and Hamming windows using Fractional Fourier Transform

P.V.Muralidhar,Venkata.L.Nsastry D, and S.K.Nayak.

***Abstract*—AnApproximated Exponential Fractional Fourier Transforms(FrFT) Mathematical derivation for Dirichletwindow, Bartlettwindow,Hanning window and Hamming window is proposed. By control the parameter of FrFT,it is possible to control the Spectral parameters of above windows like Half Bandwidth (HBW), Maximum Side Lobe Attenuation(MSLA) and Side Lobe Fall Of Ratio(SLFOR).This proposed derivations are also holds good for generalization of FrFT with Fourier Transform(FT).**

***Index Terms*—Fractional Fourier transform, Dirichlet window, Bartlett window, Hamming window ,Hanning window.**

NOMENCLATURE

FT: Fourier Transform

FrFT: Fractional Fourier Transform

## 1.INTRODUCTION

In order to reduce the effects of spectral leakages in Harmonic analysis, windows are used [1]. window functions successfully used in the areas like interpolation factors to design Anti-Imaging filters, speech processing systems, digital filter design and beam forming [2]-[3].windows are also useful to solve reconstructive errors which are objective functions to design the prototype filters [4].windows are essentially Applicable in spectral analysis of signals[5]-[6].According to [3],as the parameter of FrFTi.e$\alpha = \frac{\pi}{2}$ which could not holds good for generalization of FrFT to FT [7].In this propsedDerivatrion of FrFT,An attempt is made to study the variations of window parameters like HBW,MSLA and MSLFOR by different values of fluid parameter of FrFT to FT at $\alpha = \frac{\pi}{2}$ This paper is organized as follows. ::section-II gives an overview of FT,and mathematical model of windows by using FT. section-III gives an overview of FrFT,and mathematical model of windows by using FRFT. Later conclusive remarks are discussed in section-IV.

P.V.Muralidhar:He is currently working as Associate professor in Electronics and Communication Engineering at Aditya Institute of Technology and management Sciences,Tekkali,Andhra Pradesh, India.(email id:pendyalamurali@yahoo.com)

Venkata.L.N.Sastry.D:He is currently working as Assistant professor in Electronics and Instrumentation Engineering at Aditya Institute of Technology and management Sciences,Tekkali,Andhra Pradesh, India (email id:dvlnsastri@gmail.com)

S.K.Nayak:Reader Department of Electronic Science,BerhampurUniversity,Berhampur,Odisha,India.(email id:sknayakbu@)rediffmail.com)

## 2.Fourier Transform

FT is a mathematical model used for frequency analysis of continuous signals[8]-[9].and it finds applications in Harmonic calculations of the signal spectrum[10].

### 2.1Dirichlet window function

The mathematical analysis of Dirichlet window using FT is carried out in the following section. theMathematical characteristic equation of Dirichlet window is

$$x(t) = 1 \quad for\ |t| < 0.5$$

$= 0$ otherwise ------(1)

The Fourier transform of above equation is

$$X(\omega) = \int_{t_1}^{t_2} x(t)\exp(-j\omega t)\,dt --- (2)$$

Substitute equation-(1) in equation-(2) then, we get

$$X(\omega) = \int_{t_1}^{t_2} 1.\exp(-j\omega t)\,dt --- (3)$$

By solving above equation and applying limits, then

$$X(\omega) = \frac{\exp(-j\omega t_2)}{-j\omega} - \frac{\exp(-j\omega t_1)}{-j\omega} --- (4)$$

Thus equation-(4) can be seen that FT of Dirichlet window function and it's spectral Responses are shown in Fig:1.

### 2.2 Bartlett window function

According to [11] Bartlett window is obtained by convoluting the two Dirichlet windows which is derived above. and it's spectral Responses are shown in Fig:2.

### 2.3Hanning and Hamming Window functions

The expression for generalized Hanning and Hamming window functions are [11]

$$x(t) = \beta + (1-\beta)\cos(2\pi t) \ |t| < 1 --- (5)$$
$$= 0 \text{ otherwise.}$$

For Hanning window function $\beta = 0.5$ and $\beta = 0.54$ for Hamming window.

Then FT of equation-(5) is

$$X = \int_{t_1}^{t_2} (\beta + (1-\beta)\cos(2\pi t)\exp(-j\omega t))\,dt --- (6)$$

By writing $\cos(2\pi t)$ in terms of exponential function[12],then

$$X = \int_{t_1}^{t_2} (\beta + (1-\beta)(\frac{\exp(j2\pi t) - \exp(-j2\pi t)}{2})\exp(-j\omega t))\, dt --- (7)$$

The equation-(7) can be divided into three terms $X_1, X_2, X_3$
Where

$$X_1 = \beta \int_{t_1}^{t_2} \exp(-j\omega t)\, dt --- (8)$$
$$X_2 = \frac{1-\beta}{2} \int_{t_1}^{t_2} \exp(j2\pi t) \exp(-j\omega t)\, dt --- (9)$$
$$X_3 = \frac{1-\beta}{2} \int_{t_1}^{t_2} \exp(-j2\pi t) \exp(-j\omega t)\, dt --- (9)$$

Solving for $X_1$

The expression for $X_1$ is similar to Drichlet window function which was derived in section II.I , i.e

$$X_1 = \beta\left(\frac{\exp(-j\omega t_2)}{-j\omega} - \frac{\exp(-j\omega t_1)}{-j\omega}\right) --- (10)$$

Solving for $X_2$

$$X_2 = \frac{1-\beta}{2} \int_{t_1}^{t_2} \exp(j2\pi t) \exp(-j\omega t)\, dt ----(11)$$

Integrating equation-(11) and applying limits we will get

$$X_2 = \frac{1-\beta}{2}\left(\frac{\exp(j2\pi t_2 - j\omega t_2)}{j2\pi - j\omega} - \frac{\exp(j2\pi t_1 - j\omega t_1)}{j2\pi - j\omega}\right) --- (12)$$

Solving for $X_3$

$$X_3 = \frac{1-\beta}{2} \int_{t_1}^{t_2} \exp(-j2\pi t) \exp(-j\omega t)\, dt ----(13)$$

Integrating equation-(13) and applying limits we will get

$$X_2 = \frac{1-\beta}{2}\left(\frac{\exp(-j2\pi t_2 - j\omega t_2)}{-j2\pi - j\omega} - \frac{\exp(-j2\pi t_1 - j\omega t_1)}{-j2\pi - j\omega}\right) --- (13A)$$

Finally

$$X = X_1 + X_2 + X_3 --- (14)$$

Thus equation-(14) can be seen that FT of Hanning and Hamming window functions and it's spectral Responses are shown in Fig:3 and Fig:4 Respectively.and it's Spectral parameters are Tabulated in Table-1.

## 3.Fractional Fourier Transform

Fractional Fourier Transform widely used in quantum mechanics and quantum optics [13].Fractional Fourier Analysis can obtain the mixed time and frequency components of signals[14].it finds various applications like pattern recognisition with some spatial distortion, Image representation , compression and noise removal in signal processing [15]-[17].FrFT used for Interpretation of sinusoidal signalsand design of Digital FIR Filters[18]-[19].
The continuous –time Fractional Fourier Transform of a signal $\omega(t)$is defined through an interval [3]

$$\omega_\alpha(u) = \int_{-\infty}^{\infty} \omega(t) K_\alpha(t,u) dt --- (15)$$

Where the transform kernel $K_\alpha(t,u)$ of the FRFT is Given by

$$K_\alpha(t,u) = \sqrt{\frac{1-jcot(\alpha)}{2\pi}}\, exp\left[i\left(\frac{t^2+u^2}{2}\right)\cot(\alpha) - iutcosec(\alpha)\right] \quad if\ \alpha\ is\ multiple\ of\ \pi$$
$$= \partial(t-u) \quad if\ \alpha\ is\ multiple\ of\ 2\pi$$
$$= \partial(t+u) \quad if\ \alpha + \pi\ is\ a\ mulpiple\ of\ 2\pi --- (16)$$

Where $\alpha$ indicates rotation of angle of the Transformed signal for FrFT.

### 3.1Dirichlet window function

The mathematical analysis of Dirichlet window using FT is carried out in the following section. the mathematical characteristic equation of Dirichlet window is

$$\omega(t) = 1 \quad for\ |t| < 0.5$$
$$= 0 \quad \text{otherwise} ------(17)$$

Now solving FRFT for equation-(17) yields

Substitute equation-(17) in equation-(15 ) and applying limits results to

$$\omega_\alpha(u) = \int_{t_1}^{t_2} \omega(t) \sqrt{\frac{1-jcot(\alpha)}{2\pi}}\, exp\left[i\left(\frac{t^2+u^2}{2}\right)\cot(\alpha) - iutcosec(\alpha)\right] -----( 18 )$$

Let

$$p = \sqrt{\frac{1-jcot(\alpha)}{2\pi}}\, e^{\frac{ju^2\cot(\alpha)}{2}} ------(19)$$

Then equation-(18) becomes

$$\omega_\alpha(u) = p\int_{t_1}^{t_2} \omega(t) \exp\left(i\frac{t^2}{2}\cot(\alpha) - iutcosec(\alpha)\right) dt ----(20)$$

Substitute equation-(17) in equation-(20) results to

$$\omega_\alpha(u) = p\int_{t_1}^{t_2} 1. \exp\left(i\frac{t^2}{2}\cot(\alpha) - iutcosec(\alpha)\right) dt ----(21)$$

Now applying limits $\int_{t_1}^{t_2} \exp(-i\frac{t^2}{2}\cot(\alpha))$on both sides

$$\int_{t_1}^{t_2} \exp(-i\frac{t^2}{2}\cot(\alpha))\,\omega_\alpha(u) = p\int_{t_1}^{t_2}(\int_{t_1}^{t_2}(\exp(-i\frac{t^2}{2}\cot(\alpha))\exp\left(i\frac{t^2}{2}\cot(\alpha) - iutcosec(\alpha)\right)dt)dt --(22)$$

$$\int_{t_1}^{t_2} \exp(-i\frac{t^2}{2}\cot(\alpha))\,\omega_\alpha(u) = p\int_{t_1}^{t_2}(\int_{t_1}^{t_2}(\exp(-iutcosec(\alpha)))dt)dt ---(23)$$

$$\omega_\alpha(u) = p\frac{\int_{t_1}^{t_2}(\int_{t_1}^{t_2}(\exp(-iutcosec(\alpha)))dt)dt}{\int_{t_1}^{t_2}\exp\left(-i\frac{t^2}{2}\cot(\alpha)\right)} ---(24)$$

Now equation-(24 ) divided into two parts $I_1$ *and* $I_2$ where

$$I_1 = \int_{t_1}^{t_2}(\int_{t_1}^{t_2}(\exp(-iutcosec(\alpha)))dt)dt --(25)$$

$$I_2 = \int_{t_1}^{t_2}\exp\left(-i\frac{t^2}{2}\cot(\alpha)\right)dt ---(26)$$

Now solving for $I_1$

Integrating and applying limits on equation-(25) ,we get

$$I_1 = \int_{t_1}^{t_2}(\frac{\exp(-iut_2cosec(\alpha))}{-iucosec(\alpha)} - \frac{\exp(-iut_1cosec(\alpha))}{-iucosec(\alpha)})dt ---(26)$$

$$I_1 = (\frac{\exp(-iut_2cosec(\alpha))}{-iucosec(\alpha)} - \frac{\exp(-iut_1cosec(\alpha))}{-iucosec(\alpha)})(t_2 - t_1) ---(27)$$

Now solving for $I_2$

$$I_2 = \int_{t_1}^{t_2}\exp\left(-i\frac{t^2}{2}\cot(\alpha)\right)\mathrm{dt}$$

According to [17]

$$\exp\left(-i\frac{t^2}{2}\cot(\alpha)\right) = (1 + i\frac{t^2}{2}\cot(\alpha)) ---(28)$$

Substitute equation-(28) in equation-(26) and integrating we get

$$I_2 = \int_{t_1}^{t_2}((1 + i\frac{t^2}{2}\cot(\alpha))\,dt ---(29)$$

$$I_2 = \int_{t_1}^{t_2}1dt + \frac{icot(\alpha)}{2}\int_{t_1}^{t_2}t^2dt ---(30)$$

$$I_2 = (t_2 - t_1) + \frac{icot(\alpha)}{6}(t_2^3 - t_1^3) — (31)$$

Now substitute equations (27) and (31) in equation (24) we get

$$\omega_\alpha(u) = p.\frac{I_1}{I_2} ---(32)$$

Thus equation-(32) is the FRFT of Dirichliet window.

When substitute $\alpha = a\frac{\pi}{2}$ *where* $a = 1$ in equation-(32) results to generalized Fourier Transform given by equation-(4) [APPENDIX-A]. By adjusting $\alpha$ for different values of a we obtain different spectral responses for Dirichlet window which are shown from figure5 to figure8.and its spectral parameters are tabulated in Table2.

**3.2 Bartlett window function**

According to [16] Bartlett window is obtained by convoluting the two Dirichlet windows which is derived above. and it's spectral responses are shown in figure-9 to figure-13.and its spectral parameters are tabulated in Table3

**3.3Hanning and Hamming window functions**

The expression for generalized Hanning and Hamming window functions are [16]

$$\omega(t) = \beta + (1 - \beta)\cos(2\pi t) \;\; |t| < 1 ---(33)$$
$$= 0 \text{ otherwise.}$$

For Hanning window function $\beta = 0.5$ and $\beta = 0.54$ for Hamming window.

$$\omega_\alpha(u) = \int_{t_1}^{t_2}\omega(t)\sqrt{\frac{1 - jcot(\alpha)}{2\pi}}\,exp\left[i\left(\frac{t^2 + u^2}{2}\right)\cot(\alpha) - iutcosec(\alpha)\right] ---(34)$$

$$p = \sqrt{\frac{1 - jcot(\alpha)}{2\pi}}\,e^{\frac{ju^2\cot(\alpha)}{2}} ------(35)$$

Then equation-(35) becomes

ISSN 2229-5518

$$\omega_\alpha(u) = p\int_{t_1}^{t_2} \omega(t)\exp\left(i\frac{t^2}{2}\cot(\alpha) - iutcosec(\alpha)\right)dt \quad ----(36)$$

Substitute equation-(33) in equation-(36) then

$$\omega_\alpha(u) = p\int_{t_1}^{t_2}\left(\beta + (1-\beta)\cos(2\pi t)\exp\left(i\frac{t^2}{2}\cot(\alpha) - iutcosec(\alpha)\right)\right)dt \; --- (37)$$

According to [17]

$$\cos(2\pi t) = \frac{\exp(i2\pi t) + \exp(-i2\pi t)}{2} --- (38)$$

Substitute equation-(38) in equation-(37) then

$$\omega_\alpha(u) = p.\int_{t_1}^{t_2}\left(\beta + \left(\frac{1-\beta}{2}\right)\exp(i2\pi t) + \left(\frac{1-\beta}{2}\right)\exp(-i2\pi t)\left(\beta + (1-\beta)\cos(2\pi t)\exp\left(i\frac{t^2}{2}\cot(\alpha) - iutcosec(\alpha)\right)\right)\right)dt --- (39)$$

Now equation –( 39) divided into three parts $I_3, I_4$ *and* $I_5$ Where

$$I_3 = \beta\int_{t_1}^{t_2}\left(\exp\left(i\frac{t^2}{2}\cot(\alpha) - iutcosec(\alpha)\right)\right)dt --- (40)$$

The derivation for $I_3$ is similar to derivation of Dirichlet window i.e from equation-(22) to equation-(31)

Thus $I_3 = \beta\frac{I_1}{I_2} --- (41)$

Now solving for $I_4$

$$I_4 = \left(\frac{1-\beta}{2}\right)\int_{t_1}^{t_2}\exp(i2\pi t)\exp\left(i\frac{t^2}{2}\cot(\alpha) - iutcosec(\alpha)\right)dt --- (42)$$

Multiplying $\int_{t_1}^{t_2}\left(\exp(-i\frac{t^2}{2}\cot(\alpha))\right)dt$ on both sides, we get

$$\int_{t_1}^{t_2}\left(\exp(-i\frac{t^2}{2}\cot(\alpha))\right)dt\, I_4 = \frac{1-\beta}{2}\int_{t_1}^{t_2}\left(\int_{t_1}^{t_2}\left(\exp\left(-i\frac{t^2}{2}\cot(\alpha)\right).\exp\left(i\frac{t^2}{2}\cot(\alpha) - iutcosec(\alpha)\right).\exp(i2\pi t)\right)dt\right)dt — (43)$$

$$\int_{t_1}^{t_2}\left(\exp(-i\frac{t^2}{2}\cot(\alpha))\right)dt\, I_4 = \frac{1-\beta}{2}\int_{t_1}^{t_2}\left(\int_{t_1}^{t_2}(\exp(i2\pi t - iutcosec(\alpha)))dt\right)dt --- (44)$$

Now

$$I_4 = \left(\frac{1-\beta}{2}\right)\frac{\int_{t_1}^{t_2}\left(\int_{t_1}^{t_2}(\exp(i2\pi t - iutcosec(\alpha)))dt\right)dt}{\int_{t_1}^{t_2}\left(\exp(-i\frac{t^2}{2}\cot(\alpha))\right)dt} --(45)$$

Again equation-(45) divided into two parts $I_{41}$ *and* $I_{42}$

Where $I_{41} = \int_{t_1}^{t_2}\left(\int_{t_1}^{t_2}(\exp(i2\pi t - iutcosec(\alpha)))dt\right)dt ---(46)$

$I_{42} = \int_{t_1}^{t_2}\left(\exp(-i\frac{t^2}{2}\cot(\alpha))\right)dt — (47)$

Now solving for $I_{41}$

Integrating and applying limits on equation-(46) we get

$$I_{41} = \int_{t_1}^{t_2}\left(\frac{\exp(i2\pi t_2 - iut_2 cosec(\alpha))}{i2\pi - iucosec(\alpha)} - \frac{\exp(i2\pi t_1 - iut_1 cosec(\alpha))}{i2\pi - iucosec(\alpha)}\right)dt — (48)$$

$$I_{41} = \left(\frac{\exp\left(i2\pi t_2 - iut_2 cosec(\alpha)\right)}{i2\pi - iucosec(\alpha)} - \frac{\exp\left(i2\pi t_1 - iut_1 cosec(\alpha)\right)}{i2\pi - iucosec(\alpha)}\right)(t_2 - t_1) --- (49)$$

Now solving for $I_{42}$

substitute equation-(28) in equation –(47) and apply integrating and limits, then we get

$$I_{42} = \int_{t_1}^{t_2} \left(1 + \frac{it^2}{2}\cot(\alpha)\right) dt --- (50)$$

$$I_{42} = \int_{t_1}^{t_2} 1dt + \frac{icot(\alpha)}{2}\int_{t_1}^{t_2} (t^2)dt --- (51)$$

$$I_{42} = (t_2 - t_1) + \frac{icot(\alpha)}{6}(t_2^3 - t_1^3) — (52)$$

Finally substitute equations-(49) and (52) in equation-(45)
We will get $I_4$

$$I_4 = \left(\frac{1-\beta}{2}\right)\frac{I_{41}}{I_{42}} --- (53)$$

Now solving for $I_5$

$$I_5 = (\frac{1-\beta}{2})\int_{t_1}^{t_2} \exp(-i2\pi t)\exp\left(i\frac{t^2}{2}\cot(\alpha) - iutcosec(\alpha)\right) dt --- (54)$$

Multiplying $\int_{t_1}^{t_2}\left(\exp(-i\frac{t^2}{2}\cot(\alpha))\right) dt$ on both sides, we get

$$\int_{t_1}^{t_2}\left(\exp(-i\frac{t^2}{2}\cot(\alpha))\right) dt\, I_5 = \frac{1-\beta}{2}\int_{t_1}^{t_2}\left(\int_{t_1}^{t_2}\left(\exp\left(-i\frac{t^2}{2}\cot(\alpha)\right).\exp\left(i\frac{t^2}{2}\cot(\alpha) - iutcosec(\alpha)\right).\exp(-i2\pi t)\right) dt\right) dt — (55)$$

$$\int_{t_1}^{t_2}\left(\exp(-i\frac{t^2}{2}\cot(\alpha))\right) dt\, I_5 = \frac{1-\beta}{2}\int_{t_1}^{t_2}\left(\int_{t_1}^{t_2}(\exp(-i2\pi t - iutcosec(\alpha)))dt\right) dt --- 56)$$

Now

$$I_5 = (\frac{1-\beta}{2})\frac{\int_{t_1}^{t_2}\left(\int_{t_1}^{t_2}(\exp(-i2\pi t - iutcosec(\alpha)))dt\right) dt}{\int_{t_1}^{t_2}\left(\exp(-i\frac{t^2}{2}\cot(\alpha))\right) dt} --(57)$$

Again equation-(57) divided into two parts $I_{51}$ *and* $I_{52}$

Where

$$I_{51} = \int_{t_1}^{t_2}\left(\int_{t_1}^{t_2}(\exp(-i2\pi t - iutcosec(\alpha)))dt\right) dt ---- (58)$$

$$I_{52} = \int_{t_1}^{t_2}\left(\exp(-i\frac{t^2}{2}\cot(\alpha))\right) dt — (59)$$

Now solving for $I_{51}$

Integrating and applying limits on equation-(58) we get

$$I_{51} = \int_{t_1}^{t_2}\left(\frac{\exp(-i2\pi t_2 - iut_2cosec(\alpha))}{-i2\pi - iucosec(\alpha)} - \frac{\exp(-i2\pi t_1 - iut_1cosec(\alpha))}{-i2\pi - iucosec(\alpha)}\right) dt — (60)$$

$$I_{51} = \left(\frac{\exp\left(-i2\pi t_2 - iut_2cosec(\alpha)\right)}{-i2\pi - iucosec(\alpha)} - \frac{\exp\left(-i2\pi t_1 - iut_1cosec(\alpha)\right)}{-i2\pi - iucosec(\alpha)}\right)(t_2 - t_1) --- (61)$$

Now solving for $I_{52}$

substitute equation-(28) in equation –(59) and apply integrating and limits, then we get

$$I_{52} = \int_{t_1}^{t_2}\left(1 + \frac{it^2}{2}\cot(\alpha)\right) dt --- (62)$$

$$I_{52} = \int_{t_1}^{t_2} 1dt + \frac{icot(\alpha)}{2}\int_{t_1}^{t_2}(t^2)dt --- (63)$$

$$I_{52} = (t_2 - t_1) + \frac{icot(\alpha)}{6}(t_2^3 - t_1^3) — (64)$$

Finally substitute equations-(61) and (64) in equation-(54)
We will get $I_4$

$$I_5 = \left(\frac{1-\beta}{2}\right)\frac{I_{51}}{I_{52}} --- (65)$$

Now substitute equations –(41),(53) and (65) in equation-(39),we will get

$$\omega_\alpha(u) = p.(I_3 + I_4 + I_5) --- (66)$$

Thus equation-(66) is the FRFT equation for Hanning and Hamming windows. and if $\alpha = \frac{a\pi}{2}$ *where* $a = 1$ then the equation-(66) becomes hanning and hamming windows based on conventional Fourier transform given by equation-(14)[APPENDIX-A].And Hanning and Hamming windows spectral responses are shown from figure-(14) to figure-(25) for different values of $\alpha$.and their spectral parameters are given in table-4 and table-5 Respectively.

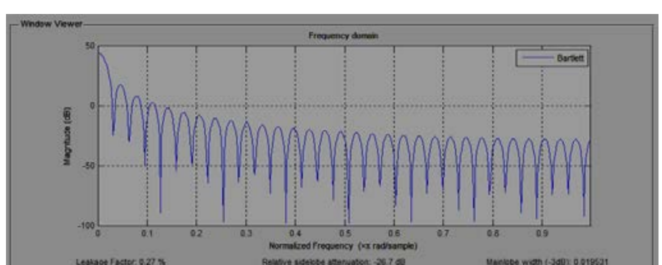

Fig1:Spectral parameters of FTbased Dirichlet window

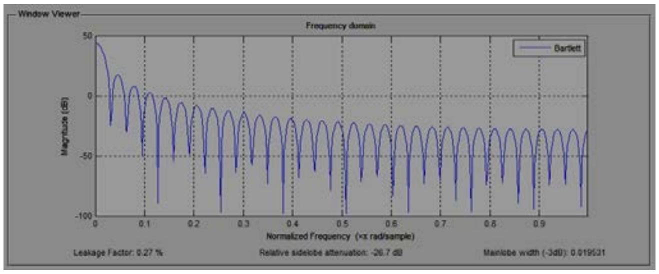


Fig2:Spectral parameters of FT based Bartlett window

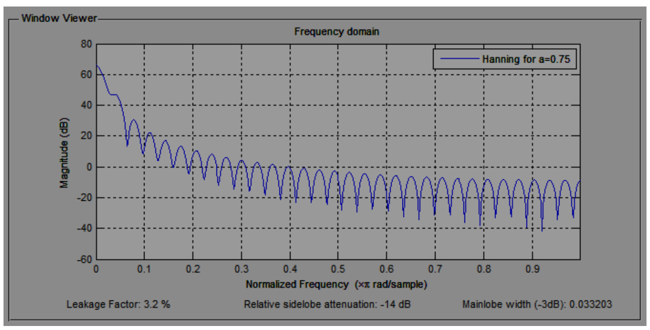


Fig3:Spectral parameters of FT based Hannning window

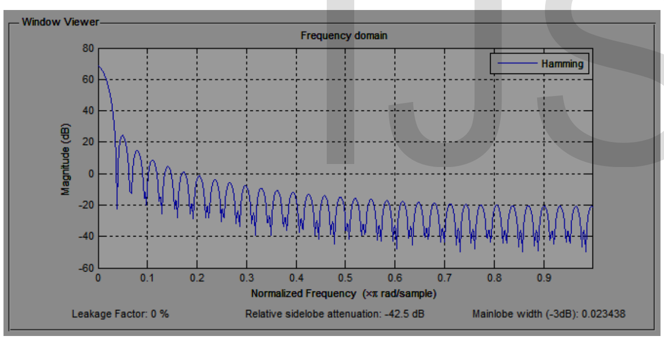


Fig4:Spectral parameters of FT based Hamming window

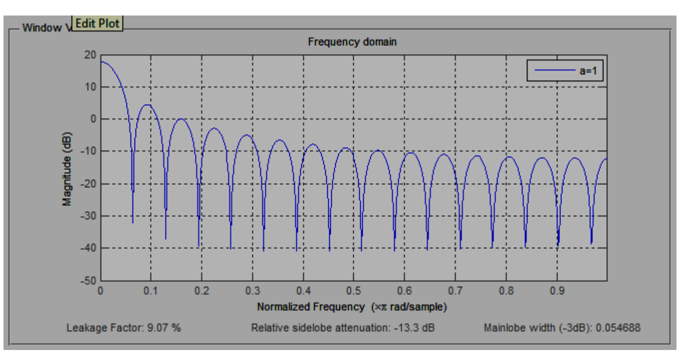


Fig5:Spectral response of FrFT based Dirichlet window for a=1

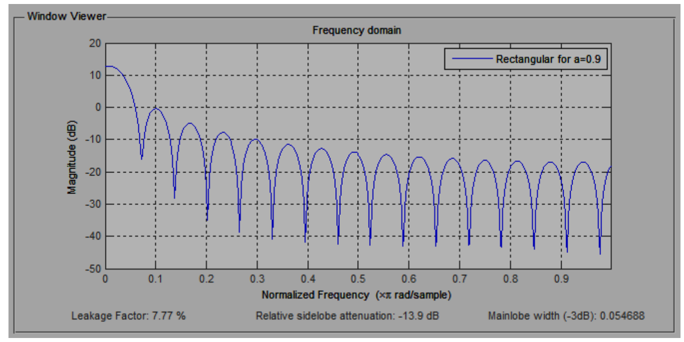


Fig6:Spectral response of FrFT based Dirichlet window for a=0.9

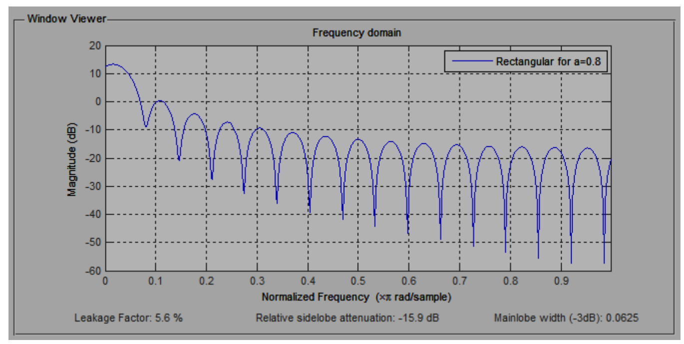


Fig7:Spectral response of FrFT based Dirichlet window for a=0.8



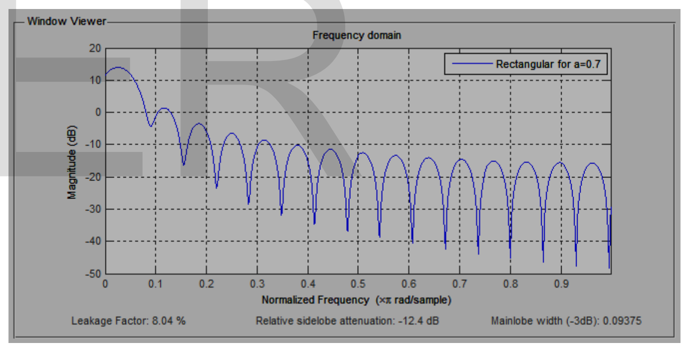


Fig8:Spectral response of FrFT based Dirichlet window for a=0.7

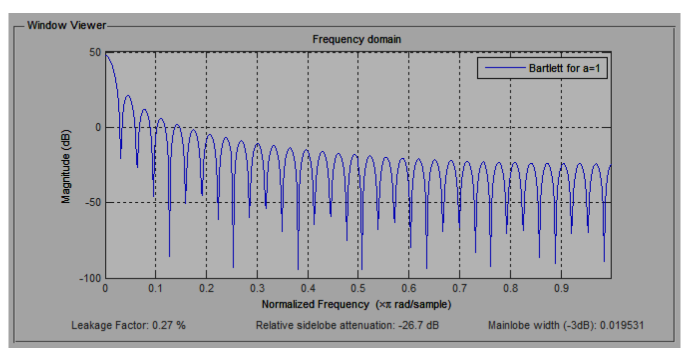


Fig9:Spectral response of FrFT based Bartlett window for a=1

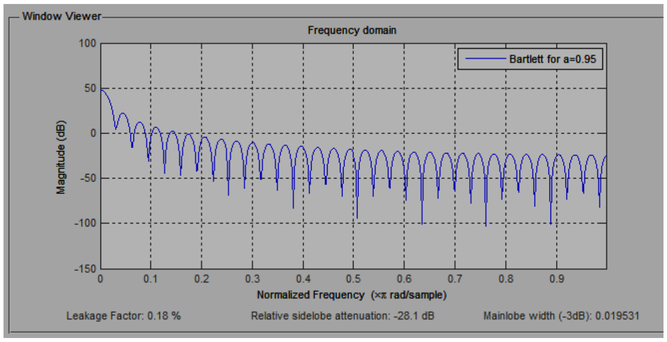


Fig10:Spectral response of FrFT based Bartlett window for a=0.95

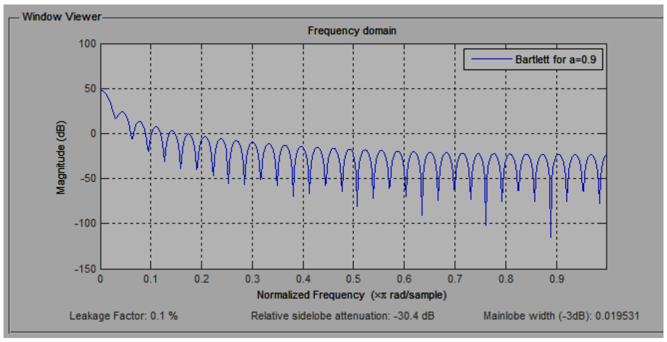


Fig11:Spectral response of FrFT based Bartlett window for a=0.9

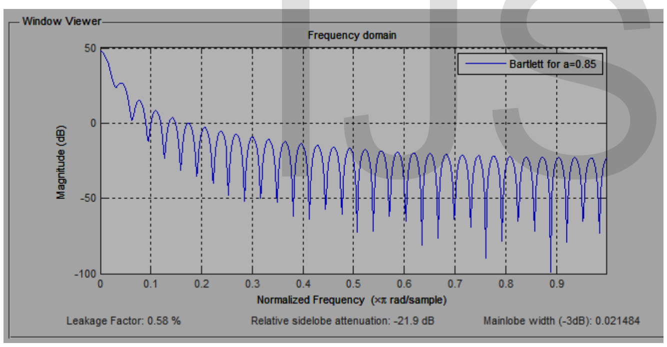


Fig12:Spectral response of FrFT based Bartlett window for a=0.85

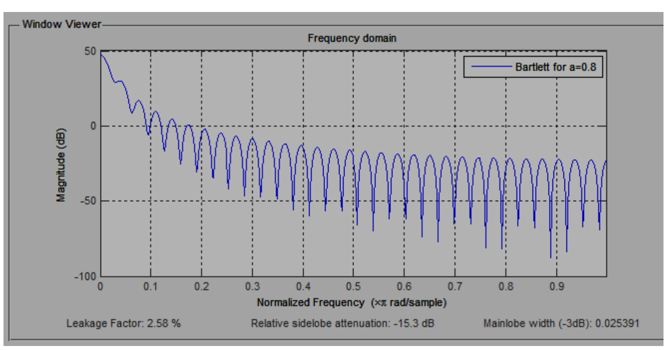


Fig13:Spectral response of FrFT based Bartlett window for a=0.8

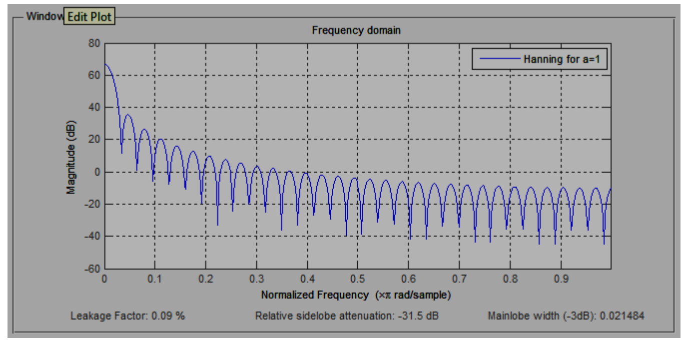


Fig14:Spectral response of FrFT based Hanning window for a=1

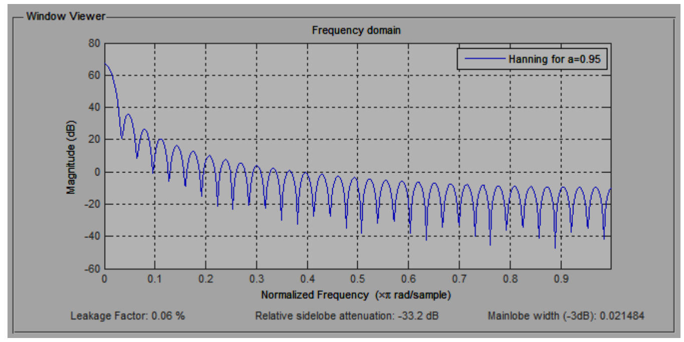


Fig15:Spectral response of FrFT based Hanning window a=0.95

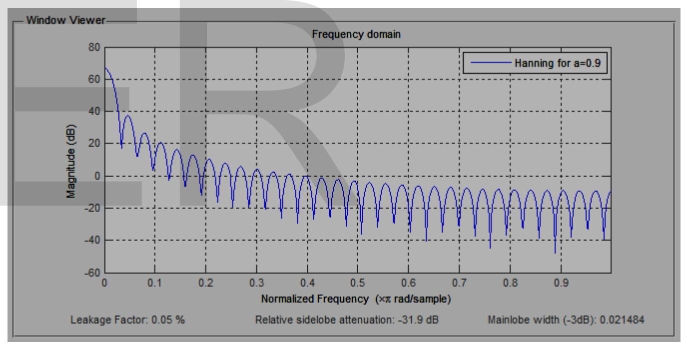


Fig16:Spectral response of FrFT based Hanning window a=0.9

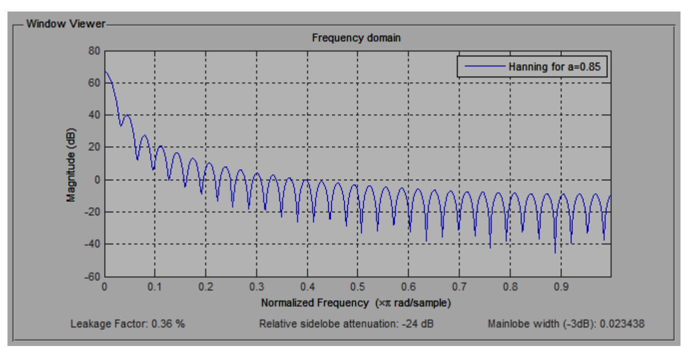


Fig17:Spectral response of FrFT based Hanning window a=0.85

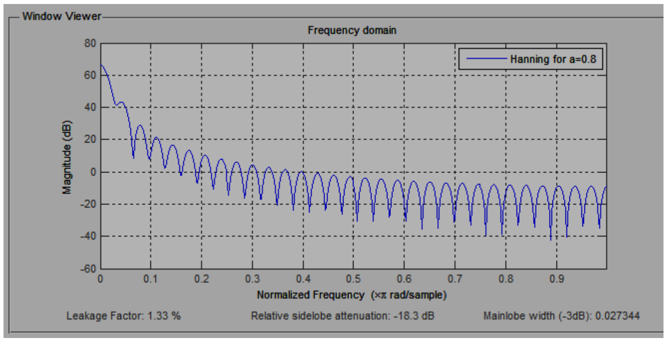


Fig18:Spectral response of FrFT based Hanning window a=0.8

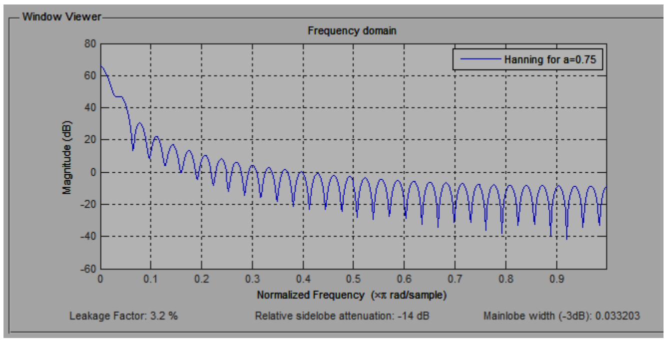


Fig19:Spectral response of FrFT based Hanning window a=0.75



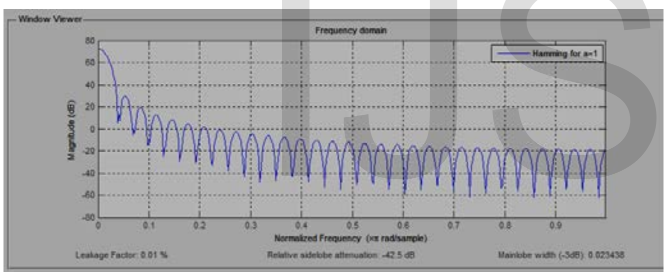


Fig20:Spectral response of FrFT based Hamming window a=1

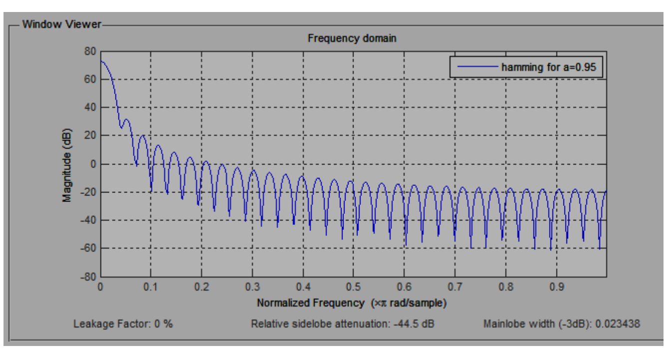


Fig21:Spectral response of FrFT based Hamming window a=0.95

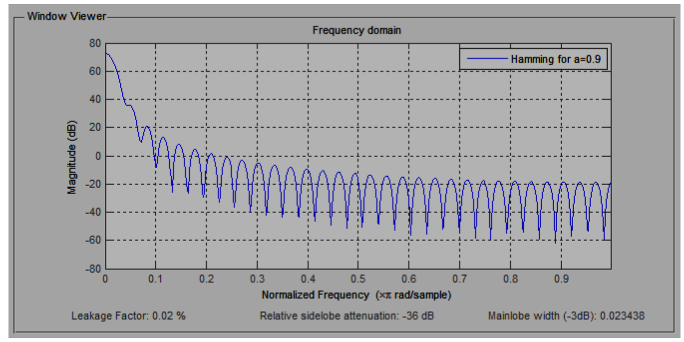


Fig22:Spectral response of FrFT based Hamming window a=0.9

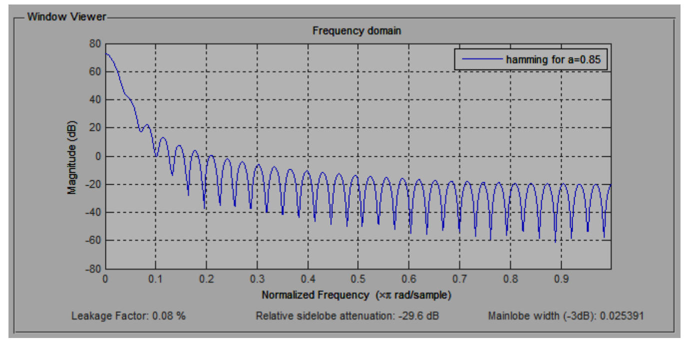


Fig23:Spectral response of FrFT based Hamming window a=0.85

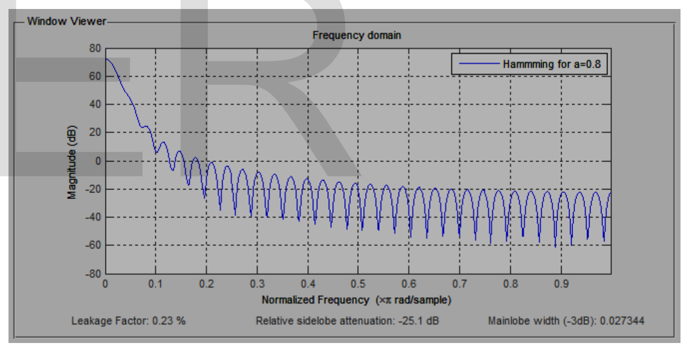


Fig24:Spectral response of FrFT based Hamming window a=0.8

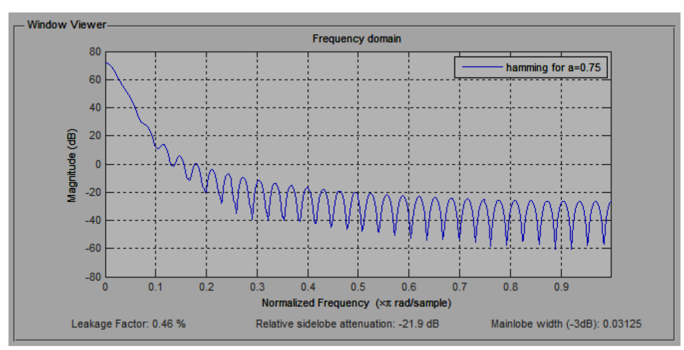


Fig25:Spectral response of FrFT based Hamming window a=0.75

Table1:Spectral Parameters of FT Based Windows

| window | HBW in dB | MSLA in dB | SLFOR |
|---|---|---|---|

| | | | in dB |
|---|---|---|---|
| Dirichlet | 0.0546 | -13.3 | -16.41 |
| Bartlett | 0.0195 | -27.3 | -44.81 |
| Hanning | 0.0195 | -31.5 | -44.86 |
| Hamming | 0.0234 | -42.5 | -45.14 |

Table2:Spectral Parameters of FrFT Based Dirichlet window for variations in a

| a | HBW in dB | MSLA in dB | SLFOR in dB |
|---|---|---|---|
| 1 | 0.0546 | -13.3 | -16.58 |
| 0.9 | 0.0546 | -13.9 | -17.34 |
| 0.8 | 0.0625 | -15.9 | -16.72 |
| 0.7 | 0.0937 | -12.4 | -17.13 |

Table3:Spectral Parameters of FrFT Based Bartlett window for variations in a

| a | HBW in dB | MSLA in dB | SLFOR in dB |
|---|---|---|---|
| 1 | 0.0195 | -26.7 | -45.29 |
| 0.95 | 0.0195 | -28.1 | -45.85 |
| 0.9 | 0.0195 | -30.4 | -48.25 |
| 0.85 | 0.0214 | -21.9 | -49.09 |
| 0.8 | 0.0253 | -15.3 | -52.61 |

Table4:Spectral Parameters of FrFT Based Hanning window for variations in a

| a | HBW in dB | MSLA in dB | SLFOR in dB |
|---|---|---|---|
| 1 | 0.0214 | -31.5 | -45.37 |
| 0.95 | 0.0214 | -33.2 | -45.27 |
| 0.9 | 0.0214 | -31.9 | -46.77 |
| 0.85 | 0.0214 | -24.0 | -49.16 |
| 0.8 | 0.0273 | -18.3 | -52.25 |
| 0.75 | 0.0332 | -14.0 | -55.84 |

Table5:Spectral Parameters of FrFT Based Hamming window for variations in a

| a | HBW in dB | MSLA in dB | SLFOR in dB |
|---|---|---|---|
| 1 | 0.0234 | -42.5 | -48.73 |
| 0.95 | 0.0234 | -44.5 | -50.06 |
| 0.9 | 0.0234 | -36.0 | -54.47 |
| 0.85 | 0.0253 | -29.6 | -42.2 |
| 0.8 | 0.0273 | -25.1 | -46.95 |
| 0.75 | 0.0312 | -21.9 | -40.05 |

**4: Conclusion:**

From the study of Exponential derivation of FrFT for Drichletwindow,Barlettwindow,Hanning window and Hmmingwindow, The controllability of window parameters like HBW,MSLA and SLFOR is possible..i.e. for Dirichlet window The MSLA increases from -13.3 dB to -13.9 dB for a= 0.95 and decreases for other values of a[Table-2].similarly for Bartlett window the MSLA increases from -26.7dB to -28.1 dB and -30.4dB for a= 0.95 and 0.9 Respectively[Table-3].For Hanning window The MSLA increases from -31.5 dB to -33.2 dB for a= 0.95 and decreases for other values of a [table-4], Finally for Hamming window The MSLA increases from -42.5 dB to -44.5 dB for a= 0.95 and decreases for other values of a[Table-5]), and also one of the property of FrFT is Generalization of FrFTto FT i.e. when $\alpha = \frac{a\pi}{2}$ where a=1;then The FrFT should equals to FT.(By comparing the Table-1 with Tables-2,3,4,5. Respectively for a=1)This proposed Mathematical derivation of FrFTfulfillsthe Property of FT [ APPENDIX-A].

**5:References**

[1].Fedric J Harris-“On the use of windows for harmonic Analysis with the Discrete Fourier Transform.”-Proceedings by IEEE,vol-66,No:1January-1978.

[2].AlokJain,RajivSaxena,S.C.Saxena.” Anti-Image FIR Filters for large Interpolation factors-signal processing 86(2006) 3240-3245.-science direct-Elsevier publications.

[3]. Sanjay Kumar,KulbirSingh, RajivSaxena-“Analysis of Dirichlet and Generalized Hamming window functions in the Fractional Fourier Transform Domains”.-signal processing 91(2011) 600.606-science direct-Elsevier publications.

[4]. AshutoshDatar,AlokJain,Pramod Chandra Sharma.”Design and performance Analysis of adjustable window functions based cosine modulated filter banks”-Digital Signal Processing 23(2013) 412-417-Elsevier publications.

[5] .Kemal Avsi,ArifNacaroglu-“cosh window family and it’s applications to FIR filter design”.-Int.J.Electron.Commun.-(AEU) 63(2009)907-916.

[6] IEEE Transactions on Accoustics,Speech,and Signal Processing,vol-37,No:2,February 1989.

[7].P.V.Muralidhar,D.Nataraj,V.Lokeshraju,Dr.S.K.Nayak-“Implementation of different FIR High pass filters using FractionalKaiserwindow”,vol:2,DOI:10.1109/ICSPS.2010.5555740,Publishing year:2010,pages(V2-651)-(V2-655).

[8]ArtyumM.Grigoryan-“FourierTransform representation by Frequency-Time Wavelets”-IEEE Transactions on signal processing,vol-53,No:7,,July-2005

[9] John G. Proakis,DimitrisG.Manolakis-“Digital signal processing,principles,Algorithms and Applications”-Pearson prentice Hall.ISBN:978-81-317-1000-5.

[10] Miguel Angel Platas-Garzan,JoseAntoniodela o Serena-“Dynamic harmonic Analysis Through Taylor-Fourier Transform”-IEEE Transactions on Instrumentation and Measurment,Vol:60,No:3,march-2011

[11] L.R. Rabiner, B. Gold, Theory and Applications of Digital Signal Processing, Prentice-Hall, 1975.

[12] Dr.B.S.Grewal-“Engineering Mathematics”,ISBN NO:81-7409-2196,25th edition, October 2006,Khanna Publications

[13]Namias.V-“The FrFT and Time Frequency representation”-J.Inst.Math.Applications,Vol:25,pp.241-265,1980.

[14]Soo-Chang pei,Mon Hung Yeh,TzyyliangLuo-

"Fractional Fourier Series Expansion for Finite signals and Dual Extension to Discrete-Time Fractional Fourier Transform"-IEEE Transactions on signal processing,vol:47,No:10,1999.

[15] Soo-Chang pei – "Two-Dimensional Affine generalized Fractional Fourier Transform"-IEEE Transactions on signal processing,vol: 49,No:4,April 2009

[16] I.SamilYetik,M.AlperKutay, Haldun .M.Ozaktas- "Image representation and compression with the Fractional Fourier Transform", Optics communications 197(2001),275-278,Elsevier publications.

[17]V.AshokNarayanan,K.M.M.Prabhu-"The FrFT:Theory,Implementation and error Analysis", Microprocessor and Microsystems 27(2003),511-521,Elsevier Publications.

[18]Ran Tao, Kiang-Yi meng,Yue Wang-"Image encryption with multi orders Fractional Fourier Transforms",IEEE transactions on Information Forensics and Security,vol:05,No:04,December 2010

.[19].P.V.Muralidhar,A.S.SrnivasaRao,Dr.S.K.Nayak- "Spectral Interpretation of Sinusoidal wave using Fractional Fourier Transform based FIR window functions",vol:04,N0:6,PP:652-657,International Review on compuers and software, November 2009.

**6:Author's Profile:**

First Other:P.V.Muralidhaar obtained M. Tech from JNTU, Hyderabad, pursuing PhD form Berhampur University. He is having an experience more than 10 years and also having more number of both national and international journals, conferences. His area of interest is signal processing,presently working with AITAM , Tekkali, Srikakulam, A.P



Second Author:Venkata L.N.. Sastry.D,Obtainedbachelors degree under JNTU Hyderabad and M.Tech from JNTU Kaikinada. His areas of interest are adaptive PID controllers, fractional PID controllers used in various batch processes and flow control loops and digital PID controllers and design of digital filters by windows.

Third Author: Dr.S.K.Nayak,completedM.Tech from IISC Bangalore, and PhD from Berhampur University.Presently he working as a Reader in Electronics Sciences Dept. Berhampur University,Odissa.His research areas are Embedded Systems and Micro controllers,Digital Signal Processing.

## APPENDIX-A

### I.Dirichlet window function

The Mathematical Equation for FrFT basedDirichlet window is

$$\omega_\alpha(u) = p.\frac{I_1}{I_2}$$

Where

$$p = \sqrt{\frac{1 - jcot(\alpha)}{2\pi}}\, e^{\frac{ju^2\cot(\alpha)}{2}} --- (67)$$

ISSN 2229-5518

$$I_1 = (\frac{\exp(-iut_2 cosec(\alpha))}{-iucosec(\alpha)} - \frac{\exp(-iut_1 cosec(\alpha))}{-iucosec(\alpha)})(t_2 - t_1) --- (68)$$

$$I_2 = (t_2 - t_1) + \frac{icot(\alpha)}{6}(t_2^3 - t_1^3) --- (69)$$

Substitute $\alpha = \frac{\pi}{2}$ in equation-(67),it becomes

$$p = \sqrt{\frac{1}{2\pi}} --- (70)$$

Substitute $\alpha = \frac{\pi}{2}$ in equation-(68),it becomes

$$I_1 = \left(\frac{\exp(-iut_2)}{-iu} - \frac{\exp(-iut_1)}{-iu}\right)(t_2 - t_1) --- (71)$$

Substitute $\alpha = \frac{\pi}{2}$ in equation-(69),it becomes

$$I_2 = t_2 - t_1 --- (72)$$

$$\therefore \omega_\alpha(u) = \sqrt{\frac{1}{2\pi}}\left[(t_2 - t_1)\left(\frac{\exp(-iut_2)}{-iu} - \frac{\exp(-iut_1)}{-iu}\right)\right] --- (73))$$

The Equation-(73) is the FT based Dirichlet window shown in equation-(4) of given in section-II.ThusFrFT is the generalized FT at $\alpha = \frac{\pi}{2}$.

**II.Hanning and Hamming window functions**

$$\omega_\alpha(u) = p.(I_3 + I_4 + I_5) --- (74)$$

Where

$$p = \sqrt{\frac{1 - jcot(\alpha)}{2\pi}} e^{\frac{ju^2 \cot(\alpha)}{2}}$$

The equation for $I_3$ is similar to Dirichlet window i.e. equation-(73)

$$\omega_\alpha(u) = \beta\left(\sqrt{\frac{1}{2\pi}}(t_2 - t_1)\left(\frac{\exp(-iut_2)}{-iu} - \frac{\exp(-iut_1)}{-iu}\right)\right) — (75)$$

$$I_4 = \left(\frac{1-\beta}{2}\right)\frac{I_{41}}{I_{42}} --- (76)$$

Where

$$I_{41} = \left(\frac{\exp(i2\pi t_2 - iut_2 cosec(\alpha))}{i2\pi - iucosec(\alpha)} - \frac{\exp(i2\pi t_1 - iut_1 cosec(\alpha))}{i2\pi - iucosec(\alpha)}\right)(t_2 - t_1) --- (77)$$

Substitute $\alpha = \frac{\pi}{2}$ in equation-(77),it becomes

$$I_{41} = \frac{\exp(i2\pi t_2)}{i2\pi} - \frac{\exp(i2\pi t_1)}{i2\pi}(t_2 - t_1) --(78)$$

$$I_{42} = (t_2 - t_1) + \frac{icot(\alpha)}{6}(t_2^3 - t_1^3) — (79)$$

Substitute $\alpha = \frac{\pi}{2}$ in equation-(79),it becomes

$$I_{42} = (t_2 - t_1) --- (80)$$

$$\therefore I_4 = \left(\frac{1-\beta}{2}\right)\sqrt{\frac{1}{2\pi}}[\left(\frac{\exp(i2\pi t_2)}{i2\pi} - \frac{\exp(i2\pi t_1)}{i2\pi}\right)] — (81)$$

Similarly

$$\therefore I_5 = \left(\frac{1-\beta}{2}\right)\sqrt{\frac{1}{2\pi}}[\left(\frac{\exp(-i2\pi t_2)}{-i2\pi} - \frac{\exp(-i2\pi t_1)}{-i2\pi}\right)] — (82)$$

$$\therefore \omega_\alpha(u) = \beta\sqrt{\frac{1}{2\pi}}[\left((t_2 - t_1)\left(\frac{\exp(-iut_2)}{-iu} - \frac{\exp(-iut_1)}{-iu}\right)\right) + \left(\frac{1-\beta}{2}\right)(\frac{\exp(i2\pi t_2)}{i2\pi} - \frac{\exp(i2\pi t_1)}{i2\pi}) + \left(\frac{1-\beta}{2}\right)(\frac{\exp(-i2\pi t_2)}{-i2\pi} - \frac{\exp(-i2\pi t_1)}{-i2\pi})] --(83)$$

ISSN 2229-5518

The Equation-(83) is the FT based Hanning and Hamming windows shown in equation-(14) of given in section-II.ThusFrFT is the generalized FT at $\alpha = \frac{\pi}{2}$.

Hence it is proved that fractional Fourier Transform is the generalized Fourier Transform for $\alpha = \frac{\pi}{2}$.