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Approximating Positive Solutions of Non-Linear Two Point Functional Boundary Value of Second Order Differential Equations

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Abstract - We prove the existence as well as approximations of the positive solutions for a boundary value problem of second order non-linear quadratic differential equation. An algorithm for the solutions is developed and it is shown that the sequence of successive approximations converges monotonically to the positive solution related quadratic differential equation.

Key words: Positive Solution, Quadratic differential equations, Monotonicity conditions, Successive apporoximation etc.

1. INTRODUCTION

In given closed and bounded interval $J=[a,b], a<b$ of real numbers R consider the non-linear two point functional boundary value problem (in short FBVP) of second order different equation

$$\left[\frac{x(t)}{f\left(t,x\left(u(t)\right)\right)}\right]^n = g\left(t,x\left(\sigma(t)\right),x'\left(n(t)\right)\right)$$

at each $t\in J$ (1)

$$x(a)=x'(b)=0$$

Where $f: J\times R\to R-\{0\}$,

$g: J\times R\times R\to R$

are continuous functions and

$\mu,\sigma,\eta: J\to J$

Converting this equation in first order

$$u''(t)=f(u)$$

for $u=\dfrac{x(t)}{f\left(t,x\left(u(t)\right)\right)}$ we substitute $u'(t)=v$

equation (1) becomes

$$\frac{d}{dt}\left[\frac{v(t)}{f\left(t,x\left(u(t)\right)\right)}\right]+\lambda\left[\frac{v(t)}{f\left(t,x\left(u(t)\right)\right)}\right] = g\left(t,x\left(\sigma(t)\right),x'\left(u(t)\right)\right), t\in J \qquad (1.1)$$

(This information is optional; change it according to your need.)

Assuming $x(t_0)=x_0\in R_+$

Which is initial value problem of first order non-linear quadratic differential equation (QDE).

Assuming that $f(0)=0$

$$uf(u)>0 \text{ for } u=\neq 0$$

for $\lambda\in R, \lambda>0$ where f and g are functions.

By a solution of the QDE (1.1) we mean a function $x\in AC'(J,R)$ that satisfies the equation (1.1) where $AC'(J,R)$ is the space of continuous real-valued functions defined on J.

The QDE (1.1) with $\lambda=0$ is well-known in the literature and is a hybrid differential equation with a quadratic delay of second type. Such differential equations can be tackled with the use of hybrid fixed point theory.

The special case of QDE (1.1) has been discussed at length for existence as well as other aspects of the solution under some strong Lipschitz and compactness type conditions. Very recently, the study of approximation of the solutions for the delay differential equation is initiated in Dhage.

Therefore it is of interest and new to discuss the approximation of solutions for the QDE (1.1) along the similar lines.

This is the main motivation of the present paper and it is proved that the existence of the solutions may be proved via an algorithm based on successive approximations under weaker partially continuity and partially compactness type

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conditions.

2. AUXILIARY RESULTS

Let P denote a partially ordered real normed linear space with an order relation $\leq$ and the norm $\|\cdot\|$. It is known that P is regular if $\{x_n\}_{n\in N}$ is a non-decreasing (resp. non-increasing) sequence in P such that $x_n \to x$ as $n \to \infty$, then $x_n \leq x(resp. x_n \geq x)$ for all $n \in N$. Clearly, the partially ordered Banach space $AC'(J,R)$ is regular and the conditions gives the regularity of any partially ordered normed linear space P may be found in Nieto and Lopez and Heikkila and Lakshmikantham and the reference therein.

We need the following definitions

Definition 2.1 :- A mapping $T:P\to P$ is called isotone or non-decreasing if it preserves the order relation $\leq$, i.e., if $x\leq y$ implies $Tx \leq Ty$ for all $x,y\in P$.

Definition 2.2 :- A mapping $T:P\to P$ called partially continuous at a point $a\in E$ if for $\in>0$ there exists $a\,\delta>0$ such that $\|Tx-Ta\|<\in$ whenever x is comparable to α and $\|x-\alpha\|\delta$. T is called partially continuous on P if it is continuous at every points of it. It is clear that if T is partially continuous on P, then it is continuous on every chain C contained in P

Definition 2.3 :- A mapping $T:P\to P$ is called partially bounded if $T(C)$ is bounded for every chain C in P. T is called uniformly bounded if all chains $T(C)$ in P are bounded by unique constant.

T is called bonded if *T(P)* is bounded subsets of *P*.

Definition 2.4 :- A mapping $T:P\to P$ is called partially compact if *T(C)* is a relatively compact subset of P for all totally ordered sets or chains C in P. T is called uniformly partially compact if *T(C)* is a uniformly partially bounded and partially compact on P. T is called partially totally bounded if for any totally ordered and bounded subset C of P, *T(C)* is a relatively compact subset of P. If T is partially continuous and partially totally bounded, then it is called partially completely continuous on P.

Definition 2.5 :- The order relation $\leq$ and the metric d on a non-empty set P are said to be compatible if $\{x_n\}_{n\in N}$ is a monotone, i.e., monotone non-decreasing or monotone non-increasing sequence in P and if a subsequence $\{x_n\}_{n\in N}$ of $\{x_n\}_{n\in N}$ converges to x implies that the whole sequence $\{x_n\}_{n\in N}$ converges to x. Similarly, given a partially ordered normed linear space $(P,\leq,\|\cdot\|)$, the order relation $\leq$ and the norm $\|\cdot\|$ are said to be compatible if $\leq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.

Definition2.6 :- An upper semi-continuous and non-decreasing function $\psi:R_+\to R_+$ is called a *D-function* provided $\psi(0)=0$. Let $(P,\leq,\|\cdot\|)$ be a partially ordered equation normed linear space. A mapping $T:P\to P$ is called partially nonlinear D – Lipschitz if there exists a *D-function* $\psi:R_+\to R_+$ such that

$$\|Tx-Ty\|\leq\psi(\|x-y\|) \qquad (2.1)$$

For all comparable elements $x,y\in P$. If $\psi(r)=kr, k>0$, Then T is called a partially Lipschitz with a Lipschitzk.

Let $(P,\leq,\|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$P^+=\{x\in p \,|\, x\geq 0\}$ And

$$E=\{p^+\subset p \mid pq\in p^+, \forall\, p,q\in p^+\} \qquad (2.2)$$

Where 0 is zero element of P. The elements of E are called positive vector in P.

Lemma 2.1 :- If $p_1,p_2,q_1,q_2\in E$ are such that $p_1\leq q_1$ and $p_2\leq q_2$ then $p_1p_2\leq q_1q_2$.

Definition 2.7 :- An operator $T:P\to P$ is said to be positive if the range $R(T)$ of T is such that $R(T)\subseteq K$.

Theorem 2.1 :- Let $(p,\leq,\|\cdot\|)$ be a regular partially ordered complete normed linear algebra such that the order relation $\leq$ and the norm $\|\cdot\|$ are compatible. Let $A,B:P\to K$ be two non-decreasing operators such that

(i) *A* is partially bounded and partially non-linear Lipschitz with *D-function* ψ_A

(ii) B is partially continuous and uniformly partially compact, and

(iii) $M\psi_A(r)<r, r>0$,where $M=\sup\{\|B(C)\|:C\in P_{ch}\}$ and

(iv) There exists an element $x_0\in X$ such that $x_0<Ax_0Bx_0$ or $x_0>Ax_0Bx_0$

Then the operator equation

$$AxBx=x \qquad (2.3)$$

has a positive solution x in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = Ax_n Bx_n$, $n = 0.1, 2, 3, ...$ converges monotonically to x.

3. MAIN RESULTS

The QDE(1.1) is considered in the function space $AC'(J, R)$ of continuous real valued function defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $AC'(J, R)$ by $\|x\| = \sup_{t\in J} |x(t)| \,\&\, x \leq y \Leftrightarrow x(t) \leq y(t)$ for $t \in J$ respectively. Clearly, $AC'(J, R)$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t the above partially order relation $\leq$. Clearly, the space $AC'(J, R)$ with usual order relation $\leq$ and the supremum norm $\|\cdot\|$ are compatible. It is known that the partially ordered Banach space $AC'(J, R)$ has some properties w.r.t. the above order relation in it.

We need the following definitions.

Definition 3.1:- A function $u \in AC'(J, R)$ is said to be a lower solution of the QDE (1.1) if it satisfies

$$\frac{d}{dt}\left[\frac{v(t)}{f\left(t, x\left(u(t)\right)\right)}\right] + \lambda\left[\frac{v(t)}{f\left(t, x\left(u(t)\right)\right)}\right]$$

$$\leq g\left(t, x\left(\sigma(t)\right), x'\left(u(t)\right)\right)$$

$$x(t_0) = x_0, \forall \;\; t \in J$$

Similarly a function $v \in AC'(J, R)$ is said to be QDE (1.1) if it satisfies the above inequality in greater than sign.

We consider the following set of assumptions in what follows :

(B_0) The map $x \to \dfrac{x}{f(tx)}$ is injection for each $t \in J$.

(B_1) f defines a function $f : J \times R \to R_+$.

(B_2) There exists a constant $M_f > 0$ such that $|f(t, x)| \leq M_f$ for all $t \in J$ and $x \in R$.

(B_3) There exists a D-function ϕ such that,

$$0 \leq f(t, x) - f(t, y) \leq \phi(x - y),$$

for all $t \in J$ and $x, y \in R, x \geq y$.

(H_1) g defines a function $g : \mathrm{J} \times \mathrm{R} \to \mathrm{R}_+$.

(H_2) There exists a constant $M_g > 0$ such that $|g(t, x)| \leq M_g$ for all $t \in J$ and $x \in R$.

(H_3) $g(t, x)$ is non-decreasing in x for all $t \in J$.

(H_4) The QDE (1.1) has a lower solution $\in AC'(J, R)$.

Remark 3.1 :- The hypothesis (B_0) holds in particular if the function $x \to \dfrac{x}{f(t, x)}$ is increasing for each $t \in J$.

Lemma 3.1 :- Suppose that hypothesis (B_0) holds. Then a function $x \in AC'(J, R)$ is a solution of the QDE (1.1) if and only if it is a solution of the non-linear quadratic integral equation,

$$x(t) = \left[f\left(t, x\left(\mu(t)\right)\right)\right]\left(\frac{ce^{-\lambda t}}{f\left(t_0, x\left(\mu(t_0)\right)\right)} + \int_{t_0}^{t} e^{-\lambda(t-s)} f\left(s, x(s)\right) ds\right) \quad (3.1)$$

for all $t \in J$, where $c = x_0 e^{\lambda t_0}$

Theorem 3.1 :- Assume that hypotheses $(B_0)-(B_3)$ and $(H_1)-(H_4)$ hold. Furthermore, assume that

$$\left(\left|\frac{x_0}{f\left(tx\left(\mu(t_0)\right)\right)}\right| + M_g a\right)\phi(r) < r, r > 0 \quad (3.2)$$

then the QDE (1.1) has a positive solution x defined on J and the sequence $\{x_n\}$, $n = 0, 1, 2, 3, ...$ Of successive approximations defined by

$$x_{n+1}(t) = [f\left(t, \left(x_n \mu(t)\right)\right)]\left[\frac{ce^{\lambda t}}{f\left(t_0, x\left(\mu(t_0)\right)\right)} + \int_{t_0}^{t} e^{-\lambda(t-s)} g\left(s, x_n\left(\sigma(s)\right), x_n'\left(n(s)\right)\right)\right],$$

$$t \in J \quad (3.3)$$

Where $x_0 = u$, converges monotonically to x.

Proof :- Set $P = AC'(J, R)$ and define two operators A and B on P by

$$Ax(t) = f\left(t, x\left(\mu(t)\right)\right), \; t \in J \quad (3.4)$$

and

$$Bx(t)=\left[\frac{ce^{\lambda t}}{f\left(t_0,x\left(\mu(t_0)\right)\right)}+\int_{t_0}^{t}e^{-\lambda(t-s)}g\left(s,x\left(\sigma(s)\right),\,x'\left(\eta(s)\right)\right)\right],$$
$$t\in J \qquad (3.5)$$

From the continuity of the integral , it follows that A and B define the maps $A,B:P\to P$.

Now by lemma 3.1 the QDE (1.1) is equivalent to the operator equation

$$Ax(t)Bx(t)=x(t),\ t\in J \qquad (3.6)$$

We shall show that the operators A &B satisfy all the condition of theorem 2.1 .This is achieved in the series of following step.

Step I :- A amd B are non-decreasing on *P*. Let $x,y\in P$ be such that $x\geq y$. Then by hypothesis (A),

We obtain

$$Ax(t)=f\left(t,x\left(\mu(t)\right)\right)\geq f\left(t,y\left(\mu(t)\right)\right)$$
$$=Ay(t),$$

for all $t\in J$.

This shows that A is non-decreasing operator on P into P. Similarly using hypothesis (H_2), it is shown that the operator B is also non-decreasing onP into itself.Thus, A & B are non-decreasing positive operators into it self.

Step II :- A is partially bounded &partially *D-Lipsschitz* on P.

Let $x\in p$ be arbitrary. Then by (B_2)

$$\left|Ax(t)\right|\leq f\left(t,x\left(\mu(t)\right)\right)\leq M_f,$$

for all $t\in J$.

Taking supremum over *t*, we obtain

$$\left\|A_x\right\|\leq M_f$$

and so, A is bounded.

This further implies that A is partially bounded on P.

Let $x,y\in p$ be such that $x\geq y$ then

$$\left|Ax(t)-Ay(t)\right|=\left|f\left(t,x\left(\mu(t)\right)\right)-f\left(t,x\left(\mu(t)\right)\right)\right|$$
$$\leq\phi\left(\left|x(t)-y(t)\right|\right)\leq\phi\left(\left\|x-y\right\|\right)$$

for all $x,y\in p$.

Hence A is a partially contraction on *P* which further implies that *A* is a partially continuous on *P*.

Step III :- *B* is a partially continuous on *P*. Let $\{x_n\}_{n\in N}$ be a sequence such that $x_n\to x$ for all $n\in N$. Then by dominated convergence theorem, we have

$$\lim_{n\to\infty}Bx_n(t)=\lim_{n\to\infty}\frac{ce^{-\lambda t}}{f\left(t_0,x\left(\mu(t_0)\right)\right)}+$$
$$\lim_{n\to\infty}\int_{t_0}^{t}e^{-\lambda(t-s)}g\left(s,x_n\left(\sigma(s)\right),\,x_n'\left(n(s)\right)\right)ds$$
$$=\frac{ce^{-\lambda t}}{f\left(t_0,x\left(\mu(t_0)\right)\right)}+\int_{t_0}^{t}e^{-\lambda(t-s)}$$
$$\left[\lim_{n\to\infty}g\left(s,x_n\left(\sigma(s)\right),\,x_n'\left(n(s)\right)\right)ds\right]$$
$$=\frac{ce^{-\lambda t}}{f\left(t_0,x\left(\mu(t_0)\right)\right)}+\int_{t_0}^{t}e^{-\lambda(t-s)}g\left(s,x\left(\sigma(s)\right),\,x'\left(n(s)\right)\right)ds$$
$$=Bx(t)$$

For all $t\in J$.

This shows that Bx_n converges monotonically to Bx pointwise on J .

Next, we will show that $\{Bx_n\}_{n\in N}$ is an equicontinuous sequence of functions in *P*. Let $t_2,t_2\in J$ with $t_1<t_2$.

Then

$$\left|Bx_n(t_2)-Bx_n(t_1)\right|\leq\left|\frac{ce^{-\lambda t_2}}{f\left(t_0,x\left(\mu(t_0)\right)\right)}-\frac{ce^{-\lambda t_2}}{f\left(t_0,x\left(\mu(t_0)\right)\right)}\right|$$
$$+\left|\int_{t_0}^{t_2}e^{-\lambda(t_2-s)}g\left(s,x_n\left(\sigma(s)\right),\,x_n'\left(\eta(s)\right)\right)ds\right.$$
$$\left.-\int_{t_0}^{t_2}e^{-\lambda(t_2-s)}g\left(s,x_n\left(\sigma(s)\right),\,x_n'\left(\eta(s)\right)\right)ds\right|$$
$$+\left|\int_{t_0}^{t_2}e^{-\lambda(t_2-s)}g\left(s,x_n\left(\sigma(s)\right),\,x_n'\left(\eta(s)\right)\right)ds\right.$$
$$\left.-\int_{t_0}^{t_2}e^{-\lambda(t_2-s)}g\left(s,x_n\left(\sigma(s)\right),\,x_n'\left(\eta(s)\right)\right)ds\right|$$
$$\leq\left|\frac{ce^{-\lambda t_1}}{f\left(t_0,x\left(\mu(t_0)\right)\right)}-\frac{ce^{-\lambda t_2}}{f\left(t_0,x\left(\mu(t_0)\right)\right)}\right|$$
$$+\left|\int_{t_0}^{t_2}\right|\left|e^{-\lambda(t_2-s)}\right.$$
$$-e^{-\lambda(t_2-s)}\left|g\left(s,x_n\left(\sigma(s)\right),\,x_n'\left(\eta(s)\right)\right)ds\right|$$

$$+\left|\int_{t_0}^{t_2}\left|g\left(s,x_n\left(\sigma(s)\right),\ x_n'\left(\eta(s)\right)\right)ds\right|\right|$$

Therefore,

$$\le\left|\frac{ce^{-\lambda t_2}}{f\left(t_0,x\left(\mu(t_0)\right)\right)}-\frac{ce^{-\lambda t_2}}{f\left(t_0,x\left(\mu(t_0)\right)\right)}\right|$$

$$+M_g\int_{t_0}^{t_0+a}\left|e^{-\lambda(t_2-s)}-e^{-\lambda(t_2-s)}\right|ds+M_g\left|t_1-t_2\right|$$

$$\to 0\ as\ t_2-t_1\to 0$$

Uniformly for all $n\in N$. This shows that convergence $Bx_n\to Bx$ is uniform and hence *B* is partially continuous on *E*.

Step IV :- B is uniformly partially compact operator on P.

Let *G* be an arbitrary chain in P .We show that $B(G)$ is a uniformly bounded & equicontinuous set in *P*. First we show that $B(G)$ is uniformly bounded. Let $y\in B(G)$ be any element. Then there is an element $x\in G$ be such that $y=Bx$.

Now by hypotheses (H_1)

$$\left|y(t)\right|\le\left|\frac{ce^{-\lambda t_1}}{f\left(t_0,x\left(\mu(t_0)\right)\right)}-\frac{ce^{-\lambda t_2}}{f\left(t_0,x\left(\mu(t_0)\right)\right)}\right|$$

$$+\left|\begin{array}{l}\int_{t_0}^{t_2}e^{-\lambda(t_1-s)}g\left(s,x_n\left(\sigma(s)\right),\ x_n'\left(\eta(s)\right)\right)ds\\-\int_{t_0}^{t_2}e^{-\lambda(t_2-s)}g\left(s,x_n\left(\sigma(s)\right),\ x_n'\left(\eta(s)\right)\right)ds\end{array}\right|$$

$$\le\left|\frac{x_0}{f\left(t_0,x\left(\mu(t_0)\right)\right)}\right|+\int_{t_0}^{t_{a+a}}\left|g\left(s,x_n\left(\sigma(s)\right),\ x_n'\left(\eta(s)\right)\right)ds\right|$$

$$\le\left|\frac{x_0}{f\left(t_0,x\left(\mu(t_0)\right)\right)}\right|+M_g a=r\,.$$

for all $t\in J$. Taking supremum over t we obtain $\|y\|=\|B\ x\|\le r$, for all $r\in B(G)$. Hence $B(G)$ is a uniformly bounded subset of *P*. Moreover $\|B\quad(G)\|\le r$, for all chains *G* in *P*.

Hence *B* is a uniformly partially bounded operator on *E*.

Now we will show that $B(G)$ is an equicontinuous set in *E*.

Let $t_1,t_2\in J$ with $t_1<t_2$.

Then for any $y\in B(G)$ one has

$$\left|y(t_2)-y(t_1)\right|=\left|Bx(t_2)-Bx(t_1)\right|$$

$$=\left|e^{-\lambda t_2}\int_{t_0}^{t_1}e^{-\lambda s}g\left(s,x\left(\sigma(s)\right),\ x'\left(\eta(s)\right)\right)ds\right.$$

$$\left.-e^{-\lambda t_1}\int_{t_0}^{t_1}e^{-\lambda s}g\left(s,x\left(\sigma(s)\right),\ x'\left(\eta(s)\right)\right)ds\right|$$

$$\le\left|\frac{ce^{-\lambda t_1}}{f\left(t_0,x\left(\mu(t_0)\right)\right)}-\frac{ce^{-\lambda t_2}}{f\left(t_0,x\left(\mu(t_0)\right)\right)}\right|$$

$$+M_g\int_{t_0}^{t_1}\left|e^{-\lambda(t_1-s)}-e^{-\lambda(t_2-s)}\right|ds+M_g\left|t_1-t_2\right|$$

$$\to 0\ as\ t_2-t_1\to 0$$

Uniformly for all $y\in B(G)$.

Hence $B(G)$ is an equicontinuous subset of *P*. Now $B(G)$ is a uniformly bounded &equicontinuous set of functions in *P*, so it is compact. Consequently, B is a uniformly partially compact operator on *E* into itself.

Step V :- w satisfies the operator inequality $w\le AwBw$.

By hypothesis (H_3), the QDE (1.1) has a lower solution *u* defined on *J*. Then, we have

$$\frac{d}{dt}\left[\frac{w(t)}{f\left(t,x\left(u(t)\right)\right)}\right]+\lambda\left[\frac{w(t)}{f\left(t,x\left(u(t)\right)\right)}\right]$$

$$\le g\left(t,w\left(\sigma(t)\right),w'\left(\eta(t)\right)\right)\qquad(3.7)$$

$$x(t_0)=x_0,\forall\ \ a\ \in J$$

Multiplying the above inequality(3.7) by the integrating factor $e^{\lambda t}$, we obtain

$$\left[e^{\lambda t}\frac{w(t)}{f\left(t,x\left(u(t)\right)\right)}\right]\le e^{\lambda t}\,g\left(t,w\left(\sigma(t)\right),w'\left(\eta(t)\right)\right)\qquad(3.8)$$

or all $t\in J$. A direct integration of (3.8) from t_0 to t yields

$$u(t)\le\left|f\left(t,w\left(u(t)\right)\right)\right|\left(\frac{ce^{-\lambda t}}{f\left(t_0,x\left(u(t_0)\right)\right)}\right.$$

$$\left.+\int_{t_0}^{t}e^{-\lambda(t-s)}\,f\left(s,w_n\left(\sigma(s)\right),w_n'\left(\eta(s)\right)\right)ds\right)\qquad(3.9)$$

for all $t\in J$. From definitions of the operators A and B it follows that

$w(t)\le Aw(t)Bw(t)$ for all $t\in J$.

Hence $w \le AwBw$.

Step VI :- Finally, the *D-function* ϕ of the operator A satisfies the inequality given in hypothesis(iv) of theorem 2.1 ,viz.

$$M\psi_A(r) \le \left(\left| \frac{x_0}{f(t, x(\mu(t_0)))} \right| + M_g a \right) \phi(r) < r$$

for all $r > 0$.

Thus A and B satisfy all the conditions of theorem 2.1 and we apply it to conclude that the operator equation $AxBx = x$ has a solution. Consequently the integral equation and the QDE (1.1) has a solution x defined on J. Furthermore, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations definitions by (3.3) converges monotonically to x. This completes the proof.

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