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Applications of Nonlinear Boundary Value Problem to the Heat Conduction Equation using Fourier Integrals

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Abstract— Old Fashioned definitions of mathematics, as a science of numbers and magnitude are no longer valid. Now-a-days it has many applications in many branches for solving physical problem including geometrical configuration**.** Partial differential equation plays an important role in mathematics. The aim of this paper is to present various types of partial differential equations with applications. Some partial differential equations almost entirely to a kind of boundary value problems which enters modern applied mathematics at every term have been included and solved by using Fourier transform. Laplace transform and separation of variables method. I have explained the physical problems on the conduction of heat and solved by different methods. Fourier series and its applications in Boundary value problem have also been discussed

Index Terms Partial Differential Equation, Homogeneity of PDE's, Fourier series and its application to the heat conduction problem.

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1 INTRODUCTION

Partial differential equations are used to formulate, and thus aid the solution of, problems involving functions of several variables; such as the propagation of sound or heat, wave, fluid flow, and elasticity. In this paper we shall discuss such equations as they commonly arise in applied mathematics. I shall begin by defining what a partial differential equation is and then examining in some detail the derivation from physical principles of several important partial differential equations, and knowing the forms of most common occurrence. It can have solutions involving arbitrary functions. We shall investigate methods of solution and their application to specific problems.

2.1 Partial differential equation:

A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a partial differential equation.

$$\frac{\partial v}{\partial x}+y^2\frac{\partial v}{\partial y}=0$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

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2.2 Linear partial differential equation:

A partial differential equation is called linear if every dependent variable and every derivative involved occurs in the first degree only and no products of dependent variables or derivatives occur. For Example

$$f(x,y)\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+3F=0$$

Here the derivatives occur only in the first power only.

2.3 Nonlinear partial differential equation:

A differential equation which is not linear is called nonlinear.

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial x\,\partial y}+\left(\frac{\partial F}{\partial y}\right)^2=F^2$$

2.4 Homogeneity of PDES:

We now consider an important case of second order linear partial differential equations so called homogeneous linear equations of the second order with constant coefficients.

$$a\frac{\partial^2 F}{\partial x^2}+b\frac{\partial^2 F}{\partial x\,\partial y}+c\frac{\partial^2 F}{\partial y^2}=0$$

Where a, b, c are constants. The ward homogenous refers to

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the fact that all terms in this equation contain derivatives of the same order (the second).

An example of non-homogeneous linear equation is

$$\frac{\partial F}{\partial t} = \alpha^2 \frac{\partial^2 F}{\partial x^2}$$

2.5 Types of second order PDEs:

There are three types of second order PDEs whose solutions have distinctly different behaviors. These are parabolic, hyperbolic and elliptic PDEs. Parabolic equations are diffusion like, while hyperbolic equations are typified by the wave equation. Laplace's equation belongs to the category of elliptic PDEs, namely

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \text{: Parabolic type}$$

$$\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2} \qquad \text{: Hyperbolic type}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{: Elliptic type}$$

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

where A,B,C,D,E,F,G are given functions of x and y which is second order constant coefficient partial differential equation in two variables(x and y).This equation is said

Parabolic if B2-4AC=0
Hyperbolic if B2-4AC>0
Elliptic if B2-4AC<0

The equation is homogeneous if G=0.

Example :

The equation

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial u}{\partial y} = 0 \qquad (1)$$

is parabolic, since A = 1, B = C = 0, and $B^2 - \overline{4AC} = 0$. Equation (1) is a special case of the so-called one –dimensional heat equation, which is satisfied by the temperature at a point of a point of a homogeneous rod.

3 Fourier Series and its Application in Boundary Value Problem

Fourier series are often used to represent the response of a system to a periodic input and the response often depends directly on the frequency content of the input. The series are used in a wide variety of such physical situations including the vibrations of a finite strings, the scattering of light has a diffraction grating and transmission of a input signal by an electric signal.

Fourier series:

Let f(x) be defined in the interval (-L,L) and outside of this interval by f(x+2L)=f(x).i.e. it is assumed that f(x) is periodic with period 2L.

The Fourier series or Fourier expansion corresponding to f(x) is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where a_n, b_n are called Fourier co- efficients and given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\mathrm{Cos}\,\frac{n\pi}{L}\,xdx$$

n=0,1,2…………

$$a_n = \frac{1}{L}\int_{C}^{C+2L} f(x)\mathrm{Cos}\,\frac{n\pi}{L}\,xdx,$$

$$b_n = \frac{1}{L}\int_{C}^{C+2L} f(x)\mathrm{Sin}\,\frac{n\pi}{L}\,xdx$$

4 Application to the heat conduction problem:

PDE: $u_t=a^2u_{xx}$, $0<x<L$, $t>0$ **(1)**
BCs: $u(0,t)=0$, $u(L,t)=0$, $t>0$ **(2)**
IC: $u(x,0)=f(x)$, $0<x<L$ **(3)**

In particular, consider the case, f(x)=200

We found that the general solution to the PDE which satisfies BCs is given by

$$U(x,t) = \sum_{n=1}^{0} T_n(0) e^{-(\frac{n\pi}{l})^2 a^2 t}\, Sin\frac{n\pi}{L}x\ldots\ldots..0<x<L\ldots..(a)$$

where the constants $T_n(0)$ are yet to be determined from IC. Setting t=0 in (a) and setting u(x,0)=f(x),we arrive at

$$U(x,t) = \sum_{n=1}^{0} T_n(0) Sin\frac{n\pi}{L}x\ldots\ldots\ldots\ldots..0<x<L\ldots\ldots\ldots\ldots(b)$$

Therefore $T_n(0)$ is the Fourier sine co-efficient of f(x) and is given by (a) as

$$T_n(0) = a_n = \frac{2}{L}\int_0^L f(x) Sin\frac{n\pi}{L}xdx$$

for f(x)=200,we know that

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$$a_n = \begin{cases} 0 \quad if\ n\,is\,even \\ \dfrac{800}{n\pi} if\ n\,is\,odd \end{cases}$$

thus finally, the solution of the above PDE problem is

$$\sum_{n=0}^{\infty} \frac{800}{\pi} \frac{1}{2k-1} e^{-(2k-1)^2 (\frac{t}{t_e})} \sin \frac{2k-1}{L} nx ......0 < x < L.........(c)$$

Where

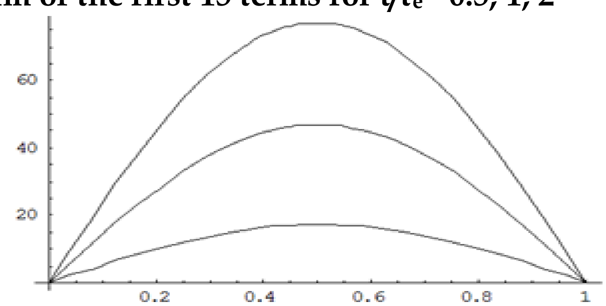
$$t_e = (\frac{L}{a\pi})^2 \quad and \quad N \to \infty$$

the solution in (c) is plotted in fig.(1) for different values of t/t_e . it turns out that unless t/t_e is very small, only a few terms are needed in the sum in (c).in fig.(2),we show that the solution can be represented to a high degree of accuracy by the first two only now do we have the actual coefficients a_1 and a_3 calculated explicitly.

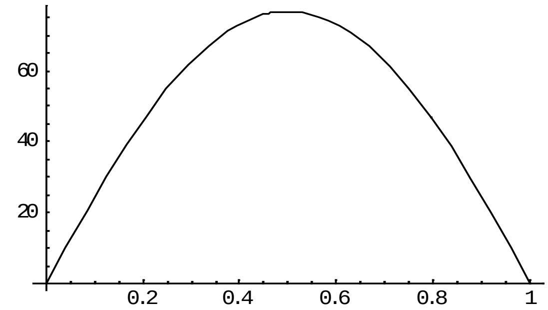
$$U(x,t) \cong \frac{800}{\pi} e^{-t/t} \sin \frac{\pi x}{l} + \frac{800}{3\pi} e^{-t/t} \sin \frac{3\pi x}{l} \quad for\, t \geq t_e$$

Only Now do we have the actual coefficients a_1 and a_3 calculated explicitly

Sum of the first 15 terms for t/t_e =0.5, 1, 2



Sum of first 2 terms when t/t_e=.5



Example:

Represent f(x)=200 in the form of a Fourier sine series over te interval 0<x<L:

$$f(x) = \sum_{n=1}^{\infty} (a_n \sin \frac{n\pi x}{L}), \quad 0 < x < L$$

the Fourier coefficients, a_n, are given by

$$a_n = \frac{2}{L} \int_0^L Lf(x) \sin \frac{n\pi}{L} xdx$$
$$= \frac{2}{L} \int_0^L 200 \sin \frac{n\pi}{L} xdx$$
$$= \begin{cases} 0 \quad if\ n\,is\,even \\ \dfrac{800}{n\pi} if\ n\,is\,odd \end{cases}$$

so we say that in 0<x<L, we have

$$200 = \frac{800}{\pi} [\frac{\sin(\pi x/L)}{1} + \frac{\sin(3\pi x)/L}{3} + \frac{\sin(5\pi x)/L}{5} + ...] \quad (1)$$

(1) is rather strange; it says that a constant, 200 can be represented by a sum of sine. let us see what we will get if we add up the sine in the right hand side of (1).in fig 1,we plot the term in the sum (i.e (800/п)sin(п x/L)).in fig 2,we plot two terms, (i.e(800/ п)[sin(п x/L)+sin(3 п x/L)/3].in fig 3,we plot 3 terms, etc. By the time we have included enough terms, we see that the r.h.s. of (1) approaches the constant value of 100 in the interior of the interval,0<x<L.(Near the edges x=0 and x=L),the oscillations get increasingly confined to the edges, where the sum of sine's tries very hard to approach 100 in the interior of the domain,0<x<L, while being identically zero at x=0 and x=L.A discontinuity created at the edges. there is also the so called Gibbs phenomenon present near the edges, where just within the boundaries; there is an overshoot of the true value of 200,by 36%

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<u>Figure:</u>

Sum of first 1 term

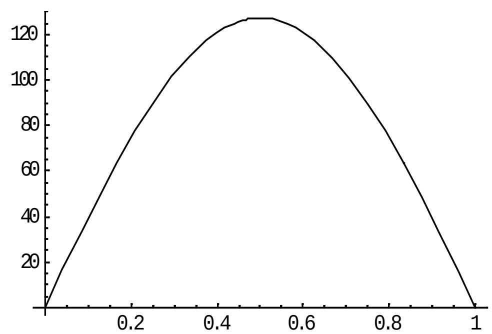


fig.1: plot of the first term in the Fourier sine expansion of 200

Sum of first 2 term

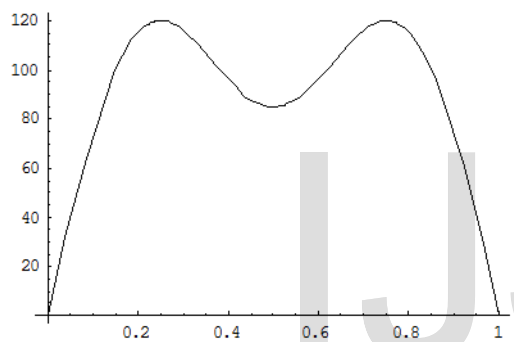


fig.2: plot of the first 2 term in the Fourier sine expansion of 200

Sum of first 3 term

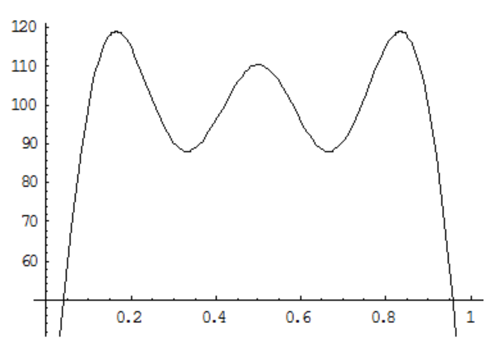


fig.3: plot of the first 3 term in the Fourier sine expansion of 200

Sum of first 6 term

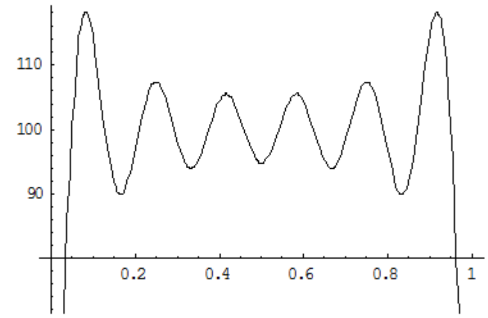


fig.4: plot of the first 6 term in the Fourier sine expansion of 200

Sum of first 15 term

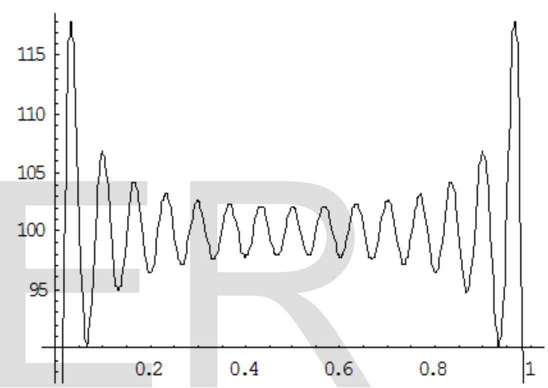
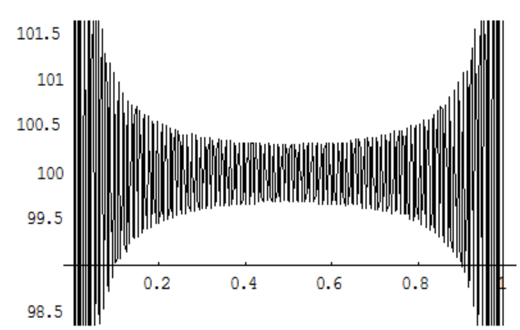


fig.5: plot of the first 15 term in the Fourier sine expansion of 200

Sum of first 100 term



5 Comments:

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What the example demonstrates is that the Fourier sine series can indeed represent f(x) in the interval indicated. We can do this for other functions, and will find that the Fourier sine series may be the one we have just done, f(x)=constant,0<x<L. This is because the sine all go to zero at x=0 and x=L, but they have add up to a nonzero constant slightly inside the boundaries. Many more terms in the sum are required to create this near discontinuity. It is reasonable to expect that the sine can represent a function which blows up (i.e. attain infinite values) in the domain 0<x<L. Such unphysical functions are excluded in our consideration. The following mathematical result can be stated in a theorem (a more general form is called Dirichlet theorem): The Fourier sine series representation of f(x) converges to f(x) for is point x in a domain where f(x) is continuous. At those points where f(x) jumps, the series converges to a value which is average of the left and right hand limits of f(x) a those points, where f(x) is discontinuous.

CONCLUSION

Partial differential equations forms the starting point for the study of Conduction of heat, transmission of electric waves also in physics & physical chemistry .A sufficiently through study of this problem would lead us eventually into every plane of Classical mathematics almost contact with special function of modern quantum theory .The form of the problem that we insist in this paper, will help to solve any sort of practical problem in physical Science..

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