# Incompressible Viscoelastic Flow of a Generalised Second Grade Fluid in a Porous Medium between Two Infinite Parallel Plates

Dhiman Bose, Uma Basu

**Abstract**— The flow of a viscous incompressible fluid of generalized second grade type between two infinite parallel plates embedded in a porous medium has been studied here. An analytical solution for the velocity field has been obtained utilizing integral transforms technique in series form in terms of Mittage-Leffler function. The affect of fractional calculus and porosity parameters on the velocity field have been illustrated graphically. The two limiting cases have been discussed as the results of the velocity field of the generalized second grade fluid.

**Index Terms**— Fractional derivative, finite Fourier sine transform, Laplace transform, Mittage-Leffler function, porous medium, Riemann-Liouville fractional calculus operator, second grade fluid

## 1 Introduction

The viscoelastic fluid flow between two parallel plates through a porous medium have attracted the attention of number of researchers due to its vast applicability in different fields such as filtration and purification of crude oil, petroleum industry, agriculture engineering etc. But it is difficult to suggest a single model which exhibits all the properties of viscoelastic fluids. For this reason many models of constitutive equations have been proposed by Mathematicians. During the last few decades fractional calculus has encountered much success in describing the viscoelasticity. In fractional calculus approach the time derivative of integer order in the constitutive equation is replaced by Riemann-Liouville fractional calculus operator. Bose et al [1] have studied unsteady incompressible flow of a generalized Oldroyed-B fluid between two infinite parallel plates. Rajagopal and Gupta [2] have investigated an exact solution for the flow of a non-Newtonian fluid past an infinite porous plate. Tan and Xu [3] have considered unsteady flows of a generalized second grade fluid with fractional derivative model between two parallel plates. Bose and Basu [4] have studied incompressible viscoelastic flow of a generalized Oldroyed-B fluid through porous medium between two infinite parallel plates in a rotating system. Fetecau et al [5] have studied unsteady flow of a second grade fluid between two side walls perpendicular to a plate. Khan et al [6] discussed exact solutions for some oscillating flows of a second grade fluid with a fractional derivative model.

In the present paper the viscoelastic flow of a generalized second grade fluid through a porous medium has been considered. The exact solution for the velocity field is obtained with the help of integral transform technique in series form in terms of well known Mittage-Leffler function. The affect of fractional calculus parameter and porosity parameter on the velocity field have been illustrated graphically.


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## 2. Generalised Fluid Model and Basic Equations

The extra stress tensor $\boldsymbol{T}$ for second grade fluid is given by the constitutive equation

$$\boldsymbol{T} = -\bar{p}\boldsymbol{I} + \mu \boldsymbol{A_1} + \alpha_1 \boldsymbol{A_2} + \alpha_2 \boldsymbol{A_1^2} \tag{1}$$

where, $\boldsymbol{T}$ is the Cauchy stress tensor, $\bar{p}$ is the hydrostatic pressure, $\boldsymbol{I}$ is the identity tensor, $\mu$ is the coefficient of viscosity, $\alpha_1$ and $\alpha_2$ are normal stress moduli, $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ are kinematical tensors defined by

$$\boldsymbol{A_1} = \nabla V + (\nabla V)^T \tag{2}$$

$$\boldsymbol{A_2} = \frac{d\boldsymbol{A_1}}{dt} + \boldsymbol{A_1}\nabla V + (\nabla V)^T \boldsymbol{A_1} \tag{3}$$

where $\frac{d}{dt}$ is the material derivative, $\nabla V$ is the velocity gradient. and $\mu, \alpha_1, \alpha_2$ satisfy the following conditions

$$\mu \geq 0, \alpha_1 \geq 0 \text{ and } \alpha_1 + \alpha_2 = 0 \tag{4}$$

For a generalized second grade fluid the equation (1) remains same but $\boldsymbol{A_2}$ is defined by

$$\boldsymbol{A_2} = D_t^{\beta}\boldsymbol{A_1} + \boldsymbol{A_1}\nabla V + (\nabla V)^T \boldsymbol{A_1} \tag{5}$$

$D_t^{\beta}$ is the Riemann-Liouville fractional calculus operator and defined by

$$D_t^{\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{d}{dt} \frac{f(\tau)}{(t-\tau)^{\beta}} d\tau \quad (0 < \beta < 1) \tag{6}$$

where $\Gamma(.)$ is Gamma function

If $\beta = 1$ then equation (5) reduces to equation (3) and the constitutive relationship corresponds to the case for ordinary second grade fluid. If $\alpha_1 = 0, \beta = 0$, the constitutive relationship corresponds to the case for Newtonian fluid.

The Equation of motion in absence of body forces is given by

$$\rho \frac{D\boldsymbol{V}}{Dt} = \nabla . \boldsymbol{T} \tag{7}$$

where $\rho$ is the fluid density and D/Dt is the material derivative.

The equation of continuity for incompressible fluid is given by

$$\nabla . \boldsymbol{V} = 0 \tag{8}$$

## 3 Mathematical Formulation And Solution

Suppose that a generalized second grade fluid be initially at rest and occupies the space between two infinite parallel plates at a distance $d$ apart embedded in porous medium. As $t > 0$ the lower plate begins to move in its plane with velocity U and due to the shear the fluid gradually moves. The x- and y- axes are taken along and perpendicular to thedirection of the plates respectively and z-axis is taken perpendicular to the xy-plane.

Since the dimensions of the plates along the x- and z-coordinate directions are infinite, all the quantities related to the motion are functions of y and t only. For unidirectional flow the velocity field will be of the form

$$\boldsymbol{V} = u(y,t)\hat{\boldsymbol{i}} \tag{9}$$

where $u$ is the velocity in the x-coordinate direction and $\hat{\boldsymbol{i}}$ denotes the unit vector in the x-coordinate direction.

Substituting the expression of velocity $V$ from equation (9) in Equations (1), (2) and (5) we obtain the shear stress components

$$T_{xy} = T_{yx} = \mu \frac{\partial u(y,t)}{\partial y} + \alpha_1 D_t^\beta \frac{\partial u(y,t)}{\partial y} \tag{10}$$

and $T_{xx} = T_{yy} = T_{zz} = T_{yz} = T_{zx} = 0$

Substituting the expressions of $T_{xy}$ and the velocity from equations (9) and (10) respectively in Eqn (7) we get

$$\frac{\partial u(y,t)}{\partial t} = \left(\nu + \alpha\, D_t^\beta\right)\frac{\partial^2 u(y,t)}{\partial y^2} - \frac{\nu u(y,t)}{K} \tag{11}$$

where $\nu = \mu/\rho$ is the kinematic viscosity , $\alpha = \alpha_1/\rho$ , $K$ is the permeability of the porous medium.

The boundary conditions can be written as

$$u(0,t) = U, u(d,t) = 0 \quad (t > 0) \tag{12}$$

The initial condition is given by $u(y,0) = 0 \quad (0 < y < 1)$

Now we introduce the non-dimensional variables

$$u^* = \frac{u}{U}\,, y^* = \frac{y}{d}\,, t^* = \frac{\nu t}{d^2} \tag{13}$$

Then the governing equation in terms of the non-dimensional variables can be written as (Dropping the sign '*'for simplicity)

$$\frac{\partial u}{\partial t} = \left(1 + \eta\, D_t^\beta\right)\frac{\partial^2 u}{\partial y^2} - \frac{1}{\sigma^2}u \tag{14}$$

where $\eta = \alpha \frac{\mu^{\beta-1}}{(\rho d^2)^\beta}$ and $\sigma^2 = \frac{K}{d^2}$

We consider the transformation $v(y,t) = u(y,t) - (1-y)$ (15)
$t > 0$

Then the Eqn (14) takes the form

$$\frac{\partial v}{\partial t} = \left(1 + \eta\, D_t^\beta\right)\frac{\partial^2 v}{\partial y^2} - \frac{1}{\sigma^2}[(1-y) - v(y,t)] \tag{16}$$

The boundary and initial conditions in terms of new variable $v(y,t)$ can be written as

$$v(0,t) = 0 = v(1,t)$$
$$\text{and} \quad v(y,0) = y - 1 \tag{17}$$

Multiplying bothsides of the Eqn.(16) by $\sin n\pi y$ and then integrating bothsides of with respect to y from 0 to 1 and utilizing boundary conditions we obtain

$$\frac{\partial}{\partial t}V_s(n,t) = -(n\pi)^2\left(1 + \eta D_t^\beta\right)V_s(n,t) - \frac{1}{\sigma^2}\left(\frac{1}{n\pi} + V_s(n,t)\right) \tag{18}$$

where $V_s(n,t)$ is finite Fourier sine transformation defined by

$$V_s(n,t) = \int_0^1 v(y,t)\sin n\pi y\, dy \qquad (n = 1,2,3,\ldots\ldots)$$

Again Taking Laplace transformation of bothsides of the Eqn.(18) and using $V_s(n,0) = -\frac{1}{n\pi}$ we get

$$\bar{V}_s(n,p) = -\frac{1}{n\pi p}\frac{1 + \frac{1}{\xi}}{1 + \frac{1 + (n\pi)^2\sigma^2}{\xi}} \tag{19}$$

where $\xi = \sigma^2 p^\beta\left(p^{1-\beta} + \eta(n\pi)^2\right)$ and '$p$'is the Laplace transform parameter

In order to avoid the lengty procedure of contour integrals and residues we rewrite the Equn. (19) as

$$\bar{V}_s(n,p) = -\frac{1}{n\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sigma^{2k}}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}(n\pi\sigma)^{2m}\frac{p^{-\beta k-1}}{[p^{1-\beta} + \eta(n\pi)^2]^k}$$
$$-\frac{1}{n\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sigma^{2(k+1)}}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}(n\pi\sigma)^{2m}\frac{p^{-\beta k-\beta-1}}{[p^{1-\beta} + \eta(n\pi)^2]^{k+1}} \tag{20}$$

Where $\bar{V}_s(n,p) = \int_0^\infty e^{-pt}\,V_s(n,t)dt$ is the Laplace transformation of $V_s(n,t)$.

Applying the inversion formulae term by term for the Laplace transform, Eqn.(20) yields

$$V_s(n,t) = -\frac{1}{n\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sigma^{2k}}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{(n\pi\sigma)^{2m}}{\Gamma(k)}t^k E_{1-\beta,2+\beta(k-1)}^{(k-1)}\left(-\lambda t^{1-\beta}\right)$$
$$-\frac{1}{n\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sigma^{2(k+1)}}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{(n\pi\sigma)^{2m}}{\Gamma(k+1)}\; t^{k+1}E_{1-\beta,2+\beta k}^{(k)}\left(-\lambda t^{1-\beta}\right) \tag{21}$$

where $\lambda = \eta(n\pi)^2$ and $E_{\alpha,\gamma}(t) = \sum_{k=0}^{\infty} t^k/\Gamma(\alpha k+\gamma)$ denotes generalized Mittage-Leffler function. Here we have utilized the following property

$$L^{-1}\left\{\frac{n!\,p^{\delta-\mu}}{(p^\delta \mp c)^{n+1}}\right\} = t^{\delta n+\mu-1}E_{\delta,\mu}^{(n)}\left(\pm ct^\delta\right) \tag{22}$$
$$\left(Re(p) > |c|^{\frac{1}{\delta}}\right)$$

Using inversion formula for finite Fourier sine transformation we obtain from Eqn.(21)

$$u(y,t) = 1 - y - 2\sum_{n=1}^{\infty}\frac{\sin n\pi y}{n\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sigma^{2k}}\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{(n\pi\sigma)^{2m}}{\Gamma(k)}t^k$$
$$\times E_{1-\beta,2+\beta(k-1)}^{(k-1)}\left(-\lambda t^{1-\beta}\right) - 2\sum_{n=1}^{\infty}\frac{\sin n\pi y}{n\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{\sigma^{2(k+1)}}$$
$$\times\sum_{m=0}^{k}\frac{k!}{m!\,(k-m)!}\frac{(n\pi\sigma)^{2m}}{\Gamma(k+1)}\; t^{k+1}E_{1-\beta,2+\beta k}^{(k)}\;\left(-\lambda t^{1-\beta}\right) \tag{23}$$

where $\lambda = \eta(n\pi)^2$

## 4 Limiting Cases

Case I If $\beta = 1$ the equation of motion (14) takes the form

$$\frac{\partial u}{\partial t} = \left(1 + \eta \frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial y^2} - \frac{1}{\sigma^2} u \tag{24}$$

Subject to the boundary and initial conditions given by Eqn.(12) and the case corresponds to Ordinary Second Grade Fluid.
The velocity field is given by

$$u_{OSGD} = 1 - y - 2\sum_{n=1}^{\infty} \frac{\sin n\pi y}{n\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sigma^{2k}} \sum_{m=0}^{k} \frac{k!}{m!\,(k-m)!} \frac{(n\pi\sigma)^{2m}}{\Gamma(k)} t^k \times E_{0,k+1}^{(k-1)}(-\lambda) - 2\sum_{n=1}^{\infty} \frac{\sin n\pi y}{n\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sigma^{2(k+1)}} \times \sum_{m=0}^{k} \frac{k!}{m!\,(k-m)!} \frac{(n\pi\sigma)^{2m}}{\Gamma(k+1)}\, t^{k+1} E_{0,2+k}^{(k)}\,(-\lambda) \tag{25}$$

Case II If $\alpha_1 = 0, \beta = 0$ the equation of motion is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} - \frac{1}{\sigma^2} u \tag{26}$$

subject to the boundary and initial conditions given by Eqn.(12) and it corresponds to Newtonian Fluid.
The velocity field is given by

$$u_{NF} = 1 - y - 2\sum_{n=1}^{\infty} \frac{\sin n\pi y}{n\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sigma^{2k}} \sum_{m=0}^{k} \frac{k!}{m!\,(k-m)!} \frac{(n\pi\sigma)^{2m}}{\Gamma(k)} t^k \times E_{1,2}^{(k-1)}(-\lambda t) - 2\sum_{n=1}^{\infty} \frac{\sin n\pi y}{n\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sigma^{2(k+1)}} \times \sum_{m=0}^{k} \frac{k!}{m!\,(k-m)!} \frac{(n\pi\sigma)^{2m}}{\Gamma(k+1)}\, t^{k+1} E_{1,2}^{(k)}\,\left(-\lambda t^{1-\beta}\right) \tag{27}$$

## 5 Numerical Results And Discussion

In Figure 1 the velocity u is plotted against the distance from the lower plate, y for different values of the fractional calculus parameter β. The fluid velocity increases for higher values of the parameter β. The flow patterns are parabolic in nature. Figure 2 displays the velocity profile u against y for different values of the porosity parameter σ. From the figure it is evident that as σ takes higher values the fluid velocity u decreases. The parabolic flow pattern gradually becomes linear for increasing values of the parameter σ ,that is porosity produces a resistance in the flow field and the fluid velocity decreases linearly towards the upper plate from the lower one for higher values of σ. Figure 3 depicts the velocity profile u against t for different values of the parameter β. As β increases the fluid velocity decreases near the moving lower plate. The fluid velocity curve becomes almost parallel for the Ordinary Second Grade Fluid( that is the Case I we considered earlier)compare to the other velocity curves. Figure 4 depicts the velocity field against time t for different values of the porosity parameter σ. As σ increases the fluid velocity decreases. Hence it is clear that the porosity of the medium produces a resistance in the fluid velocity.

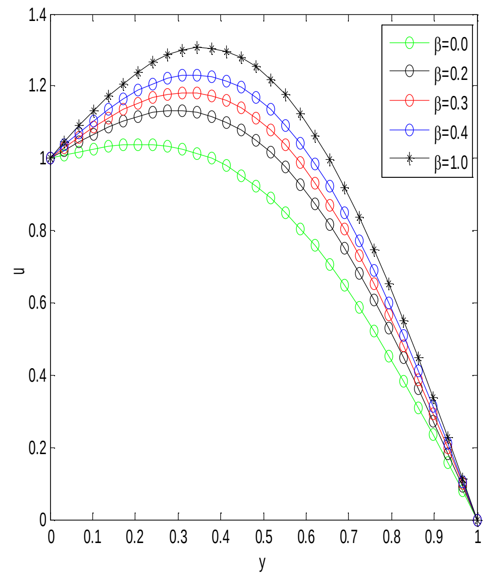


Figure1: The velocity $u$ is depicted against the distance from the lower plate y for different values of fractional calculus parameter β σ =5, t=2, η=0.3

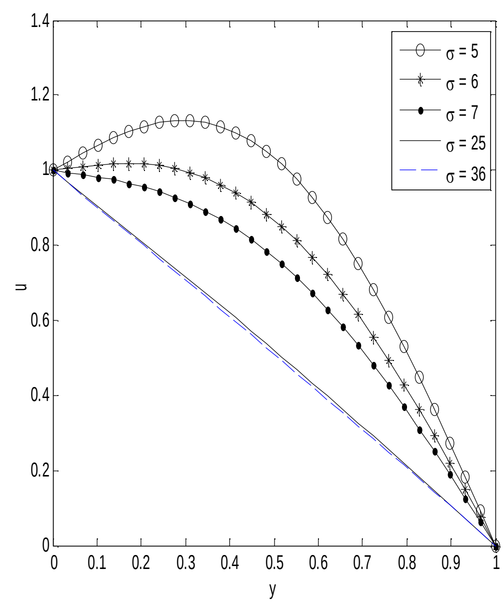


Figure 2: The velocity $u$ is depicted against the distance from the lower plate y for different values of the porosity parameter σ t=2, β=0.2, η=0.3

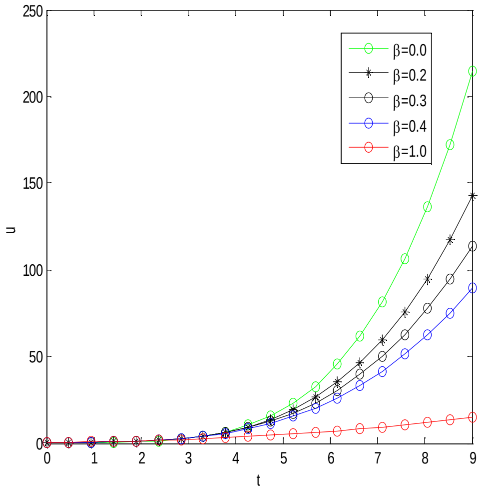


Figure 3: The velocity $u$ is depicted against time t for different values of the fractional calculus parameter β. σ=5, η=0.3, y=0.4

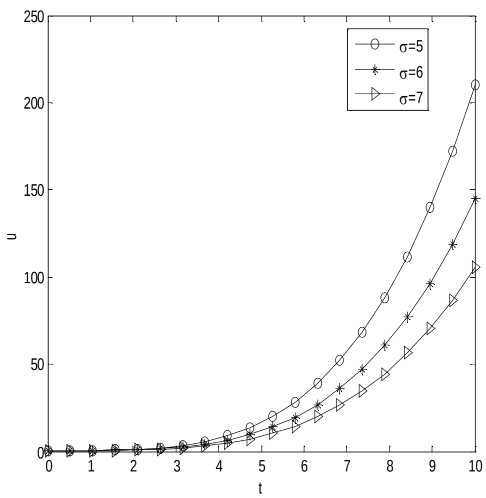


Figure 4: The velocity $u$ is depicted against time t for different values of the porosity parameter σ. β=0.2, η=0.3, y=0.4

## 6 Conclusions

The incompressible viscoelastic flow of a generalized second grade fluid in a porous medium between two infinite parallel plates is studied here. The exact solution for the velocity field is obtained in series form. The affect of the fractional calculus parameter β and the porosity parameter σ have been discussed graphically. The expressions of the velocity for two cases with $\beta = 1$ and $\alpha_1 = \beta = 0$ are derived as limiting ones from the expression of the velocity field for the generalized second grade fluid.

## 7. Acknowledgement

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## 8. References

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